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Mathematical Connections
in Art, Music, and Science

Twisted Domes

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Abstract

The most usual polyhedra with large numbers of triangular faces are geodesic domes, having non-regular triangles chosen so that the polyhedron approximates to a sphere. If the faces are equilateral triangles more interesting forms result, particularly if there are no planes of mirror symmetry, and the polyhedron has a twisted appearance. Some techniques for producing such polyhedra are described, and illustrated with examples.

Twisted Tilings

Of the various kinds of tiling that Grünbaum and Shephard[1] call not edge-to-edge, some (the right hand tilings figure 1 for example) retain rotational symmetry but lose mirror symmetry, and occur in right and left handed forms. Although they can be seen as the result of sliding faces, they can also be seen as the result of twisting the faces of an edge-to-edge tiling. This has the effect of producing additional faces at the vertices of the edge-to-edge tiling.

A dual to a tiling has a vertex corresponding to each face of the original, and a face coresponding to each vertex of the original. Vertices in the dual are joined by an edge if and only if faces in the original have an edge in common (and conversely). The new faces in the twisted tiling must be faces of the dual tiling, because of their relation to the original vertices: the edges meeting at the vertex are sides of the new face.

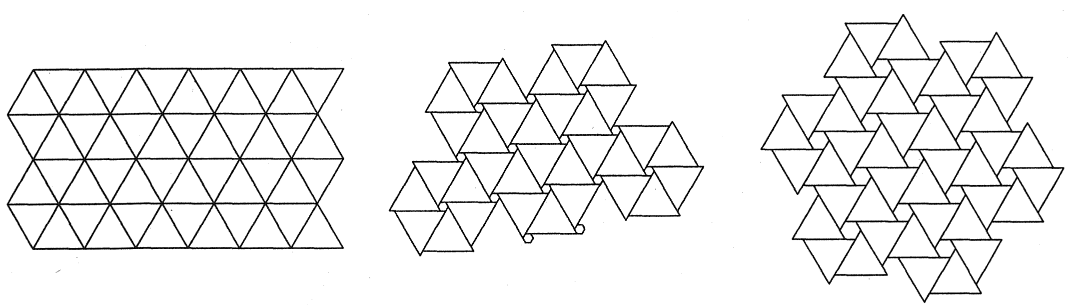

Figure 1. *Twisting the faces of a regular triangular tiling.*

Twisted Polyhedra

Applying the same idea to a polyhedron will not work because each vertex lies somewhere along an edge. Taken separately, the faces at such a vertex could hinge, but even this is prevented by the configuration at neighbouring vertices. Twisting faces forces them to lie flat.

The same difficulty does not arise with spherical tilings, which are closely related to polyhedra since the edges of a polyhedron can be projected from a point, usually the centre of symmetry, onto a sphere, producing a tiling. The spherical faces can be twisted, and many non-edge-to-edge spherical tilings are known.

People usually find twisted objects visually interesting, and various ways have been devised to generate such polyhedral structures, for example twisted spherical tilings can provide a template for tensegrity structures, and George Hart has produced a family of twisted polyhedra that he calls propellohedra[2].

Twisted Domes

One way to avoid the difficulty that happens when vertices occur somewhere along the length of an edge is to subdivide the edge, creating more vertices. In trying to reconstruct the shape of a particular type of virus[3] Stephan Werbeck [private correspondence] came up with the following construction: each face is replaced by a pyramid, having equilateral triangular faces, and each triangle subdivided into smaller triangles, the complete pyramids are then twisted relative to each other, but still keeping the smaller triangles edge to edge. This will leave holes that must be filled with extra faces. If the hole is an equilateral triangle it can be filled with the same triangles as the rest of the polyhedron. Figure 2 illustrates the process for a tetrahedron. In the original construction the apical pyramids were left off.

