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Mathematical Connections
in Art, Music, and Science

Fractal Patterns and Pseudo-tilings Based on Spirals

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Abstract

A variety of fractal patterns and pseudo-tilings are described that are created by iterative arrangement of successively smaller spirals and spiral-shaped tiles. In some cases, these are used to create two-dimensional designs that resemble natural trees.

1. Introduction

This paper deals with fractal constructs in which the simplest building block is a spiral. There are numerous algorithmic fractals with spiral structure, but these are distinct from the designs explored here. There are also numerous spiral tilings that can be constructed from polygonal tiles, but this is again a separate topic.

Algorithmic fractals such as the Mandelbrot set have been and continue to be extensively used for creation of a wide range of imagery [1]. Graphically constructed fractals do not appear to have been explored as extensively. The work described here is an extension of earlier work on fractal tilings [2-4]. As is the case with these fractal tilings, the pseudo-tilings and patterns described here do not cover the infinite mathematical plane, but are rather finite in spatial extent. An infinite number of spiral building blocks fit into this finite area due to the rapid shrinkage of these building blocks with successive iteration.

All of the fractal structures presented here were created on a Macintosh iBook computer using the commercial drawing program FreeHand. The construction of the fractal patterns and pseudo-tilings is carried out through a sufficient number of generations that the resulting figure is a good approximation to the eye to an infinitely detailed fractal.

2. Fractal Patterns Based on Spirals

2.1. "Protospirals" The term "protospirals" is coined here to denote the spiral to which all spirals in a given pattern are similar. This terminology provides a convenient analog to the term "prototile", which denotes a tile to which either a subset of the tiles or all of the tiles in a given tiling are similar. Fractal patterns based on two different protospirals are explored here – the Golden spiral and the Plastic spiral [5]. The construction of these spirals is shown in Figure 1.

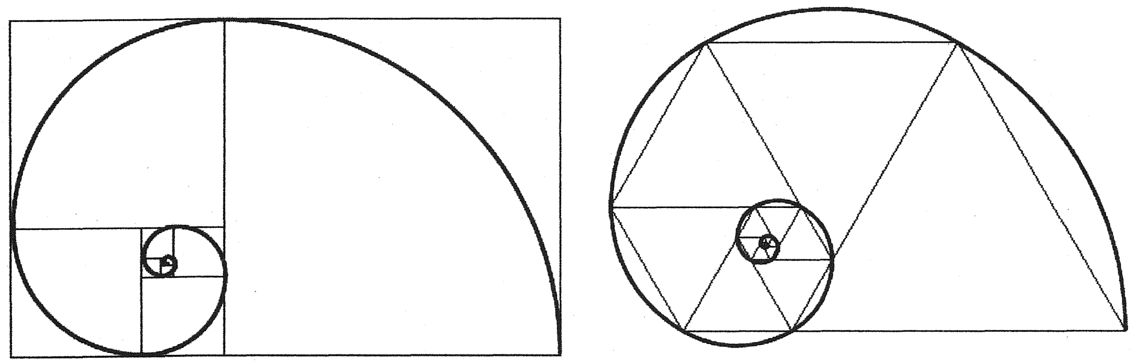

Figure 1: *Two protospirals used to generate fractal patterns. At left, a Golden spiral, formed by joining quarter-circular arcs scaled by the Golden Section ϕ (1.618...). At right, a Plastic spiral, formed by joining one-sixth-circular arcs scaled by the Plastic Number (1.3247...).*

2.2. Patterns Based on the Golden Spiral and Plastic Spiral. A number of patterns have been generated by using a Golden spiral (or Plastic spiral) as the protospiral, scaling it by $1/\phi$ (or the inverse of the Plastic Number) one or more times, attaching one or more of the scaled spirals to the larger one using some simple set of rules, and then iterating. The rules often include mirroring the spirals between successive generations. Largely for convenience, attachments are made at points where quarter-circular (or one-sixth-circular) arcs are joined in the individual spirals. This process is illustrated in Figure 2.

