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Pulling Ropes and Plumbing Lines: Geometry for the Neolithic Engineer

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Abstract

Drawing on his own early experiences as a surveyor the author presents an introduction to the history, principles and methods of ancient rope pulling, i.e., surveying geometry, as manifested in Megalithic monuments. Concentrating on some likely field techniques dating back to Neolithic builders, this paper assesses similarities and contrasts between the focus of classic geometry on axiomatic proofs and compass and straightedge construction and the propensity of rope pulling toward canonical principles of mensuration and mechanical construction.

Unlike the stone tools for which the Neolithic age is named, geometry exists only in the patterns of the building projects to which it was applied. The design tools of these ancient engineers were fashioned from fiber and wood and have long ago decayed. The sophistication of wood and fiber technology of Neolithic cultures most likely varied from simple to highly complex, though this cannot now be known. What is known can be construed from the analysis of the ancient building sites, from contemporary reconstructions of likely methodology and by extrapolating back from historical times where the evidence remains in written and physical artifacts.

The most important of these tools was the rope, actually a heavy cord, used to demarcate and extend borders and alignments. The nature of geometric systems may have first evolved from the practical methods developed for using the rope to survey land divisions in the burgeoning agrarian cultures of the Neolithic. Moreover, geometry participated in the technology and symbolism of constructing larger and larger sacred spaces for the ever increasing population fueled by the increased food supply.

The Written Record

Early writings provide some clues to these methods. The earliest and most complete written instructions for the applications of rope geometry appear in the *Sulbasutras*, which were written as appendices to early Hindu religious tracts, the *Vedangas*. The term *sulb* means to measure and *sulba* translates as measuring rope or cord. The closest English translation for *sutras* is codes [7]. The *Sulbasutras* contain the canonical codes for laying out Vedic ceremonial structures and offer the most complete evidence for early Indian geometry. The earliest extant version of the *Sulbasutras*, the *Bodhayana*, dates from about 800BCE, but original versions may have been composed as early as 1500BCE. Archaeological evidence points to a knowledge of this geometry in pre-Vedic culture, the Harappans, pushing the sophisticated application of rope geometry on the Indian subcontinent to well before 2000BCE [15].

In addition to the *Sulbasutras* Egyptian and Greek sources provide hints as to the practice of early rope geometry. Hieroglyphic depictions and a paragraph from the second book of early Greek historian Herodotus [16] provide a sketch of the profession of the rope puller. Notable in the hieroglyphic illustrations is a distinctive clasp worn by the rope pullers as a mark of their caste. The ostensible purpose of the clasp was to hold the coiled rope, but it also symbolized the special mathematical knowledge and attendant social status to which the pullers were privy. Mathematics could not be studied by just anyone and was usually a priestly reserve. Consequently, the rope puller belonged to a quasi-priestly social caste.

These rope pullers, as Herodotus referred to them, spent a sort of working holiday solving practical and hypothetical problems during the long season of the Nile's annual flooding. Until the river's waters ebbed from the inundated fields the rope pullers put pen to papyrus and stylus to clay in the professional development of their geometric methods.

The rope pullers were not full priests, due to their secular status as professional tax agents of the pharaoh. Herodotus sets the beginnings of that status at about 1400BCE when the pharaoh Sesostris assigned regular plots of land to individual farmers and had these recorded in order to systematize the process of taxation. The rope pullers performed an initial cadastral survey, but, due to the annual floods, were kept busy in perpetuity updating that survey. The Nile's flooding tended to wreak havoc on the orderly plots, scouring land from the upstream banks and adding it onto downstream banks or covering field markings with mud. This necessitated annual riparian surveys to re-establish boundaries for accurate tax assessments.