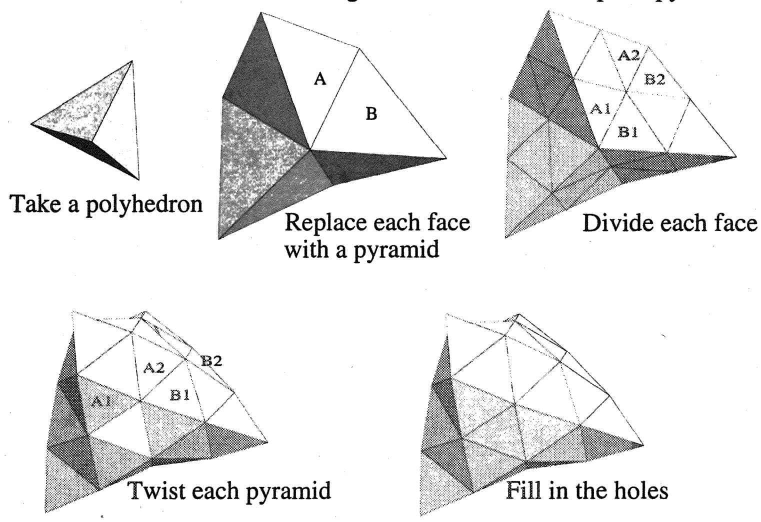

Figure 2. *The construction of a twisted dome.*

Dual Goldberg Polyhedra

I built two or three models using this idea at the conference of the UK Association of Teachers of Mathematics in Bath at Easter 2003. Eva Knoll noticed them, and pointed out that those made from only triangles could be made by subdividing faces of a suitable deltahedra. Icosahedral examples of such polyhedra were first described by Goldberg[5], although he was primarily concerned with their (topological) duals. Coxeter[6] applied the same idea to virus structures, as did Caspar and Klug[7]. The division of the triangles can be understood by placing a face of the original deltahedron on a grid of triangles from the new one (figure 3).

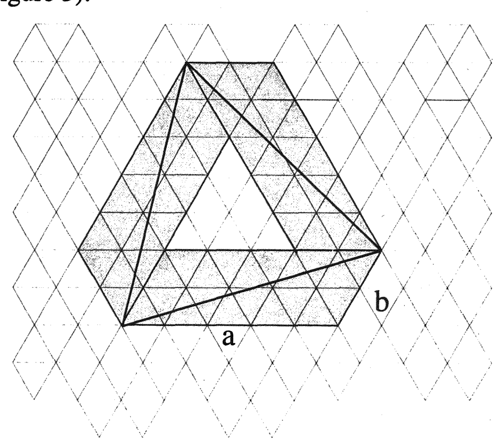

Figure 3. *A triangle from the original deltahedron on a grid of triangles from the new one.*

It is easy to see that the large triangle has an area made up from three half-parallelograms (and each parallelogram contains $2 \times a \times b$ small triangles), plus a triangle of side $a - b$, containing $(a - b)^2$ small triangles. It follows immediately that each original face is replaced by $a^2 + ab + b^2$ small triangles. I have extended the term "dual Goldberg polyhedron" to include any polyhedron produced from a deltahedron in this way (figure 4). Notice that tetrahedron 3,1 is the twisted dome in figure 2.