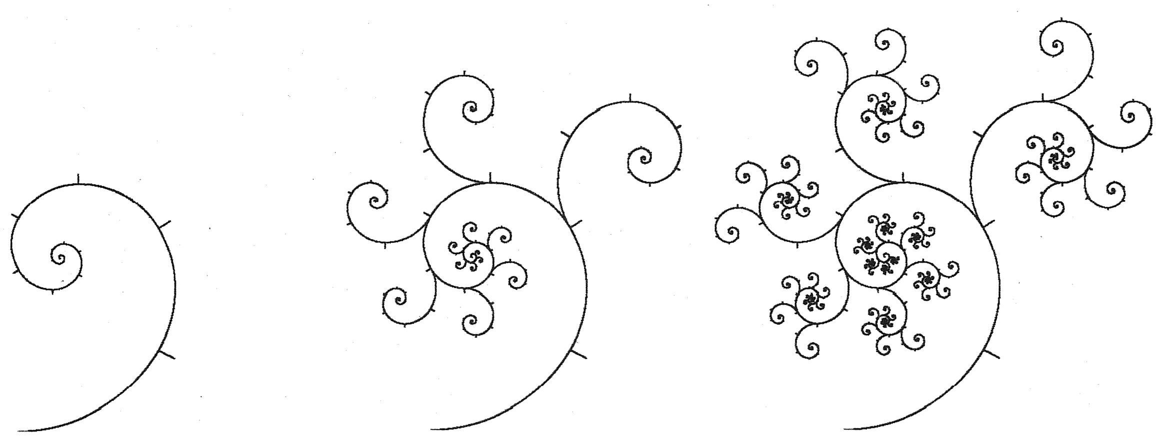

Figure 2: *The first three steps in the iterative process of generating a fractal pattern using a Plastic protospiral and a particular arrangement of smaller spirals on the first-generation spiral. The tick marks are shown to indicate the points at which the one-sixth-circular arcs making up each spiral were joined. These serve as connection points for smaller spirals. In this particular pattern, the first attachment point is skipped to prevent a cluttered pattern in which lines cross one another.*

Patterns based on these spirals that are deemed esthetically pleasing and/or mathematically interesting are shown in Figures 3 and 4, where the patterns are based respectively on the Golden spiral and the Plastic spiral. The first pattern in Figure 3 and the first three in Figure 4 are constructed by connecting smaller spirals to larger ones to form bifurcation or branching points. The second pattern in Figure 3 and the fourth through sixth in Figure 4 are constructed by joining two or more spirals at their outermost points and then nesting smaller copies of these joined spirals within larger ones.

In each case, a simple set of rules is used for attaching smaller spirals to the largest generation. The full pattern is then generated by iterating; i.e., following the same set of rules for each smaller spiral. Rules that result in overlapping lines are intentionally avoided. The resulting patterns are elegant for the most part, though the 90° angles in the Golden spiral can lead to somewhat "blocky" results, as in the first example in Figure 3.

