Bridges
Mathematical Connections
in Art, Music, and Science

Permuting Heaven and Earth: Painted Expressions of Burnside's Theorem

James Mai
School of Art
Campus Box 5620
Illinois State University
Normal, IL 61790-5620, USA
E-mail: jlmai@ilstu.edu

Daylene Zielinski
Mathematics Department
Bellarmine University
2001 Newburg Road
Louisville, KY 40205, USA
E-mail: dzielinski@bellarmine.edu

Abstract:

The creative process of the artist, James Mai, and an exploration of the mathematics within *Permutations: Astral* and *Permutations: Earthly* (hereafter referred to as *Astral* and *Earthly*) are the subjects of this paper. While the paintings lend themselves to mathematical interpretations, the artist's conception and execution were not strictly mathematical; considerations of visual aesthetics and metaphoric references were integrated with logical relationships. The creative process was often a circuitous one as the artist was challenged to understand the complete range of visually distinct figures through diagramming alone, without the benefit of mathematical abstraction.

A mathematical analysis using Burnside's Theorem provided by Daylene Zielinski lays out an orderly approach that can be implemented in this type of artistic investigation. This useful theorem can tell the artist, in advance, how many distinct forms he will discover. It is important to note, however, that Burnside's Theorem says nothing about what these forms will look like or how to find them all; it predicts only the final count of visually distinct figures which can be created under a set of rules determined by the artist.

1. Introduction

The artist James Mai aims above all for wholeness, or completeness, in his paintings. For Mai this value of completeness is both aesthetic and mathematical in nature because the seeds for metaphoric meaning are carried in the mathematical relationships themselves. The artist posits the paradoxical possibility that a work of art might "open up" to associations and metaphors only when it achieves "closure" as a complete and integral whole. These values impelled the artist's permutational investigations in the paintings, *Astral* and *Earthly* (Figures 1a and 1b). Although the procedures were largely intuitive and inductive, the purpose of each painting is to visualize all of the unique shapes within a system of variables. Unknown to the artist during the development of these paintings, Burnside's Theorem is uniquely positioned to be useful to such artistic investigations where visual systems are driven by a permutational structure and the goal is a complete set of all possible figures.

The initial stages of *Astral* and *Earthly* developed through open-ended play, pencil sketches on graph paper in a search for shapes whose geometric features permit systematic variation. This stage seeks to define those essential features of a shape that, through permutation, will generate a range of new and unique shapes. After the permutational features are defined, the complete family of forms is sought—Mai looks for closure to what started as open-ended play. The goal is an objectively related family of shapes whose features and relationships are coherent and self-revealing to an independent viewer.

Although *Astral* and *Earthly* developed simultaneously, they were not initially considered to be related to each other. Eventually the permutational process involved in the development of each painting revealed their similarities and pulled them together as a pair. From that point forward, they grew to resemble and to define each other. As we shall see, this is in large part because they are both expressions of Burnside's Theorem; first, however, we will examine the permutational nature of each painting.

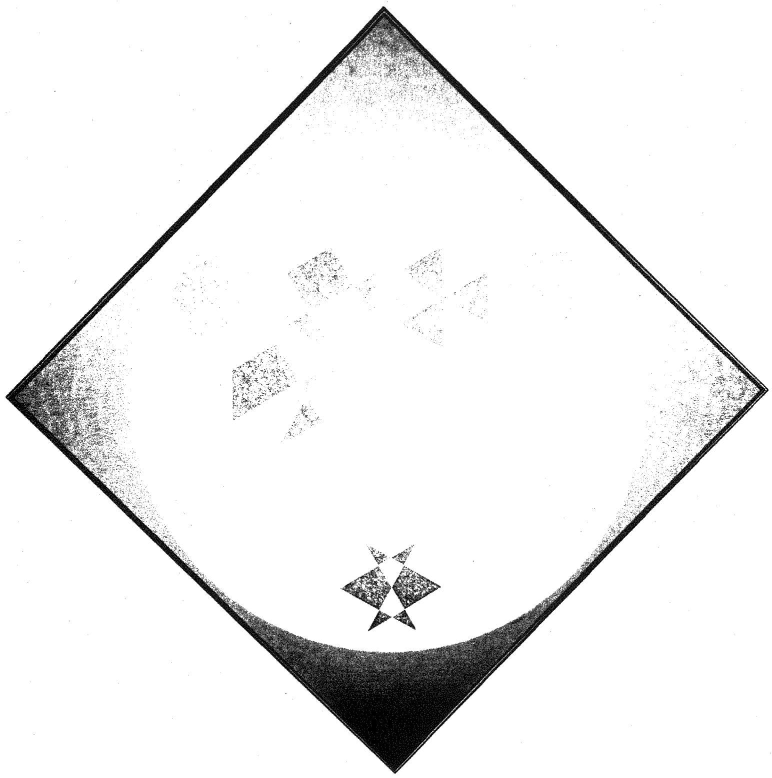

Figure 1a: *Permutations: Astral*, 42 x 42" (square), acrylic on canvas.