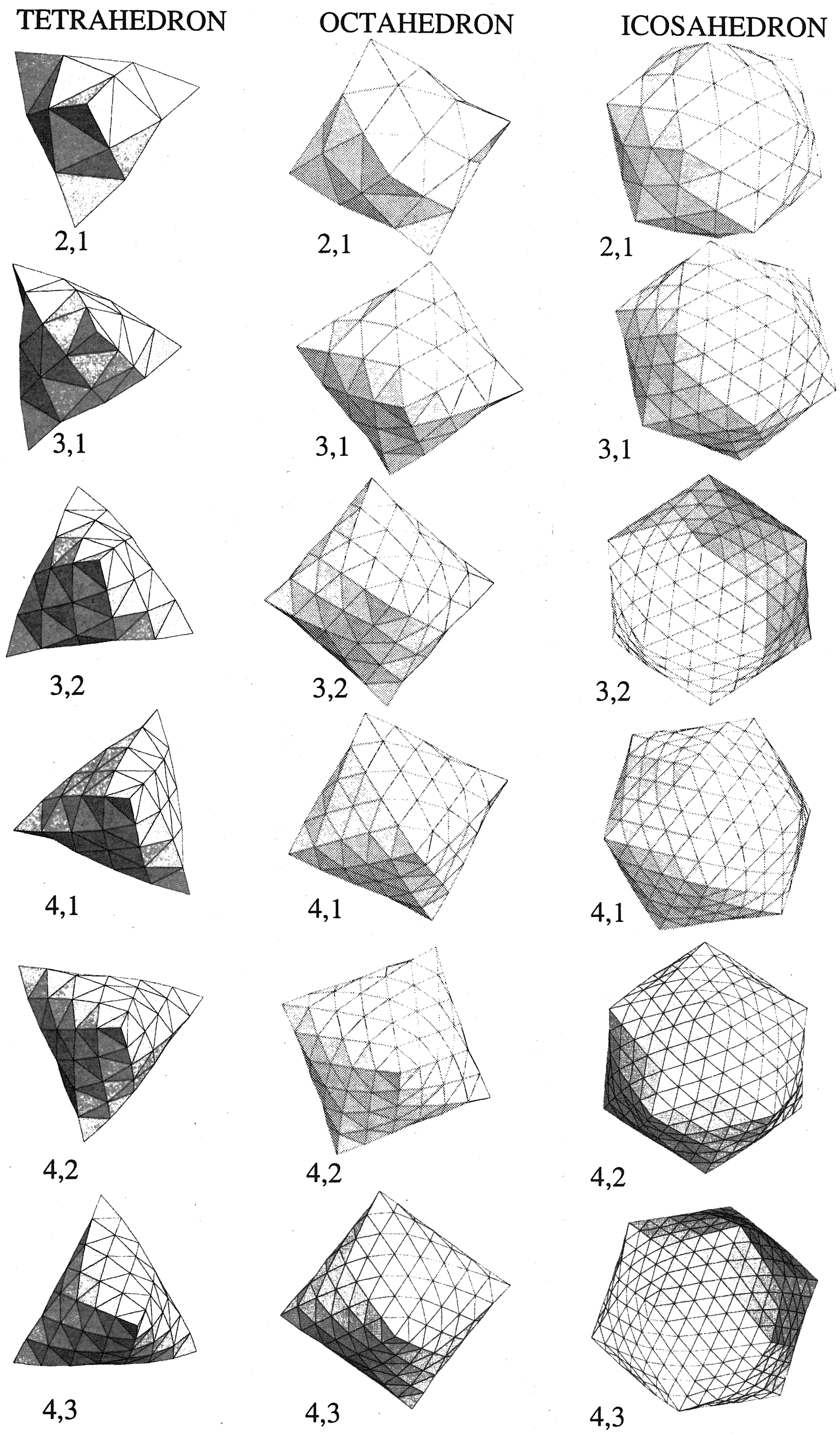

Figure 4. *Dual Goldberg polyhedra based on regular deltahedra, with a,b < 5.*

Vertex positions for these polyhedra have been generated using HEDRON[8], which uses a relaxation method to determine their coordinates, given combinatorial data about the vertices in each face. It produces VRML files as output.

Conformations

Even the simplest of these polyhedra are not completely determined by the way the triangles are connected, and Eva Knoll[9] has considered dimpled versions of some icosahedral examples. Different polyhedra made by connecting the same faces in different ways are generally known as isomers, by analogy with molecular structures in chemistry. Molecules that have the same atoms connected the same way but positioned differently are known as conformational isomers (or conformers), and it seems appropriate to use the same term for polyhedra with the same faces connected the same way but positioned differently.