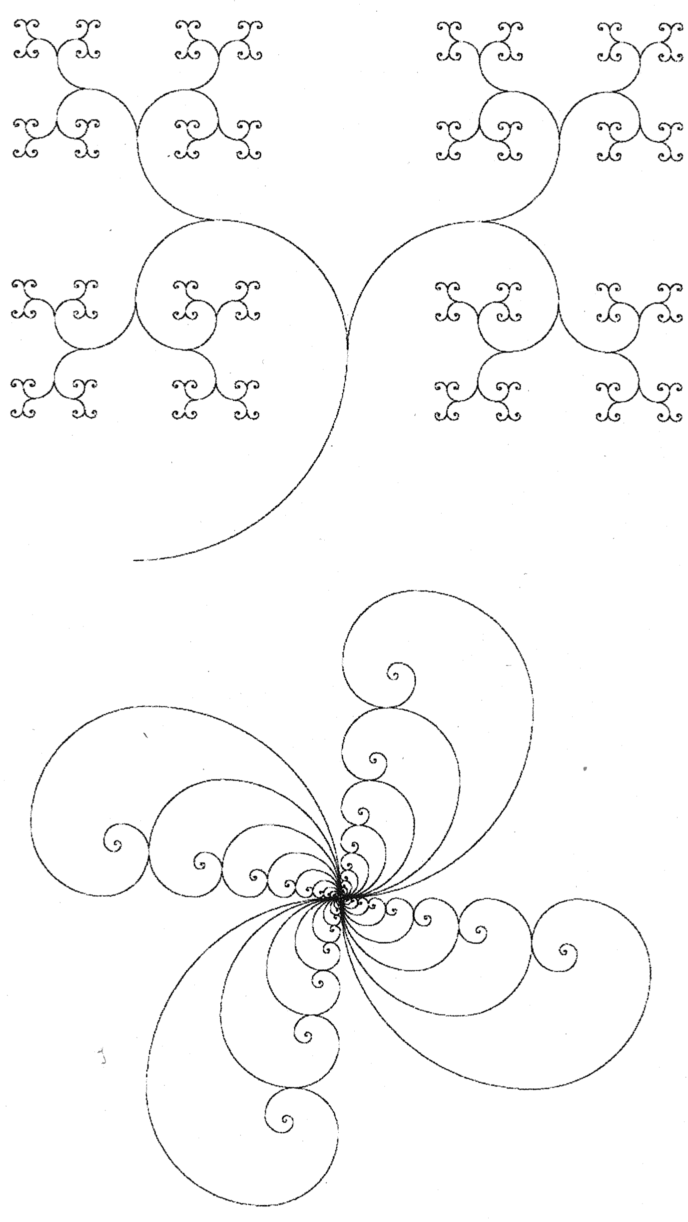

Figure 3: *Two patterns creating by iteratively joining Golden spirals that are scaled down with each successive generation.*

