The Manifold Beauty of Piano-hinged Dissections

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Abstract

A geometric dissection is a cutting of one geometric figure into pieces that we can rearrange to form another. For some dissections, it is possible to hinge the pieces together, so that we can flip the pieces one way on the hinges to form one figure, and flip them another way to form the other figure. When the hinge connects two pieces along a shared edge in both target figures, the movement corresponds to a folding. We call such dissections *piano-hinged dissections.* This paper explores design methods and properties of piano-hinged dissections.

1. Introduction

A *geometric dissection* is a cutting of a geometric figure into pieces that can be rearranged to form another figure [5, 11]. For two-dimensional figures, they are striking demonstrations of the equivalence of area, reaching back to the classical Greek era and the zenith of Arabic-Islamic culture. Dissections have become increasingly popular over the last century, after Sam Loyd and Henry Ernest Dudeney first included them in their newspaper and magazine puzzle columns [1, 12].

Some dissections have a nifty property, resulting from hinging the pieces together in a certain way. We can then form one figure by sweeping the pieces one way on the hinges, and the other figure when we sweep them the other way. Almost a century ago, Dudeney [2] introduced perhaps the best-known such dissection, of an equilateral triangle to a square. Under the spell of such hinged dissections, I wrote a whole book on the subject [6].

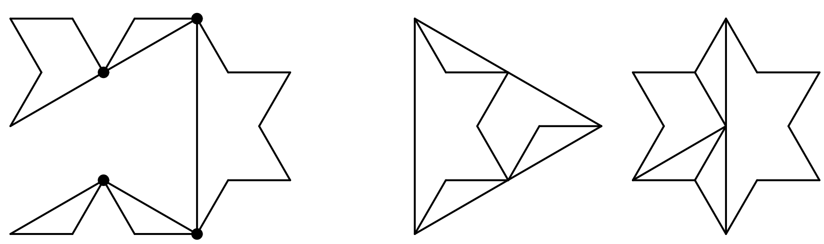

Figure 1: Hinged dissection of an {6/2} to a triangle

Figure 1 illustrates a hinged dissection of a hexagram to an equilateral triangle. The dissection without hinges is on the right. It was discovered independently by Geoffrey Mott-Smith [13] and Harry Lindgren [11]. Apparently neither realized that the pieces could hinge, as on the left.

Figure 1 is certainly lovely, and yet model-builders will quickly note that it does not lead to a very durable working model. It would be much much more stable if the hinges were along edges, rather than at points. Such hinges are called *piano hinges.*

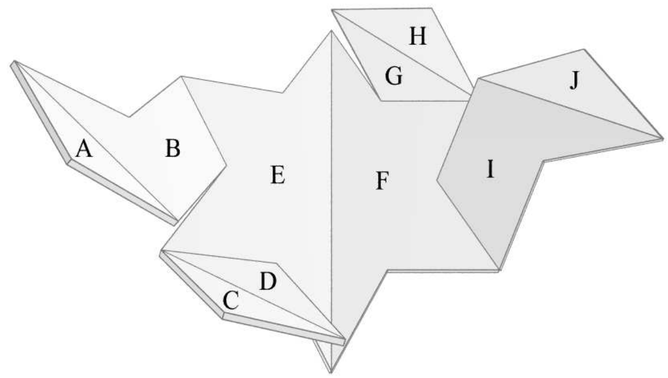

Figure 2: Perspective view of an unfolded assemblage for an equilateral triangle to a hexagram

Suppose that we treat each polygon as a shallow prism separated into two levels which we would dissect further into flat pieces. A piano-hinge could bring two flat pieces that are side-by-side on the same level to be one on top of the other. This hinging motion is essentially what you get when you fold a piece of paper. Thus people who enjoy origami, the Japanese art of paper folding, may also enjoy piano-hinged dissections.

My first such dissection was of a 10-piece dissection of a hexagram to an equilateral triangle (Figures 2 and 3). In Figure 3, I indicate a piano hinge that connects a piece on the top level to a piece on the bottom level by a line of dots next to the hinge line on each of the two levels. I indicate a piano hinge between two pieces on the same level, by a true dotted line segment. In Figure 3, piece E on the top level of the triangle hinges with piece F on the bottom level. In the hexagram, they still share a hinge, but both now sit side by side on the bottom level.