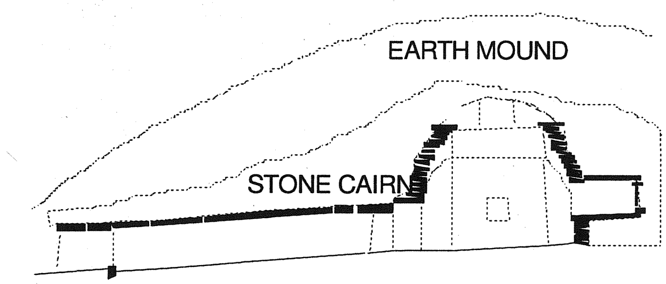
The link between the practice of classic Euclidean geometry and rope geometry is credited to the ancient Greek philosopher Thales. Returning from Egypt around 585BCE, he bore notes on the rope pullers learning. From these the early Greek mathematicians soon generalized the rope-and-stake applications of the rope pullers into a system of points, lines and arcs linked by axiomatic logic. At the same time they took geometry from the fields to the page by restricting geometric description to two drawing tools, the straightedge to describe straight lines and a pivoting tool, the compass, for executing arcs. The Greeks honored their Egyptian predecessors by dubbing their paper explorations geometry for "earth measure".

Going further into history the written record becomes thinner. Babylonian tablets from as early as 4000BCE, when Europe was still a cluster of Neolithic societies, contain records of land ownership bearing the seal of the surveyor to verify the accuracy of the land's size and location.

Tomb Geometry

This is as far back as any written record goes, but empirical evidence for rope-pulling geometry dates back at least another millennium, especially in data gathered from the analyses of the practical and ritualistic design of Neolithic tomb building. All indications are that these early European geometers were, like their Vedic counterparts, simultaneously both priest and architect.

Prehistoric graves from the European Mesolithic were typically stone lined underground cysts for burial of one or two individuals. These evolved into above ground burials smoothly mounded over with cairns of rock and earth. By the 6th millennium BCE the cysts had grown into above ground chambers, large enough for communal burial; and the cairns reach monumental proportions, refined into sculpted mounds shored by stone walls.



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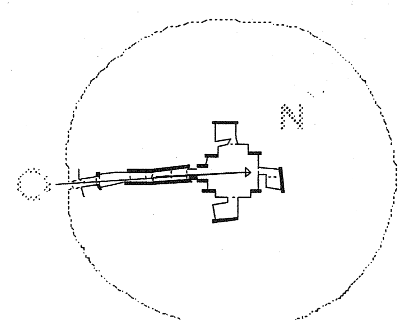

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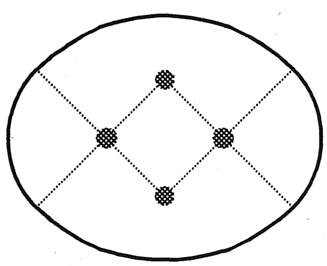
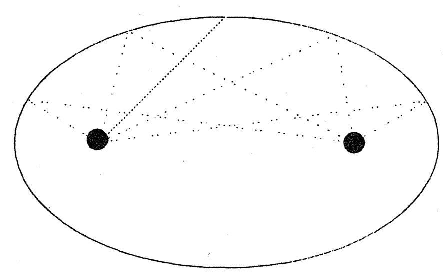
Serving as both temple and mausoleum, these structures, called dolmens, became increasingly refined and complex, as did the geometry underlying their design. By the third millennium BCE some, like the court tombs in Ireland, featured one or more entry courts; others, such as Maes Howe in Scotland, included elegantly dressed stone with ossuary chambers inset into the walls (above figure); and yet others, on the Iberian Peninsula and the western Mediterranean islands, boasted columns, murals and votive statuary. Perhaps most intriguing were the large numbers of dolmens built to align with celestial events, to link the geometry on earth with that of the heavens.

At Stonehenge, the ritualistic and burial functions separated. The famous stones' geometry became a structure of pure ritualistic purpose, while the plains surrounding it hosted numerous burial mounds. A close look at Stonehenge reveals its geometric origins in the dolmen. The stone shoring morphed into a ringed colonnade and the inner chamber became a stone ellipse.