Certainly the obvious conformations of dual Goldberg polyhedra are those that most resemble the original deltahedra, but others may have more visual interest. Some of the options in HEDRON will produce the conformation that is the nearest approximation to a sphere (which produces the same conformation if there are few faces). As the number of faces increases the form becomes more flexible, and variations occur quite readily. Figure 5 shows an octahedral example, and figure 6 an icosahedral one.

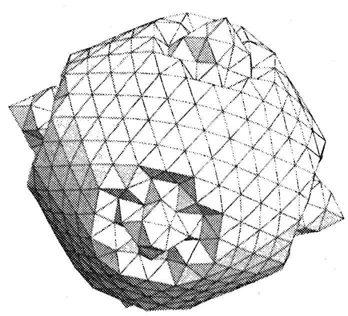

Figure 5. *Spherical conformation of dual Goldberg octahedron with a = 7, b = 5.*

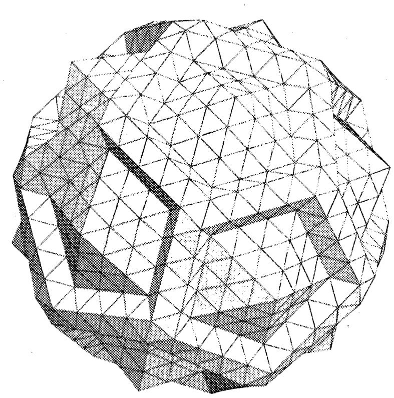

Figure 6. *Spherical conformation of dual Goldberg icosahedron with a = 5, b = 4.*

If a non-convex deltahedron is used as a starting point the derived dual Goldberg polyhedra will inherit valleys from the concave edges. A conformation that approximates a sphere will have ridges in the valleys. Figure 7 shows such conformations of the dual Goldberg polyhedra derived from the stella octangula with a = 5, compared with the most obvious alternatives, without ridges.