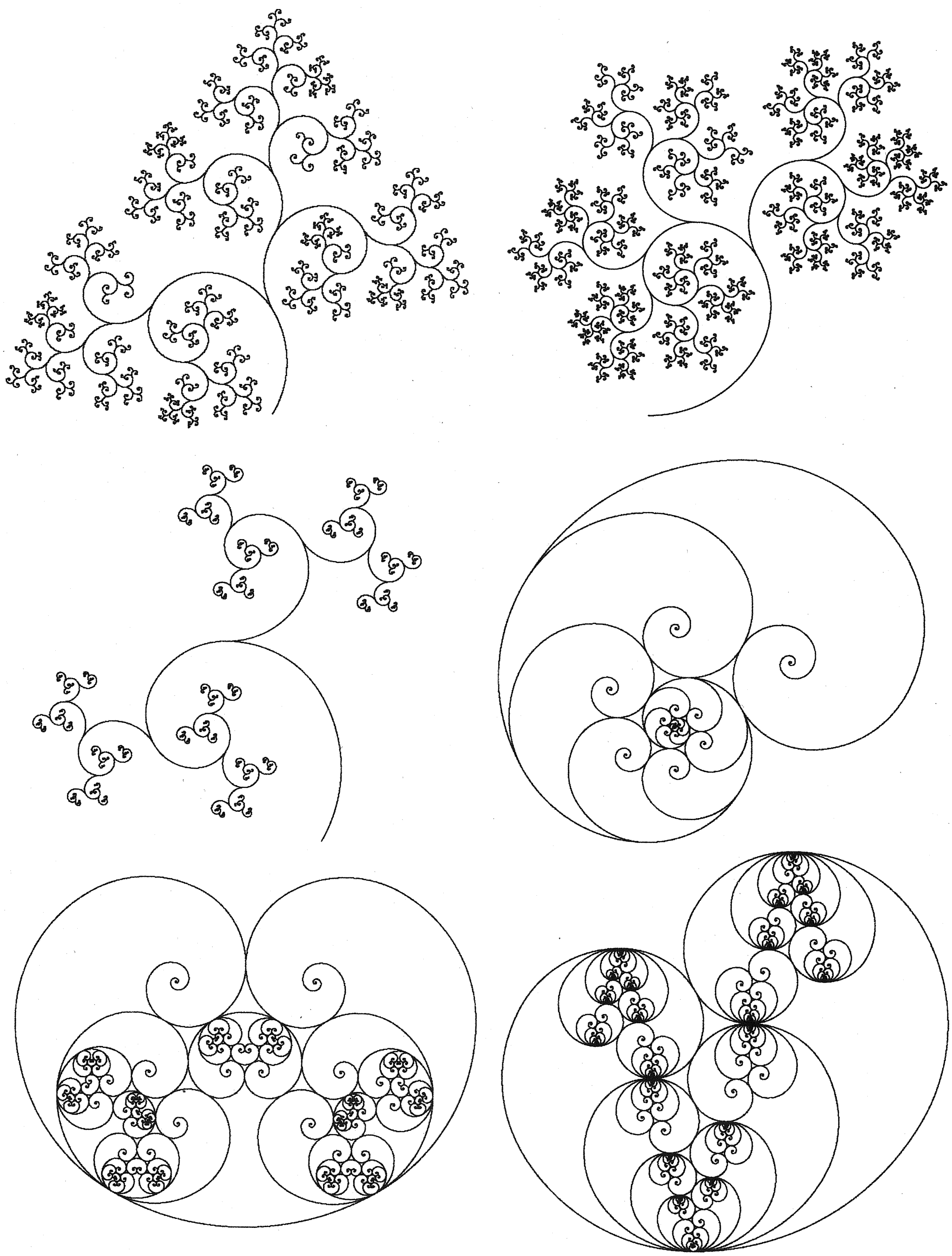

Figure 4: *Six patterns creating by iteratively joining Plastic spirals that are scaled down with each successive generation. The construction process for the upper right pattern is the one described in Figure 2.*

3. Fractal Pseudo-tilings Based on Spirals

3.1. Construction of Prototiles. In this section, spiral-shaped tiles are constructed to allow fractal pseudo-tilings as opposed to fractal patterns. First, a spiral is formed by scaling a straight-line segment by $1/\phi$, rotating it counterclockwise 72°, connecting it to the first segment, and then iterating. This entire spiral is then scaled by $1/\phi$ and rotated clockwise 27°. (This angle was selected by trial and error to give the tile an attractive width for the pseudo-tilings.) The two spirals are joined at their center points and the outermost point of the smaller spiral is pulled to meet the outermost point of the larger spiral (Figure 5). This forms a spiral prototile with inner edges and outer edges.