Principles of Rope Geometry

In theory the rope puller could execute all classical constructions. The stretched rope functions as a straight line and the rope pivoted about a point can serve as a compass. However, the geometry of rope pulling differed from classical construction in two basic ways. First, rope pulling allowed for mechanical construction. Though the only known ovals from Neolithic times were four-arc constructions (four circular arcs, two broad curves and two sharp curves fitted into an approximate oval), a rope puller could, in principle, draw an ellipse by the thread method, a mechanical construction in which a string, looped over two focal points and a pencil, serves to guide the pencil over an elliptical path. By contrast classical compass and straight edge construction only permits elliptical constructions by plotting a set of interpolated points. Mechanical construction also permits determinations impossible with classical construction, such as the trisection of an angle.





Second, the approach of the rope pullers was not axiomatic or even particularly systematic. Rather it was a series of methods, codes of mensuration, for accurately accomplishing specific layout and measuring tasks. Although the rope pullers had an excellent recognition of validity in their solutions, they did not require formal proofs and, if practical, often settled for good approximations. Methods that incorporated the value of *pi* in the solution, for example, varied widely in its approximation, from as low as 2.99 to as high as 3.20 for Vedic geometers [14]. The best-known example is the squaring of the circle for ritualistic layouts: it was, by necessity, an exercise in approximation.

Methodology

From a purely practical point of view rope pulling faced a problem not confronted by compass and straight edge construction: the difficulty of scribing a line on rocky or grassy landscapes. When marking paper or incising clay tablets a point is typically located as the intersection of two arcs. Other than through a tedious process of hunt-and-peck, a rope puller could not easily locate points by this method.

Consequently certain basic constructive methods, such as bisection or perpendicular construction, were impractical to rope pulling.

On the other hand the early rope pullers devised means that were equally effective and far more efficient in practice. These means stress the ease with which the rope, given two stakes and a puller, can tighten into a triangle. Rope pulling was especially effective at using triangles to create congruency. A keen understanding of the usefulness of right triangles also characterizes this craft suggesting a rudimentary intuition for trigonometry.

Bisections and Perpendiculars. Bisecting a line with rope is quick and easy: pull a rope between the end stakes of the line and fold in half. To bisect an angle simply drive stakes equidistant on the angle's legs. Mark the center of a length of rope by folding it in half. Fix the ends of the rope to the stakes and pull from the middle of the rope to locate a point on the bisecting line. In their reconstruction of a Roman rope, surveying students at the University of Akron fashioned bights on each end, in order to slip the rope over the stakes and also to serve as handholds [2].

Constructing a perpendicular on a line is equivalent to bisecting a straight angle. Drive two stakes on a line equidistant from the origin of the perpendicular. Fix the ends of the rope to the stakes, and then pull from the middle to determine a point on the perpendicular. Reverse this process to draw a perpendicular to a line: fix the middle of the rope to the stake from which the perpendicular is drawn and pull the ends of the rope to meet the line. Bisect the segment between the staked points to mark the point where the perpendicular meets the line.

Calibration. Procedures, such as subdividing, parallel construction and arc layout, are speeded up if the rope is demarcated into units. Knotting is most often cited as the likely method for marking off the units. To compensate for the inevitable errors in knotting, beads may have been threaded onto the thin rope or cord and abutted to the knot. Thin notches on the beads would then more accurately calibrate the distances on the rope. Contemporary reconstructions suggest that ropes were likely pre-stretched and then saturated with wax or pitch in anticipation of the rope lengthening with the fatiguing of continued use or the effects of damp weather [2].

Calibration could have been based on a standard or on a module that varied from site to site. This latter was the case during the Dark Ages after the collapse of the Roman Empire and its preservation of standard measures. Data from modern surveys of prehistoric sites indicates a possibility that many early cultures maintained standard measures across the range of their influence. Megalithic builders on both the British Isles and the Breton coast appear to have shared a common measure dubbed the Megalithic yard [21]. A similar "yard" can be found in the layout of Hopewell ceremonial mound sites of the Ohio valley and in the giant earth drawings on the Nazcan plateau in Peru. What is not clear from the statistical nature of the available data, is whether it proves a hypothetical unit or merely the averaging of the stride of various rope pullers in the region. It is a good possibility that the practice of calibrating the rope to its user's pace was widespread.