Piano-hinged dissections appear in such remarkable variety that it was easy to write two articles [7, 9] and also a book [8], from which I have excerpted this article.

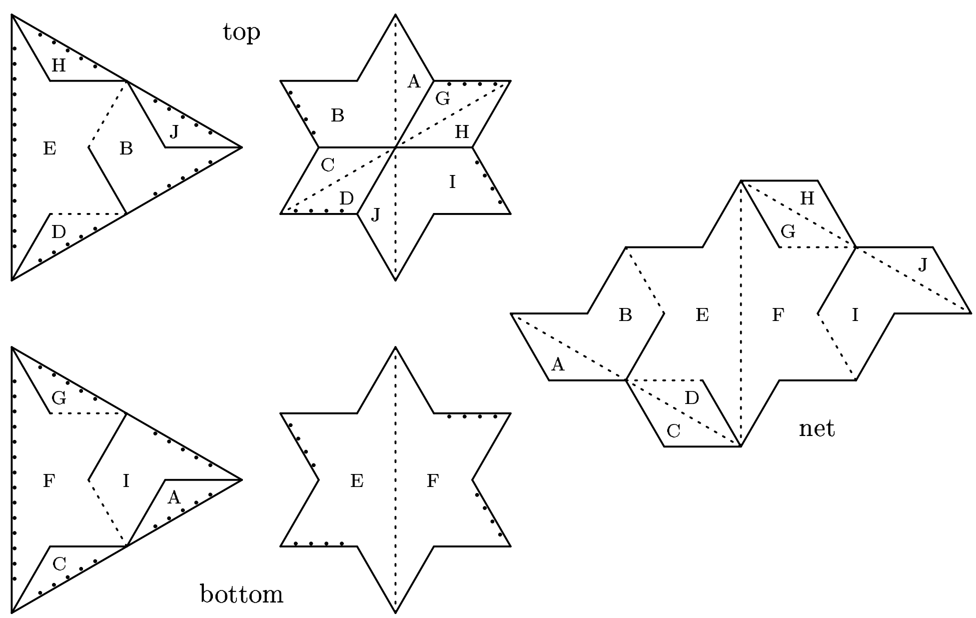

Figure 3: Folding dissection of a hexagram to a triangle

2. Piano-hinged Dissections and Their Properties

There is a simple dissection of three equal triangles to a larger triangle, described by Plato in his *Timaeus*. It is derivable by overlaying tessellations of triangles of the corresponding sizes. We can produce a piano-hinged variation of this dissection (Figure 4) by using the unhinged version on each of the two levels and then hinging the pieces appropriately. A perspective view of the three identical assemblages is in Figure 5.

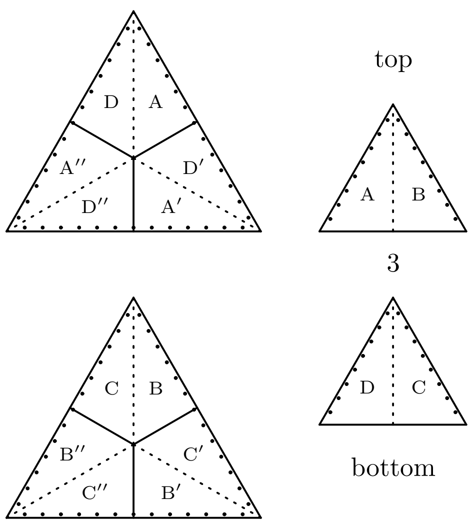

Figure 4: Piano-hinged three triangles to one

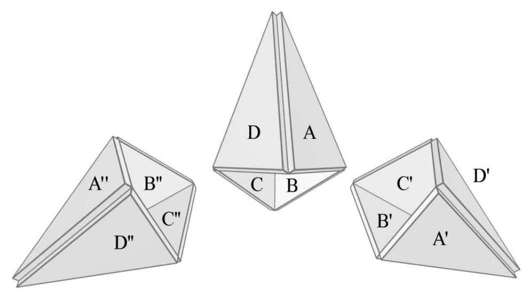

Figure 5: View of the assemblages

A dissection is *cyclicly hinged* if we can remove one of the hinges without disconnecting the pieces. There are several different ways to cyclicly hinge pieces. The first, a *vertex-cyclic hinging*, has four or more pieces that touch at a common vertex, and each piece is hinged with its predecessor and successor on the cycle. Let Σ be the sum of the angles that meet at the vertex. If $\Sigma < 360°$, then the vertex-cyclic hinging is *cap-cyclic*. In Figure 5, each assemblage is cap-cyclicly hinged.