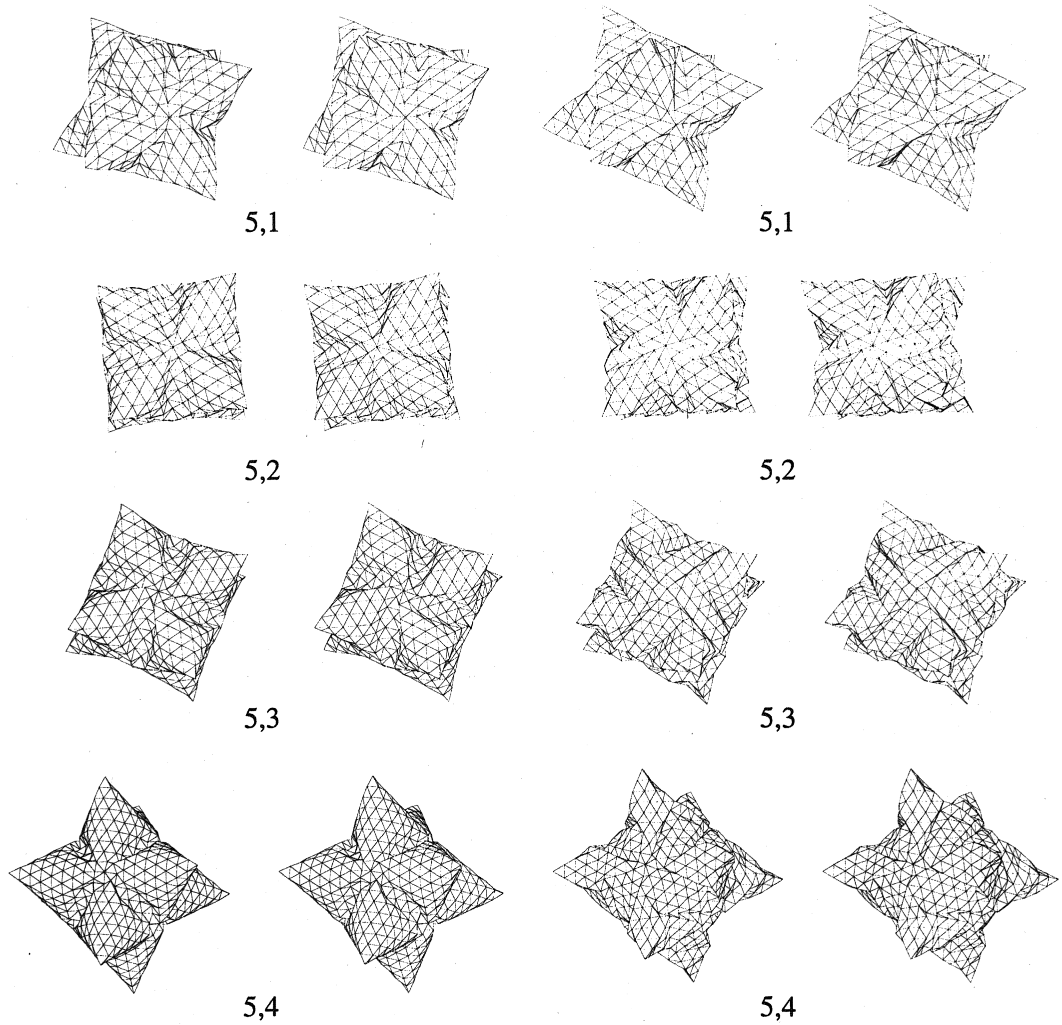

Figure 7. *Stereo pairs of dual Goldberg polyhedra derived from the stella octangula*

Other Polyhedra

Not all polyhedra constructed by the method illustrated in figure 2 are dual Goldberg polyhedra. For example, start with an octahedron, construct equilateral pyramids on its faces (producing a stella octangula), divide them and twist. The hole that is left is a square, and it is a unit square if the pyramids are twisted by one edge length. Figure 8 shows the simplest such polyhedron: the pyramid edges have been divided into 2 and twisted by 1 edge length. As in the original Werbeck construction the apical tetrahedra have been left off, emphasising the relationship of this object to a cube.

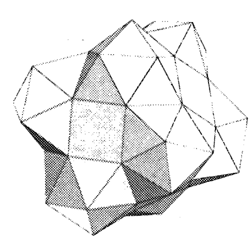

Figure 8. *A non-Goldberg twisted dome. Apical pyramids were removed from the white faces.*

Another method starts with an Archimedean polyhedron, for example a truncated tetrahedron. This has some faces that are hexagons, so it is not possible to construct equilateral pyramids on every face. Similar situations occur with other polyhedra that have octagonal or decagonal faces. One solution is to use pyramids of double size, so with a truncated tetrahedron use triangular pyramids of edge-length two on the triangular faces, and replace the hexagonal faces with triangular pyramids of edge-length four. Twist them so that adjacent pyramids have one edge-length in common, and fill the remaining holes with triangles. Figure 9 shows the result. Again the apical tetrahedra have been omitted.