Calibrated ropes were used to measure larger linear distances. In the likely event that the distances did not end in even units, evidence indicates that the rope puller carried a wooden rule of one half the length of a unit. This half unit appears frequently in Neolithic sites and Roman practice may shed light on Neolithic applications. The standard Roman unit, for example, was the *ped* or foot (11.6 inches), so the Roman *mensor's* carried a small pocket rule of one-half *ped* [5]. This rule divided into five parts, each of which then subdivided into ten parts. The rope puller used this pocket rule to add or subtract from the nearest whole unit on the rope. The Megalithic, Hopewell and Nazcan builders also used a half unit commonly referred to as a cubit.

Mensuration and Error. This appears precise, but is less accurate than it sounds. Since ropes will vary in length depending on the force of tension, or fatigue through repeated use or the dampness of the day, accuracy fell well short of modern standards. Early errors of +/- 1% were typical. This is evident in the layouts of the huge Hopewell sites. The Hopewell tribes paired circular with square enclosures, each surrounded by a perimeter mound between 900 and 1000 feet across. At the Newark Fairgrounds Earthworks the perimeters of the two enclosures are 3736.6 feet and 3712.0 feet respectively to fall within a 1% magnitude of one another [18]. This is close enough to signal the designer's intent to create symbolic equivalence between circle and square, the most common pairing in sacred geometry around the world [11]. Errors notwithstanding, this demonstrates a sophisticated and consistent approach to layout and well served its ceremonial function.

The Hopewell squares also are off true right by a similar error of 1% in arc magnitude. This error and the distance errors would be about right for an early stage of rope pulling. A stretched rope can yield an error of up to +/- 2%. Since the error will vary between too long and too short for each pull of the rope, this inaccuracy will partially average out over the long run. This plus the skill of the rope pullers kept the error down to that manifested by the Hopewell geometers.

Egyptian rope pullers fared better, cutting that error to about .05%. In general the Egyptian geometers possessed the same material technology, textile and wood, as did their earlier Neolithic counterparts. (They lacked the bronze and iron instrumentation of the Greek and Roman pullers.) Improved rope technology – the Egyptians may have imported [1] or cultivated hemp, a more dimensionally stable fiber – may account for part of that error reduction, but writing would account for much more.

With writing came computational methods, and a way to calculate errors based on recorded field data. The Neolithic engineer would have arrived at all values through field constructions and not calculations. The Egyptians value for *pi* is an example of accuracy engendered by the ability to calculate fractions. They used the fraction 22/7, 3.1428..., far closer to the actual value of 3.1415... than 3.16, the nearest constructed value in ancient times.

The square base of the Great Pyramid at Giza is a particularly good example of ancient Egyptian accuracy. The north side of the base is 8 inches smaller than the south side [3]. Two of the angles are off by just under 4 minutes of arc, the other two are within a few seconds of true. By modern standards the 8 inches of error is still relatively poor. Today's average construction engineer with a tape measure from the neighborhood hardware store would fare better. As revealed by extant papyri, Egyptian mathematics held very strong rules and formulas and rigorous checks were standard practice. In the case of the Great Pyramid the diagonals are virtually equal, so that this standard check for square would not reveal the discrepancy. If anything the equivalence of the diagonals could indicate a significant effort on the part of the rope pullers to true up the layout. However, equal diagonals produce a true square only if all of the sides are equal. The culprit for error was likely the rope.