The case when $\Sigma = 360°$ is rather simple and is omitted here. Note that there is an alternative way to piano-hinge the pieces in Figure 4, which falls into this case. Explore that on your own.

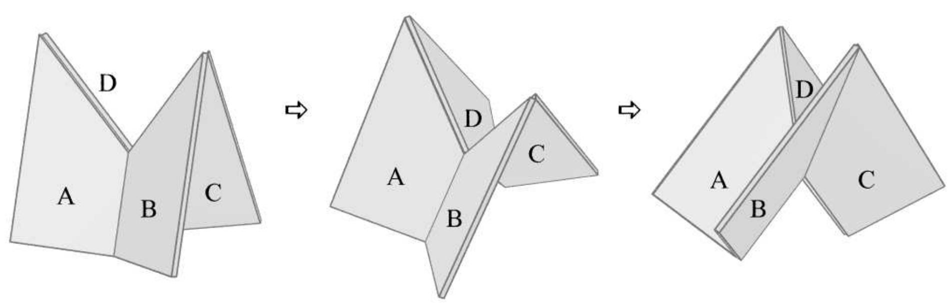

Figure 6: Saddle-cyclicly folding a mitre to a gnomon

If $\Sigma > 360°$, then the hinging is *saddle-cyclic*. The cycle will occur at a concave vertex in the dissection. Its name derives from the shape of the surface as it moves from one configuration to the other. It opens from a concave angle to a surface with a saddle point and then closes to a concave angle on the other side of the surface. As an example, we see the piano-hinged dissection of a mitre to a gnomon in Figure 7. The mitre is what remains from a square when we cut away an isosceles right triangle whose hypotenuse precisely covers a side of the square. The *gnomon* is what remains from a square when we cut away a square of one fourth the area from a corner of the larger square. Both the mitre and the gnomon have an angle of $\Sigma/2 = 270°$. On each level we split the figure into two pieces, splitting these large angles and thus creating four pieces that meet at angles whose total is 540°. We see how to fold from the mitre to the gnomon in Figure 6.

One type of cyclic hinging that is not vertex-cyclic is *tube-cyclic*, in which the cycle involves a sequence of faces that do not share a common vertex. An example is the dissection of the T-pentomino to the U-pentomino in Figure 8. The U-pentomino appears upside down in the figure, because that makes it easier to see how the folding works. Pentominoes [10] are figures formed by gluing five squares edge to edge. Here we see that pieces B, C, D, and E form a cycle, as we see in Figure 9. As we unfold and then refold the assemblage, we are able to take the center column of the T and distribute it to the two arms of the U.