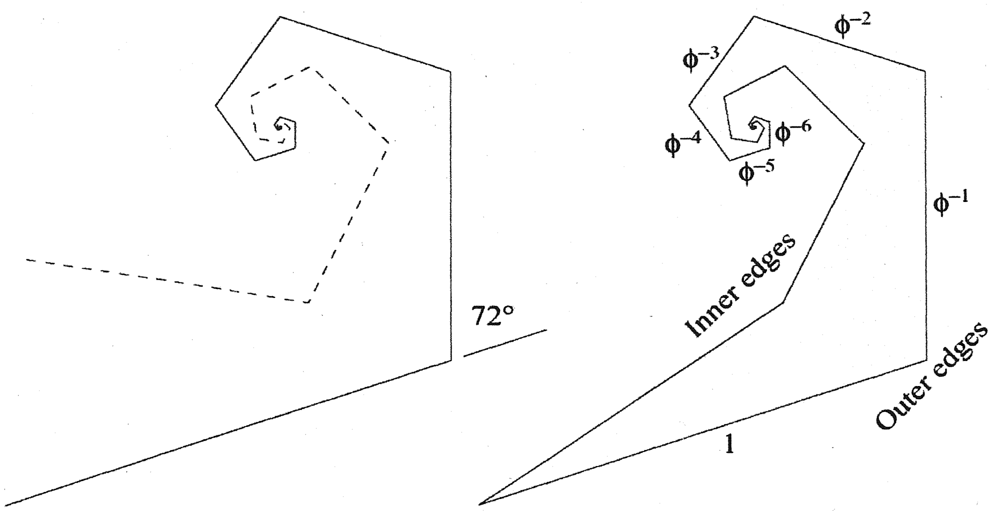

Figure 5: *Construction of a spiral prototile. The lengths of the first seven outer edes are shown in terms of ϕ.*

3.2. Construction of Pseudo-tilings. A "pseudo-tiling" is constructed using this prototile by tiling with outer edges only in an edge-to-edge fashion. The "pseudo-" prefix denotes the fact that the inner edges are not satisfied in these constructs. A fractal pseudo-tiling is formed from this prototile by scaling the tile by $1/\phi$, mirroring it, and matching the longest edge of the second tile with the second longest edge of the first tile. Scaling by $1/\phi$ again, the longest edge of the third tile is matched with the third longest edge of the first tile. This process is carried out many times to complete the first iteration, after which this process is carried out for each of the new tiles, completing the second iteration, followed by additional iterations. The resulting fractal pseudo-tiling shown in Figure 6 is strongly reminiscent of a tree, albeit a rather muscular and geometric one.

Minor modifications to the prototile lead to pseudo-tilings with significantly different appearances. In the left prototile of Figure 7, the second-longest arm of the prototile is stretched by approximately 11%. The second-generation tile that matches to this tile in edge-to-edge fashion must of course be scaled by 0.7 instead of $1/\phi$. The resulting pseudo-tiling, shown in Figure 8, has a much less regular boundary. Likened to a natural tree or bush, it has a very prickly appearance. In the right prototile of Figure 7, several modifications have been made to the prototile of Figure 5. These were determined by trial and error to lead to a pseudo-tiling (Figure 9) bearing strong resemblance to a natural tree.

The fractal pseudo-tilings presented here bear some similarity to binary trees [6], though the building blocks and construction method are quite different. In binary trees, which could also be considered fractal pseudo-tilings, squares (or more generally rectangles) and triangles are alternated, with the triangles creating the bifurcations.