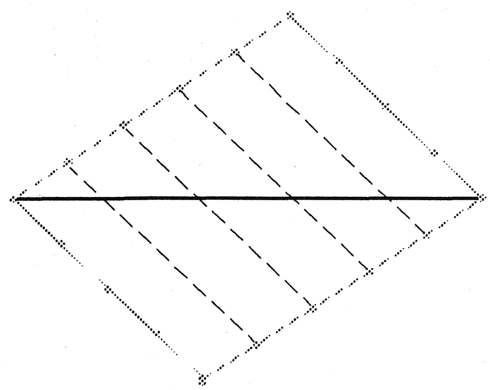
Ritual and Practice. There is no direct evidence of knotted ropes in field surveying. Evidence cited for the knotted rope as the standard surveying tool of ancient times is actually the ceremonial use of knotted ropes, where symbolism counted more than accuracy. In their ancient rituals Egyptian priests employed an extent of rope knotted thirteen times to form twelve intervals [6]. The intervals were sufficient to layout a 3,4,5 triangle and thus a right angle. Known today as the Pythagorean triangle, it has through history also borne the label of Egyptian triangle. The *Bodhayana Sulbasutras* give six Pythagorean triplets, (3,4,5), (5,12,13), (8,15,17), (7,24,25), (12,35,37), (15,36,39), for ritual application.

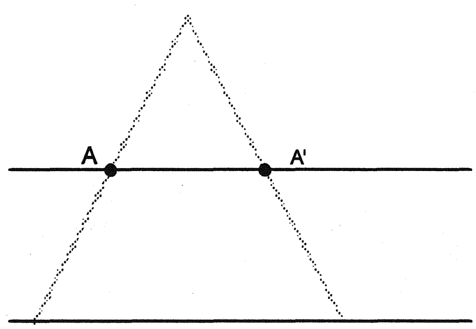
Writers on ancient geometry are fond of extrapolating from Egyptian ritual and pointing to this triangle as the most likely method for determining a right angle with a calibrated rope. Much effort has gone into trying to detect this and other Pythagorean triplets in the layout of Neolithic sites, but since the evidence is based on the same data as the claims for a standardized "yard", all claims are inconclusive.

In field practice pulling a triplet is relatively clumsy and inaccurate. Field tests made by the author revealed a dramatic increase in accuracy if pulls were limited to V's, i.e., where two rather than three legs of a triangle are pulled simultaneously. The difference was quite significant, with errors increasing by an average factor of ten in a three-leg pull. The effectiveness of the two-leg pull can be found in the following methods for constructing divisions, parallels and arcs with the calibrated rope.

Subdividing. The classic construction for dividing a given line into n equal parts is one procedure speeded up by the calibrated rope. Stake the rope at the ends of the given line and pull to describe a triangle. From one end of the rope count the same number of units as the divisions required and use that knot as the apex of the triangle. Drive a stake at each knot counted. Repeat these steps on the opposite side of the line taking care that the new triangle is congruent through rotation and not mirrored. Parallel lines stretched between the congruent stakes will cross the given line at equidistant points.

Parallels. Pull the rope from the given stake to the line, angling the rope so that the distance along the rope line is a whole number of knotted units. Extend the rope out past the given point, also by a whole number of units, and drive a stake. Turn toward the line by a whole number of units to complete an isosceles triangle. A line extended between the given point and its mirror on the opposite side of the triangle will be parallel.

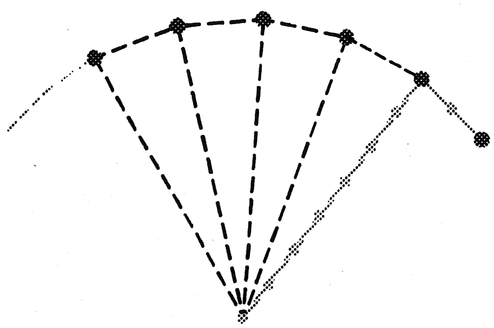


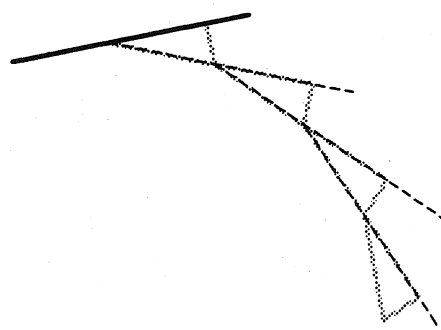


Arc Layout. One can rarely scribe an arc when surveying in the field. Grass and rocky ground are not conducive to accurate scribing. Consequently surveyors typically describe an arc with a series of equally spaced stakes. This is also the method used today for forming curved concrete walks, where stakes are interpolated with flexible metal strips to achieve curvature.