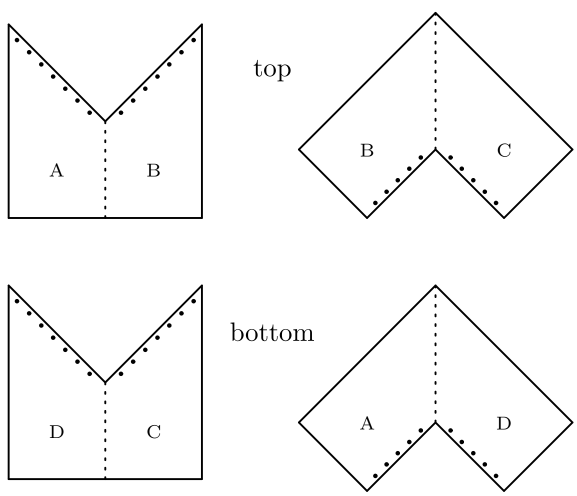

Figure 7: Saddle-cyclic mitre to gnomon

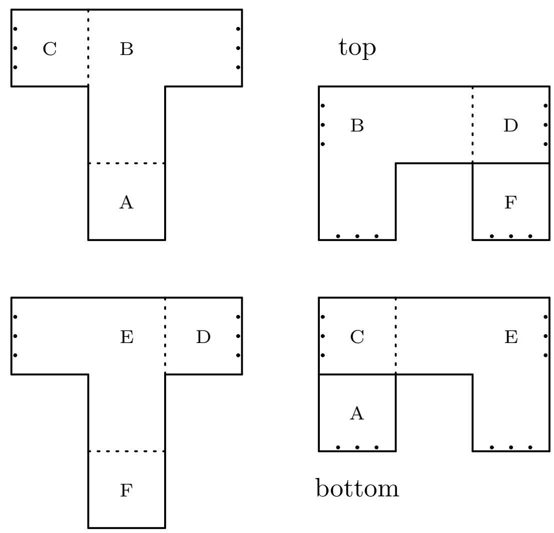

Figure 8: Tube-cyclic T- to U-pentomino

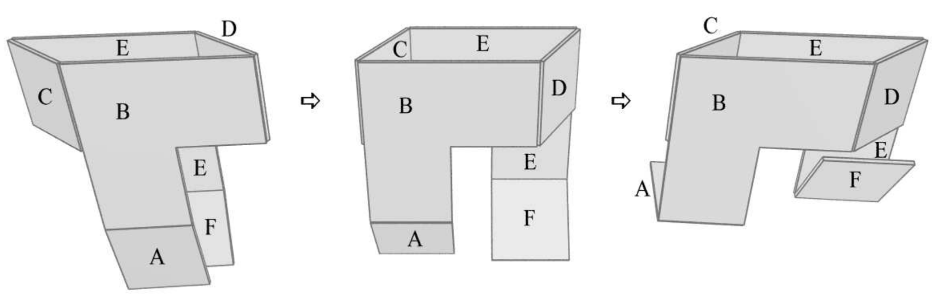

Figure 9: Tube-cyclicly folding a T-pentomino to a U-pentomino

Two more properties are worth identifying. The saddle-cyclicly hinged dissection of a mitre to a gnomon has a lovely "inside-out" property. Each piece is on only one level, so that it has two primary surfaces. When we assemble the pieces to form our figure, one surface is on the outside, and the other surface is on the inside (i.e., hidden from view). After we fold the pieces to form the other figure, all of the surfaces that were on the outside are now on the inside, and vice versa.

The reverse of the inside-out property is the *exterior-preserving* property, in which those surfaces exposed to the exterior stay on the exterior, and those on the interior stay on the interior. The cap-cyclic three triangles to one and the tube-cyclic T-pentomino to U-pentomino have this property.

3. Conversion from Twist Hinges to Piano Hinges

Are there general methods for discovering piano-hinged dissections? In fact, there is a neat method for converting a dissection that uses "twist hinges" to piano hinges. As an example, Figure 10 displays a twist-hinged dissection of an ellipse to a heart, derived from William Esser's dissection [4] of an ellipsoid to a heart-shaped object.

In this twist-hinged dissection, the ellipse has been sliced along a diagonal through its center, and one piece has been rotated 180° relative to the other, producing the heart shape. A small open circle indicates the position of a twist hinge, whose axis of rotation is perpendicular to the diagonal cut. I mark the piece that we flip with an "$*$" in the ellipse, and with a "$\star$" in the heart.