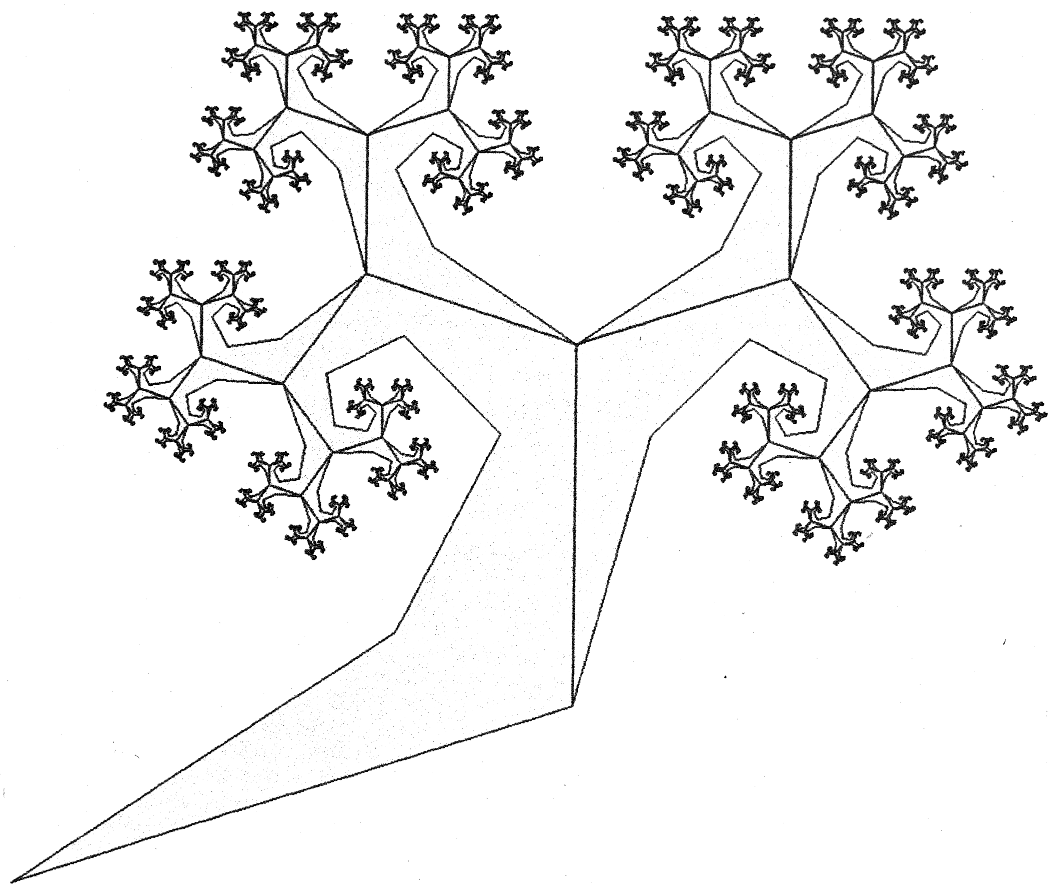

Figure 6: *A fractal pseudo-tiling constructed using the prototile of Figure 5.*

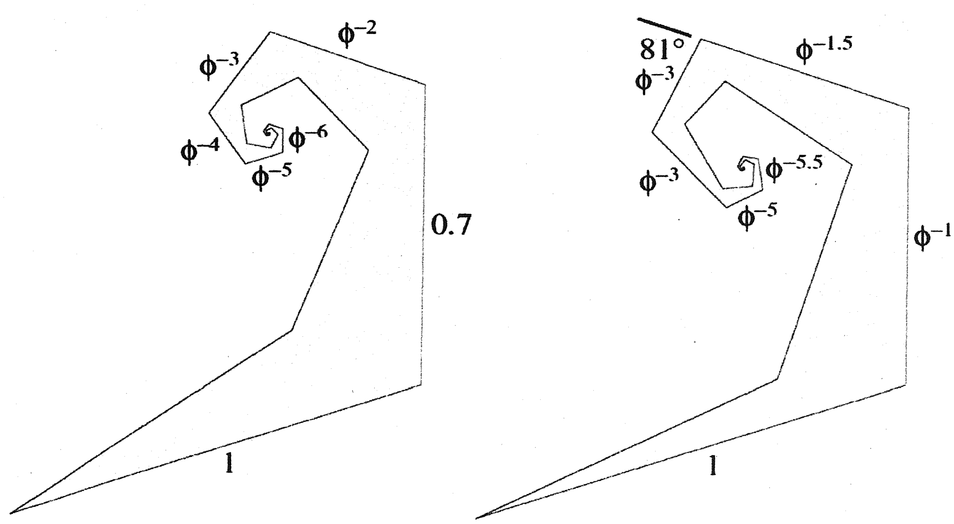

Figure 7: *Prototiles created by modifying the prototile of Figure 5. Unless otherwise noted, the angles between adjacent outer edges are 72°.*