The simplest rope method, and one that works well for smaller arcs, is to attach the end of the rope to a center stake and pull to a point at the given radius of the arc. To simultaneously draw the arc and space out the stakes the rope puller selects a point on the rope equal to the radius of the arc and a second point further on the rope by the distance desired between the stakes. Hold the further point on the previously placed stake and pull at the radius point to determine the position of the next stake. Continue by using the same two points on the rope.

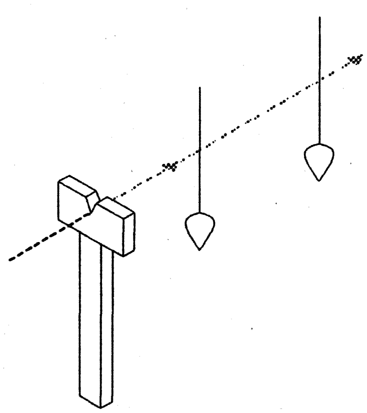
If the radius of the arc is larger than the length of the rope or the center of the arc is inaccessible, this method is not feasible. Instead the rope puller can generate an arc by staking out a series of closely spaced chords. If each chord is of equal length and if each deflects from the previous arc by the same angle, then the chords will all lie on the same circle. This method is similar in principle to that employed by modern road surveying. From a point on a line determine a specified slope to that line and measure a specified distance on that slope. Using the new line as a baseline and the measured point as the origin of the next slope, determine another line of the same slope and distance as the last. This can be carried out quickly by using the rope to copy the right triangle for which the previous chord was the hypotenuse. Continue to complete the arc.

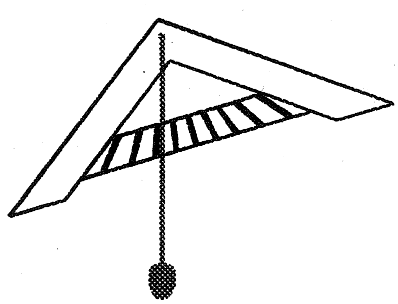




Plumbs. A weighted line is the universal measure of verticality. Early versions were weighted with stone, and later lead, thus the name plumb line. Roman and Egyptian practice provide clues as to how the plumbed line may have been used by Neolithic builders. One function was that of a level or inclinometer (see diagram below). The most frequent application of the plumbed line, though, was as a sighting device.

Given two ground points an assistant would hold the plumb line over the first point while the ancient surveyor sighted through the vertical line and the second point. A second assistant would then drive stakes situated on the sight line of the surveyor. To ensure further accuracy the Egyptian rope puller added a T-shaped device, the *meshet*. Grasping the *meshet* by its stem, he sighted through a V cut into the top bar until both plumb lines appeared as one in the notch [12]. Whether or not the Neolithic geometer used a backsight like the *meshet* to increase accuracy, ample evidence exists for sighting alignments and the plumbed line is an excellent tool for this – far better than just two stakes.





Where the rope alone allows triangulation from two given stakes by measuring two sides (equivalent to SSS congruency), the sighted line combined with the rope permits this with one angle and one side (equivalent to SSA congruency). A bearing taken from one point can be sighted through two plumbed lines, while an assistant swings a rope, marked to a given length, from the second original stake. Where the mark on the rope intersects the line of sight determines the third point of the triangulation.

Given structures like Stonehenge, the Hopewell ceremonial centers of the Ohio River valley and the huge geoglyphs of the Nazca plateau, it is clear that Neolithic, pre-literate culture had developed a sophisticated geometry for engineering some of the world's most intriguing monuments. This geometry evolved to create a form of graphic delineation of space onto the surface of the earth. Available technology of fiber and wood and the surface characteristics of the earth itself led to a geometry based on three simple pulls of the rope: a stretched line, a sighted line and a two-leg pull.

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