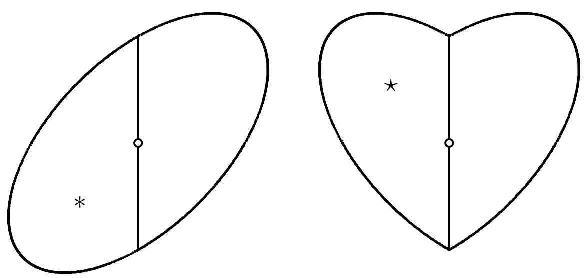

Figure 10: Twisting an ellipse to a heart

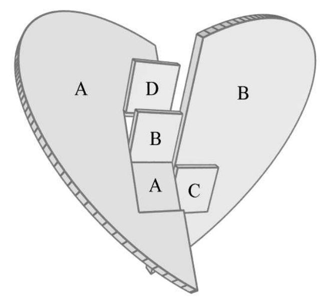

Figure 11: Folding an ellipse to a heart

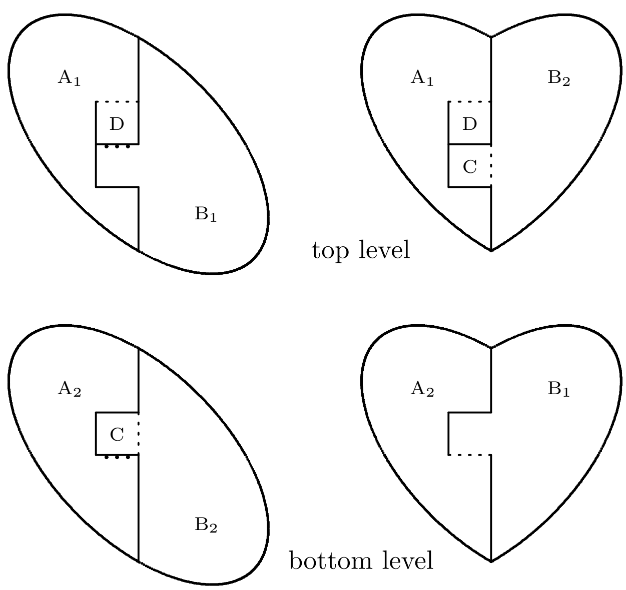

Figure 12: Piano-hinged dissection of an ellipse to a heart

Remarkably, we can simulate a twist hinge by three piano hinges, introducing two new pieces in the process. We must also allow pieces to have portions on one or both levels. In our example, the lower level of piece B (in the ellipse) will share a hinge with the upper level of piece A. We line up the axis of the hinge with the axis of rotation of the twist hinge, enabling the piano hinge to simulate the action of the twist hinge. To be able to perform the rotation along the piano hinge, I have cut pieces C and D out of piece A. Once we fold them out of the way, we can rotate piece B around, and then fold pieces C and D back into position. The 4-piece dissection is in Figure 12. We see a perspective view of the assemblage in mid-fold in Figure 11.

How does one design twist-hinged dissections in the first place? This is the topic in Chapter 22 of [6]. In addition to ad hoc techniques, there is a method to convert (almost) ordinary hinged dissections to twist-hinged dissections.

4. A Quick Sampling of Some Piano-hinging Techniques

Although the conversion technique of the last section is powerful in theory, it can lead to piano-hinged dissections in which the hinges are short and not very stable in practice. In this section we sample two techniques that produce more satisfying models.

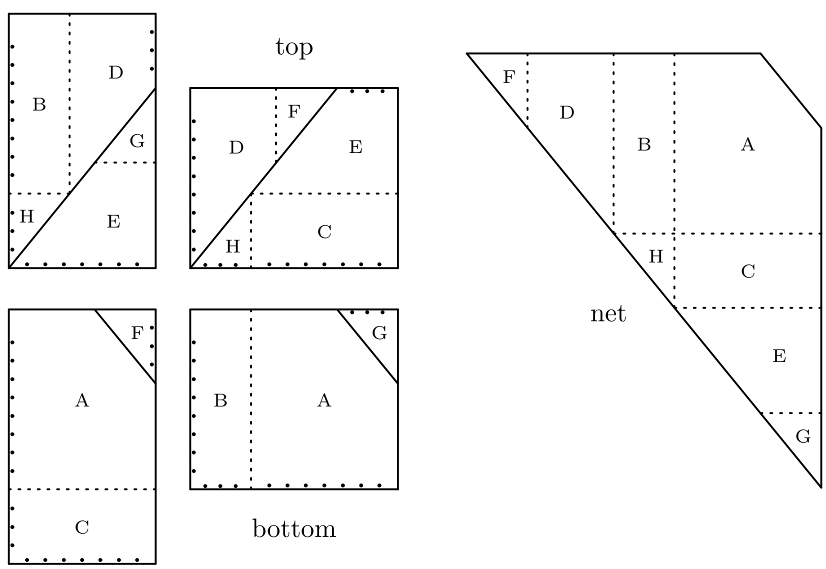

Figure 13: Folding dissection of a rectangle to another

Our first technique allows us to transform any rectangle into any other rectangle. The dissection (Figure 13) uses eight pieces as long as the lengths of the rectangles are within a factor of two of each other. Let l and w be the length and width of the first rectangle, and αl and w/α be the length and width of the second, where $1 < \alpha < 2$. Take a right triangle with legs of lengths $(\alpha+1)l$ and $(1+1/\alpha)w$, and cut from it a small right triangle, with legs of lengths $(\alpha-1)l$ and $(1-1/\alpha)w$. This produces a trapezoid whose bases are the hypotenuses of the two right triangles. Parallel to the leg of length $(\alpha+1)l$, make cuts at distances of w/α, w, and $2w/\alpha$. Parallel to the leg of length $(1+1/\alpha)w$, make cuts at distances of l, αl, and $2l$.