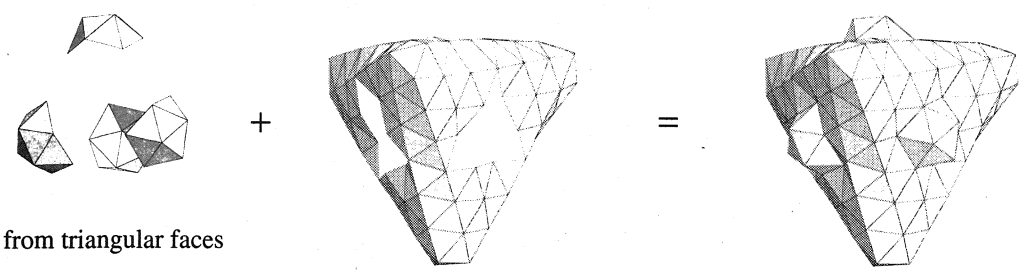

Figure 9. *A twisted dome based on the truncated tetrahedron.*

Triangulated Duals

The commonest examples of polyhedra with large numbers of triangular faces are geodesic domes, where the triangles are adjusted so that the polyhedron approximates to a sphere. The duals of these domes are also well known, and have twelve pentagonal faces, with the rest hexagons, and in general the faces are not regular. HEDRON tries to make all the faces of a polyhedron regular and it makes the pentagons and hexagons in a dual twisted dome non-planar. Triangles, however, must be planar, so proper polyhedra can be generated by triangulating the faces of the dual, then letting HEDRON find a form made from equilateral triangles. It is not difficult to show that this procedure replaces each triangle by three smaller ones (figure 10). In fact it is a Goldberg transformation with a = 2, b = -1.

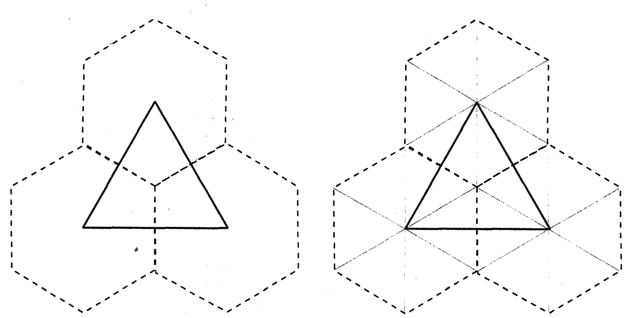

Figure 10. *Triangulating the dual replaces each triangle with six half-triangles.*

Figure 11 shows the result of this process applied to some simple examples, including Stephan Werbeck's original tetrahedral and icosahedral constructions. The 5-vertices produced by removing the apical pyramids appear as (distorted) pentagonal pyramids in the triangulated duals, and cause buckling.