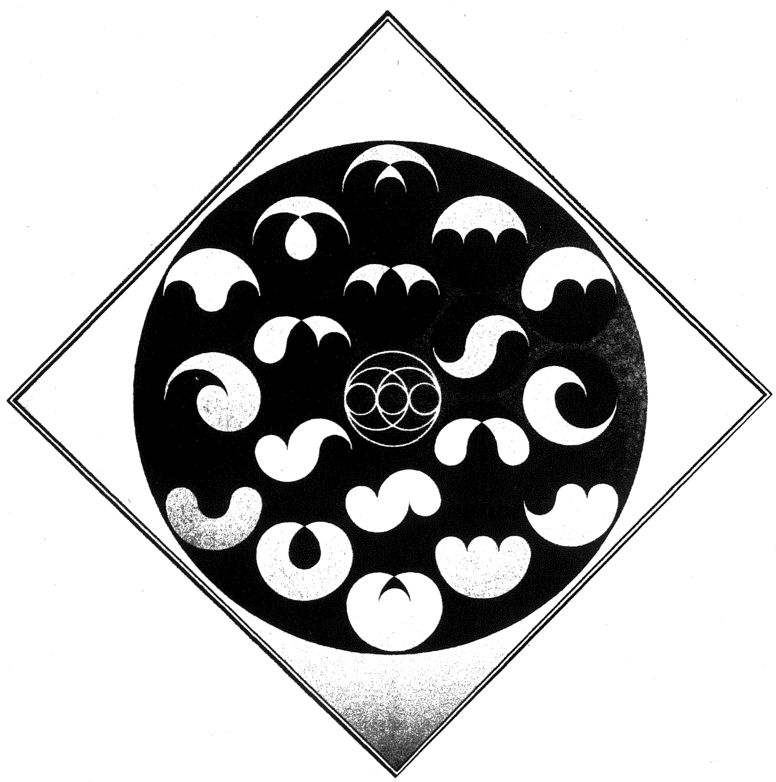

Figure 1b: *Permutations: Earthly*, 42 x 42" (square), acrylic on canvas.

2. Permutations

Astral addresses the following question: given a hexagonal array of vertices and all possible edges connecting them, how many visually distinct circuitous paths that visit each vertex exactly once (except the beginning/ending vertex) are there? Note that we will discard as redundant any shape that is merely a reflected or rotated version of another shape. After initial creative experimentation, Mai devised a symbolic shorthand of assigning numbers to vertices to work out the different shapes. Here is where permutations enter the picture. What follows is a compact and sanitized version of the creative process.

We begin with the diagram in Figure 2a. Since each path will pass through each vertex, we will choose to begin all paths at 1. Now, any path can be named by a permutation of the numbers 1 through 6, beginning with 1. The path 1, 3, 2, 4, 6, 5 is shown in Figure 2b. There are 720 permutations of the numbers 1 through 6, but since we always begin with one, we are really only permuting the five remaining numbers. This cuts the number of permutations to 120. Since a path drawn clockwise or counter-clockwise is the same, the permutations 1, a, b, c, d, e and 1, e, d, c, b, a will produce the same path. Now we have only 60 permutations to investigate.