Allowing pieces to have portions on one or both levels makes it possible to save a piece, and certain sizes for the rectangles also allow the use of fewer pieces. (See [8].)

Our second example may evoke surprise that such a dissection puzzle can have a piano-hinged solution. Henry Dudeney posed a couple of puzzles which challenged readers to produce a square from a rectangle with a rectangular hole in the middle. (See for example [3].) With favorable dimensions for the rectangle and its hole, the dissection requires just two pieces.

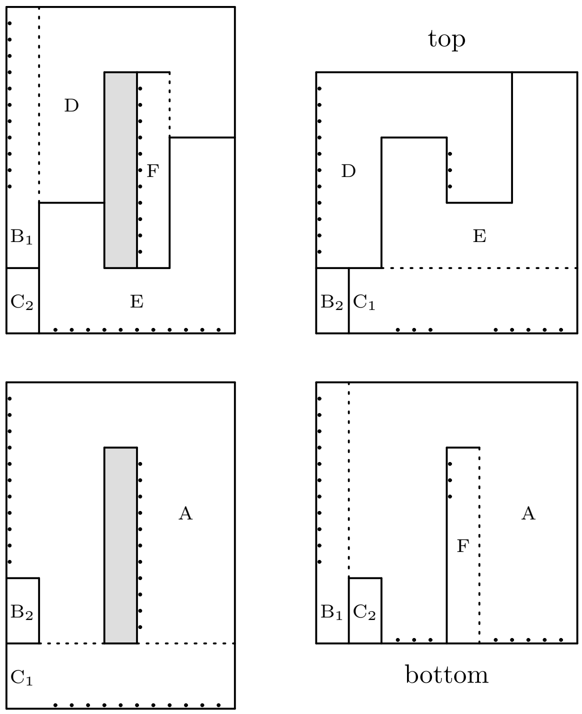

Figure 14: Rect. with a rect. hole to a square

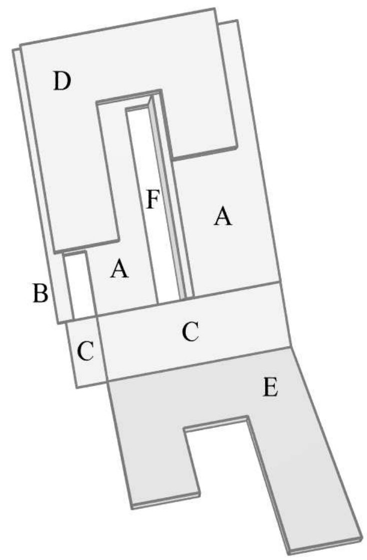

Figure 15: Tube-cyclic assemblage

If we consider cases in which the length of the hole is 4 units less than the length of its enclosing rectangle, we can often find a 6-piece rounded piano-hinged dissection. As an example, Figures 14 and 15 show a dissection of a (7×10)-rectangle with a (1×6)-rectangular hole to a square. We cut a piece F to fold over and exactly fill the (1×6)-rectangular hole on the bottom level. We then cut what remains on the top level to fit together, with pieces B and C having portions on both levels. As a bonus, pieces A, B, D, and F hinge together tube-cyclicly.

Remarkably, this piano-hinged dissection results by a reduction to an unhinged puzzle! After cutting piece F, cut piece A as large as possible so that it will occupy the bottom level of the rectangle and the bottom level of the square. Cut pieces B and C to fill out the remainder of the bottom level and some part of the top level of each figure. The remaining area of the top of the rectangle is a (6×10)-rectangle with a (2×6)-rectangular hole, and the remaining area of the top of the square is an (8×6)-rectangle. This corresponds to converting a (9×20)-rectangle with a (3×12)-rectangular hole to a 12-square. We need just two pieces (D and E) to accomplish this.

5. Conclusion

Based on high school geometry, and animated by a simple folding motion, piano-hinged dissections are both beautiful and accessible. Originating from a background of mathematical recreations, these mathematical transformer toys may well find significant application in education, particularly in mathematics enrichment activities.

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