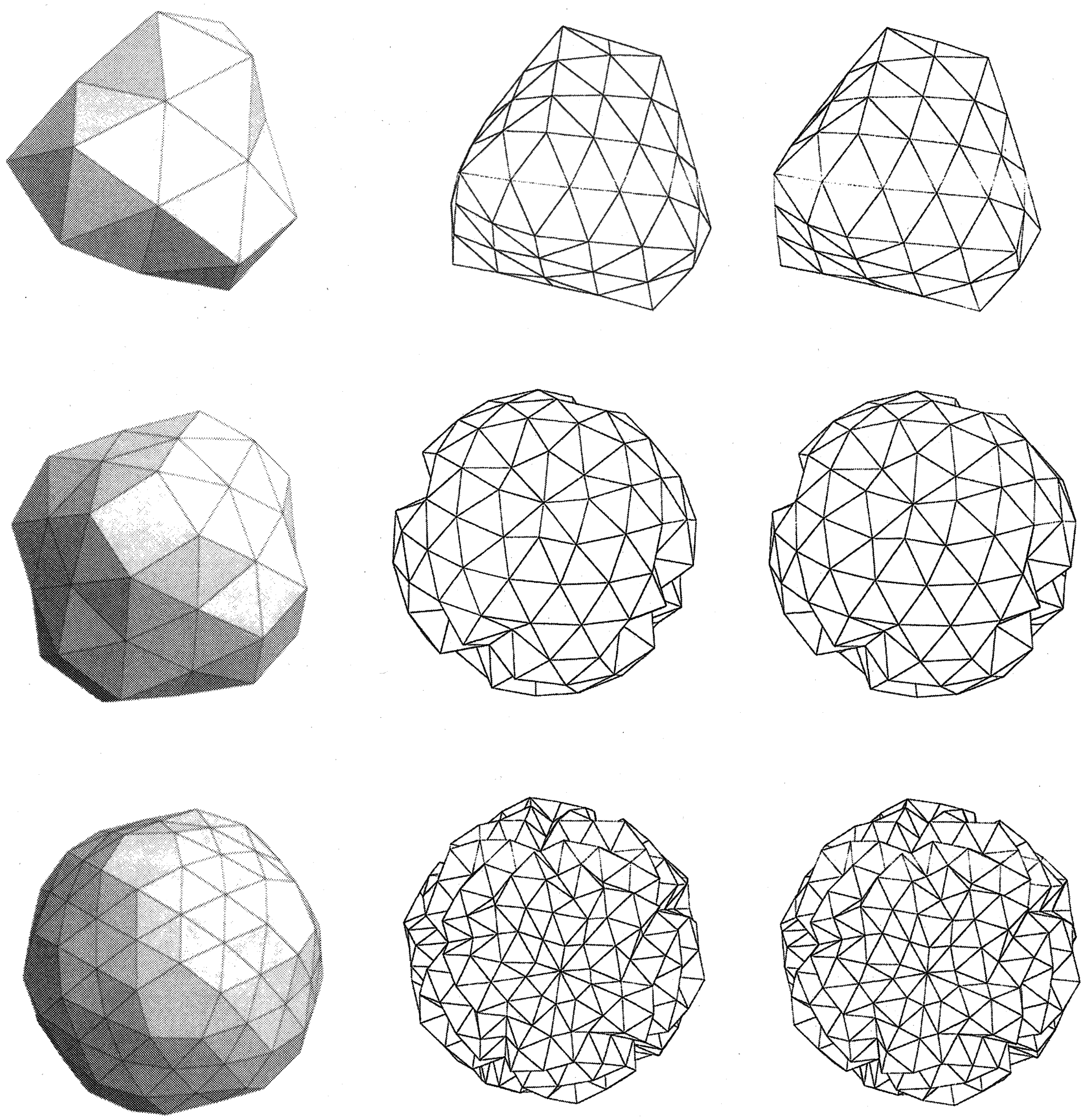

Figure 11. *Three twisted domes, tetrahedral, octahedral and icosahedral, with stereo pairs of their triangulated duals.*

If the apical pyramids are not removed, as in the dual Goldberg polyhedra, then the triangulated dual does not buckle. (figure 12).

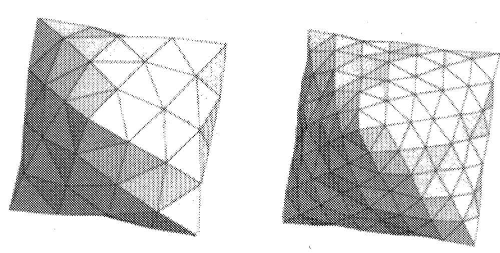

Figure 12. *The triangulated dual of a dual Goldberg octahedron ($a = 3$, $b = 1$) is another dual Goldberg octahedron ($a = 5$, $b = 2$).*

References

[1] Grünbaum B. and Shephard G.C.,*Tilings and Patterns,* W.H.Freeman and Company, 1987. p.72
[2]Hart G., "Sculpture based on Propellorized Polyhedra", Proceedings of MOSAIC 2000, Seattle, WA, August, 2000, pp. 61-70.
Available online at http://www.georgehart.com/propello/propello.html
[3] examples can be seen at www.virology.net/Big_Virology/BVRNApicorna.htm
[4] Werbeck S. private correspondence.
[5] Goldberg M., "A class of multi-symmetric polyhedra", *Tohoku Mathematics Journal,* (1937), **43,** 104–108.
[6] Coxeter, H.S.M., "Virus Macromolecules and Geodesic Domes", *A Spectrum of Mathematics Essays presented to H.G. Forder,* ed. John Butcher (Oxford, O.U.P., 1967).
[7] Caspar D.L.D. and Klug A., "Physical Principles in the Construction of Regular Viruses",*Cold Spring Harbor Symp. Quant. Biol.,* **27**, 1–24 (1962).
[8] written by Jim McNeill, available from http://web.ukonline.co.uk/polyhedra
[9] Knoll, E., Decomposing Deltahedra, International Society of the Arts, Mathematics and Architecture (ISAMA) conference, Albany, NY, 2000.