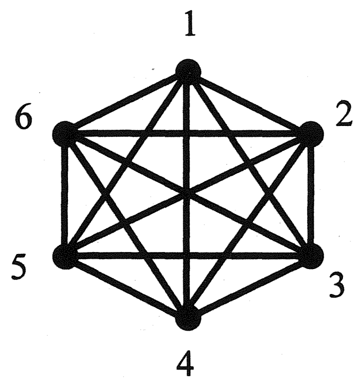

Figure 2a:
Six vertices of and all possible edges

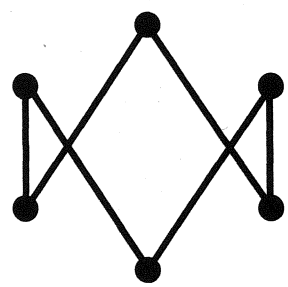

Figure 2b:
The path 1, 3, 2, 4, 6, 5

A similar methodology was used for *Earthly*. This investigation grew from a combined set of semi-circular and circular forms (Figures 3a and 3c) designed fifty years ago by Victor Flach, Universities of Oregon, Pittsburgh, and Wyoming, as "efficient little graphic combinatorial counting devices" for his tetradic studies and compositions: sequential combinatorial family groupings for 1, 2, 3, 4; 1, 2, 4, 3; 1, 3, 2, 4 (including each set's "pictorial image designation, leaf, bat, bird," respectively, shown in Figure 3b). Mai used the circular form in Figure 3c as matrix for his expanded study of upper and lower semi-circular pathways and the eighteen figures in *Earthly*.

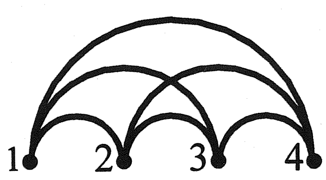

Figure 3a:
Semi-circular Form

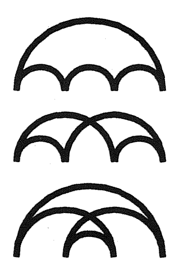

Figure 3b:
Flach's Pictorial
Image Designations

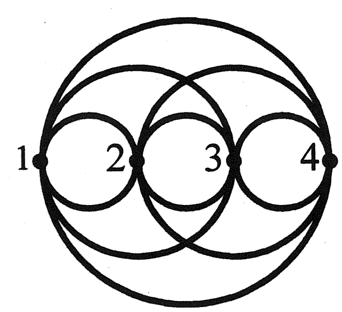

Figure 3c:
Circular Form

This time, we are working not only with permutations of the numbers 1 through 4, but also with the option of taking the upper path or the lower path when connecting each pair of vertices. Again, we will artificially sanitize the creative process of the artist. We will indicate the upper semi-circle with a u after the number and the lower semi-circle with a d. So, we begin with the problem of finding the number of arrangements of four symbols from the collection {1u, 1d, 2u, 2d, 3u, 3d, 4u, 4d} with the restriction that we always start with 1u or 1d and must include one of each of the pairs {2u, 2d}, {3u, 3d}, and {4u, 4d}. Thus, we have two options for the first symbol, six for the second symbol, four for the third, and two options for the fourth symbol. That makes 2 x 6 x 4 x 2, or 96, arrangements. However, again we see that reversing the order of the three symbols that follow the 1u or 1d at the beginning produces the same path. So, we shall dictate that the number in the second symbol in any arrangement be smaller than the number in the fourth symbol. So this leaves us with only 48 arrangements to investigate. It is interesting to note that whenever 4u or 4d ends an arrangement, we create a figure that has a natural complement, as demonstrated in Figure 4a; however, not all figures come with a natural complement, as shown in Figure 4b.

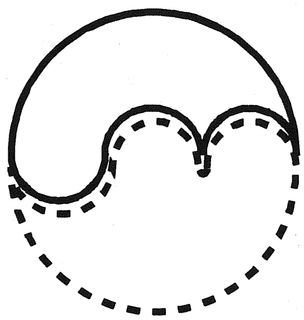

Figure 4a:
The figures 1d, 2u, 3u, 4u
and 1d, 2u, 3u, 4d

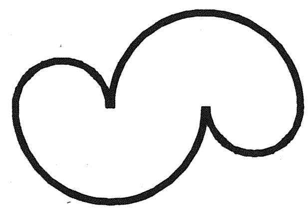

Figure 4b:
The figure 1u, 2u, 4d, 3d

3: Burnside's Theorem

During the preparatory work for both paintings, Mai worked back and forth between symbolization and visualization, but without a manageable systematic approach he became concerned that he might not recognize whether a set of figures was complete. He expected that his trial-and-error approach to both the symbolization and visualization would generate many redundancies, and he recognized that those redundancies would emerge from symmetries such as rotations and reflections. This is where Burnside's Theorem could shorten such a process and lead to predictable results for any artist working with permutational systems that are intended to generate a set of distinct figures.

Whenever we are faced with the task of finding the number of distinct shapes in a set of all variations of a particular type from some parent form, such as Figures 2a and 3c, we may apply Burnside's Theorem. In order to make its statement clear, we need some preliminary definitions. Every geometric shape has a group of symmetries. These are the reflections and rotations that fix the figure. In other words, if the original figure is reflected or rotated by one of its symmetries, we could not see the difference. For example, the group of symmetries of Mai's form in Figure 3c is relatively simple. It consists of one vertical reflection, one horizontal reflection, one 180° rotation, and the identity symmetry, which can be thought of as a 0° rotation. These symmetries are shown in Figure 5.

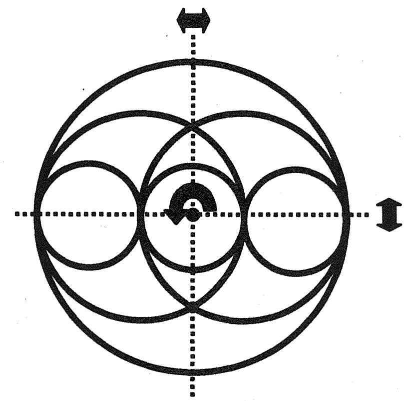

Figure 5: The two reflections and the 180°rotational symmetries of Figure 3c

Each symmetry leaves a certain number, sometimes zero, of the individual shapes unchanged just as it leaves the parent form unchanged. For example, the 180°rotation leaves not only the parent in Figure 3c unchanged, but also the shape in Figure 4b as well. The set of individual shapes that each symmetry leaves unchanged is called the fix of that symmetry. We use lower case Greek letters to name the symmetries. With this notation, the fix of a particular symmetry φ is fix(φ) and |fix(φ)| represents the number of shapes fixed by that symmetry. We will call the group of symmetries of the parent figure G and let |G| indicate the number of symmetries in G, including the identity symmetry. With this in place, we may now state Burnside's Theorem.

If G is a finite group of symmetries of a parent form S then the number of visually distinct figures in the family of shapes from S is given by $\frac{1}{|G|}\sum_{\varphi\in G}|\text{fix}(\varphi)|$ *where this sum is taken over each symmetry in the group G.*

In other words, all we need to do is find how many of the shapes in our family are fixed by each symmetry of the parent form. Sum these amounts and divide by the number of symmetries of the parent. The interested reader can find a proof and visually oriented discussion of Burnside's Theorem in the latest edition of *Contemporary Abstract Algebra* by Joseph Gallian.

To see how this theorem works in relation to *Earthly*, we will apply it to Figure 3c. In our previous discussion of this painting we realized that we had only to investigate 48 arrangements of four symbols from the set {1u, 1d, 2u, 2d, 3u, 3d, 4u, 4d} where we begin each arrangement with 1u or 1d and require the number in the second symbol to be less than the number in the fourth symbol. Now, we will find the fix of each of the four symmetries of the parent shape.

Two of the symmetries are easy to work with. Clearly the identity symmetry, which we will call R_0, leaves all circuits fixed since it moves nothing. Next, we will tackle the horizontal reflection, which we will call H. This symmetry leaves no circuit fixed because no semi-circle and its horizontal reflection appear in the same circuit.

To discuss the 180°rotation, which we will call R_{180}, we need to notice that if a circuit involves either of the largest semi-circles connecting 1 to 4 then it is not fixed by R_{180}. So, the only circuits that could be fixed by R_{180} start at 1 travel to 2 then to 4 then to 3 and back to 1, since we can't have 4 in the second symbol if 3 is in the last symbol. The only way such a path could be fixed by R_{180} is if whichever of u or d appears in the first symbol, its opposite appears in the third symbol and the same happens with the second and fourth symbols since these pairs of arcs will not only swap positions with each other but also reverse up/down orientation. This gives only the circuits 1u, 2u, 4d, 3d; 1u, 2d, 4d, 3u; 1d, 2u, 4u, 3d; and 1d, 2d, 4u, 3u. So, the R_{180} fixes only four of the 48 arrangements.

The vertical reflection, which we will call V, takes a bit more thought. We will divide our attack into the case where 4 is the number in the last symbol and where 3 is the number in the last symbol. If 4 is the number in the last symbol, then an arc connecting 1 and 4 is in the circuit and so is an arc connecting 2 and 3. All of these arcs are fixed by V. So we need only to make sure that the two remaining arcs in the figure are either both up or both down since they will trade places with each other under the vertical reflection. So whichever of u or d appears in the first symbol, must also appear in the third symbol. So, we have a choice of 1u or 1d for the second symbol, a choice of 2u, 2d, 3u or 3d for the second symbol, no choice for the third symbol since whichever number 2 or 3 not used in the second symbol must be in the third and the u or d must match the first symbol, and finally a choice of 4u or 4d for the last symbol. That's 2 x 4 x 1 x 2, or 16, circuits ending with 4 fixed by V. If the number in the last symbol is 3, then none of the individual arcs in the circuit are fixed by V, but there are two pairs of arcs that swap places. So if the u's and d's agree in the first and third and in the second and fourth symbols, the circuit will be fixed by V. That's four additional circuits. So, V fixes 20 of the 48 arrangements.

Now we are ready to apply Burnside's Theorem. The number of visually distinct circuits through Figure 3c is $\frac{1}{4}$ ($|fix(R_0)| + |fix(H)| + |fix(R_{180})| + |fix(V)|$) = $\frac{1}{4}$ (48 + 0 + 4 + 20) = 18. All eighteen of these figures can be seen arranged about the parent figure in *Earthly*. A similar calculation can be done for Astral, but since the symmetry group of its parent figure contains sixteen symmetries, that calculation would take too much space in this paper. It is not particularly taxing, just detailed.

4: Aesthetic Concerns

After the artist has the full set of visually distinct figures, he can devote all his energies to aesthetic concerns. Mai intends that his paintings transcend illustration or demonstration of permutational procedures and realize some level of aesthetic wholeness and metaphoric association. The creative process only halfway completed, Mai's next step was to forge appropriate compositional and color organizations.

Mai employs color to reinforce and codify the groupings within each permutation set. In *Astral*, the twelve distinct figures arrange themselves into groups according to the number of edges extant from the original hexagonal ring: one shape with all six outer edges; two with four edges; three with three edges; three with two edges; two with one edge; and one shape with zero outer edges (see Figure 6a for these groups). Each group is denoted by a different color in the painting. In *Earthly*, the visually most important groupings are the twelve shapes that have a natural complement and six shapes that do not (see Figure 6b for these groups). Complementary pairs are denoted by complementary colors. While Mai believed that color could clarify some of these relationships, he anticipated that compositional organization would play an even more important role in acknowledging groupings and complementary relationships.

A number of compositional possibilities exist for each painting. Since the number of visually distinct figures in each permutational set, twelve for *Astral* and eighteen for *Earthly*, is divisible by six, a hexagonal array can be employed. On the other hand, since each set of figures possesses a parent form which can be viewed as a point of origin or matrix from which all the variations originate, this parent form can be used as the center of a circular composition. Certainly, *Earthly* is related to a circle by virtue of its parent form. Further complicating the situation are the different possible orientations for the figures within a hexagonal or circular arrangement. The individual figures of each set can be arrayed so that each has a line of symmetry oriented towards the parent in the center, creating a radial organization; alternately, each set can be arrayed so that the figures' axes of symmetry are oriented vertically or horizontally, creating a parallel organization (e.g., Figures 6a and 6b). The artist's problem was that no definitive choice emerged from these options.

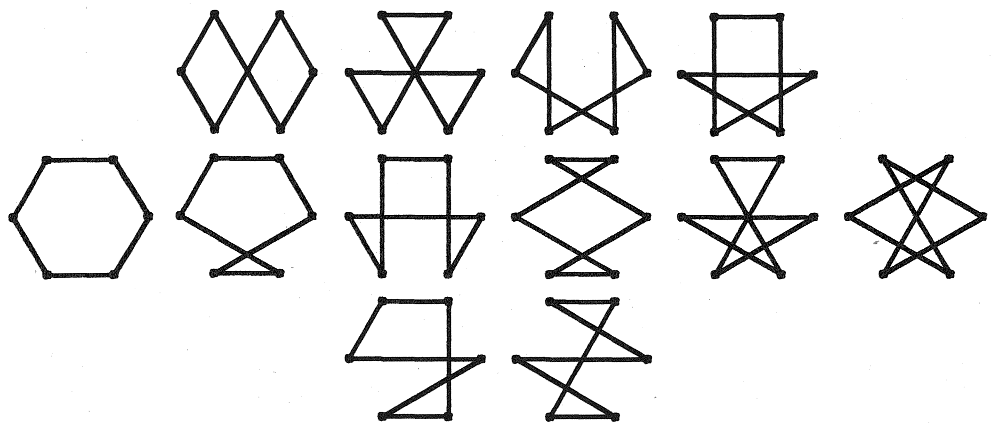

Figure 6a: Grouping of *Astral's* distinct figures by number of outer edges.

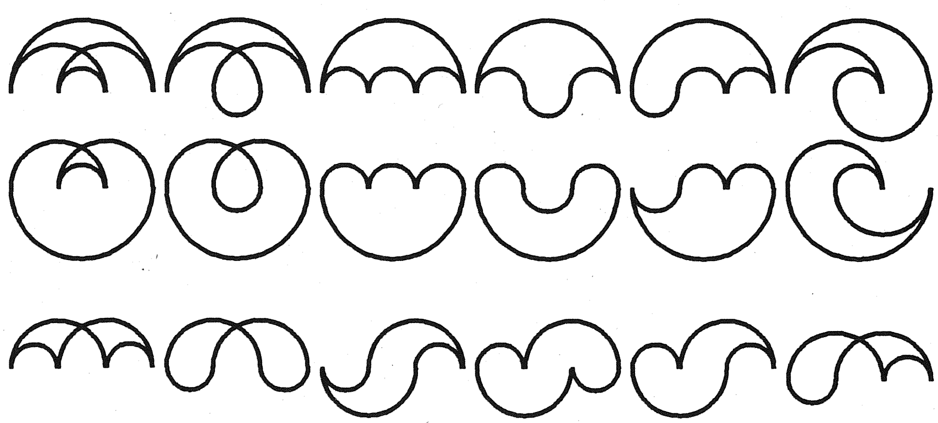

Figure 6b: Grouping of *Earthly's* six complementary pairs and six unique figures.

Figurative content, or subject-matter, began to suggest itself at this stage and to offer a solution to the question of composition. To the artist, the hexagonal figures suggest star-like bodies or constellations due to the straight edges connecting vertices and the prominent angularity of several of the figures in this set, while the semi-circular figures seem akin to living forms, such as birds, shells, fishes, and worms. The paintings defined each other by mutual contrast, and this contrast subsequently offered an answer to the compositional question. As constellation-like forms, *Astral's* shapes are organized radially around the center, acknowledging the omni-directional order of the sky. The animal-like forms of *Earthly* assume a vertical-horizontal orientation, whereby the horizontal axis that separates upward and downward semicircles acknowledges the horizon line of the terrestrial realm, and the lines of reflective symmetries are oriented vertically to evoke the vertical pull of gravity. Further, the eighteen figures of *Earthly* are grouped and arranged so that smaller, lighter shapes occupy the top of the composition and larger, heavier shapes occupy the bottom, suggesting a menagerie of animals that inhabit the air, land, and water. Within that organic arrangement, ten of the twelve natural complements are paired in vertical alignment and two are paired in horizontal alignment.

Under this figurative interpretation, Mai gives each painting a different color temperature appropriate to its subject matter. *Astral's* light and bright colors within a yellow field suggest a sunlit sky, a realm of light and warmth; *Earthly's* darker and somewhat duller colors within a field of blue suggest the dank and cool world of water and land, a realm of evolutionary activity. In *Earthly*, each individual element is colored differently, with warmer colors given to those upper forms that suggest lighter, active, air-born creatures and cooler colors given to those lower forms that remind one of heavier, slower, land- or water-dwelling creatures. The large, surrounding yellow circle of *Astral* suggests the expansive and continuous horizon of the dome of the sky; the corresponding blue circle in *Earthly* alludes to the self-contained planet of living forms and processes floating in the firmament.

5: Conclusion

It is important to remember that the paintings, *Astral* and *Earthly*, are non-objective. There is no subject matter, whether constellations or animals, directly depicted in the paintings. Mai's strategy is to allow figurative references to emerge naturally from the structure of the geometric beginnings and to imply those references metaphorically through the abstract relations of color, shape, and composition, rather than through illustrated subject matters.

Metaphors in visual art are generated from the deep correspondences between the structural aspects of colors and shapes internal to the painting and the structure of some aspects of the external world of experience. To form such structural correspondences is to relate otherwise unrelated events, to integrate previously unintegrated experiences, and to reveal new meanings in the process. Mathematical considerations are no less important to that process than aesthetic considerations and imaginative associations; indeed, the permutational sets of figures themselves were the springboard to the aesthetic organizations and metaphoric interpretations in *Astral* and *Earthly*. The applicability of Burnside's Theorem to determining the completeness of sets of permutational figures provides an important tool for the artist and promises to further bridge art and mathematics in Mai's future paintings.