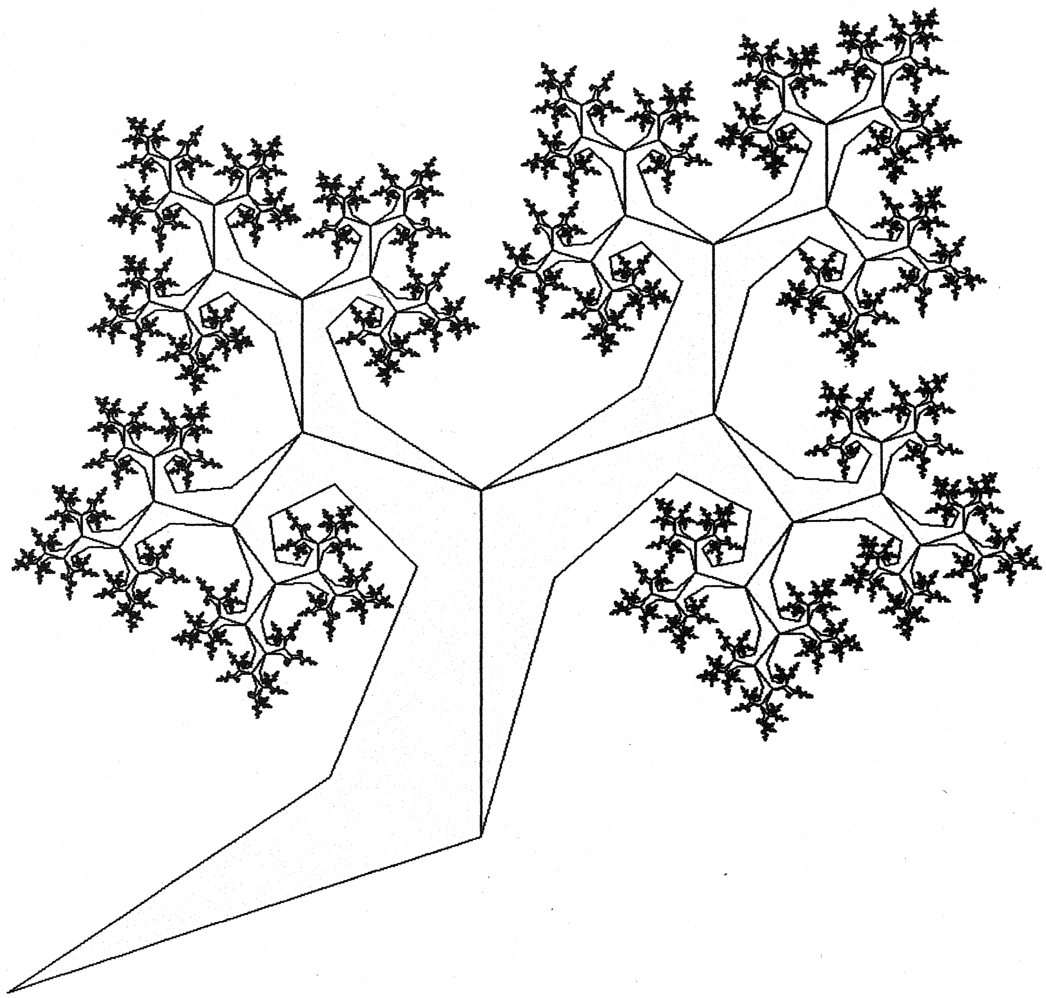

Figure 8: *Pseudo-tiling based on the left prototile of Figure 7.*

4. Conclusions

The work presented here is by no means an exhaustive treatment of the possibilities for graphical fractal constructs based on spirals. Many different types of spirals, such as Fermat's spiral or any of a variety of equiangular spirals [7] weren't explored at all. Countless different spiral prototiles could be constructed, and only a few of the possibilities for positioning smaller spirals relative to larger ones were explored here. It isn't necessary for the spirals to be connected, nor is it necessary that they not be allowed to overlap. From an artistic perspective, there is a wide range of possibilities for rendering these constructions and for choosing settings in which to place them.

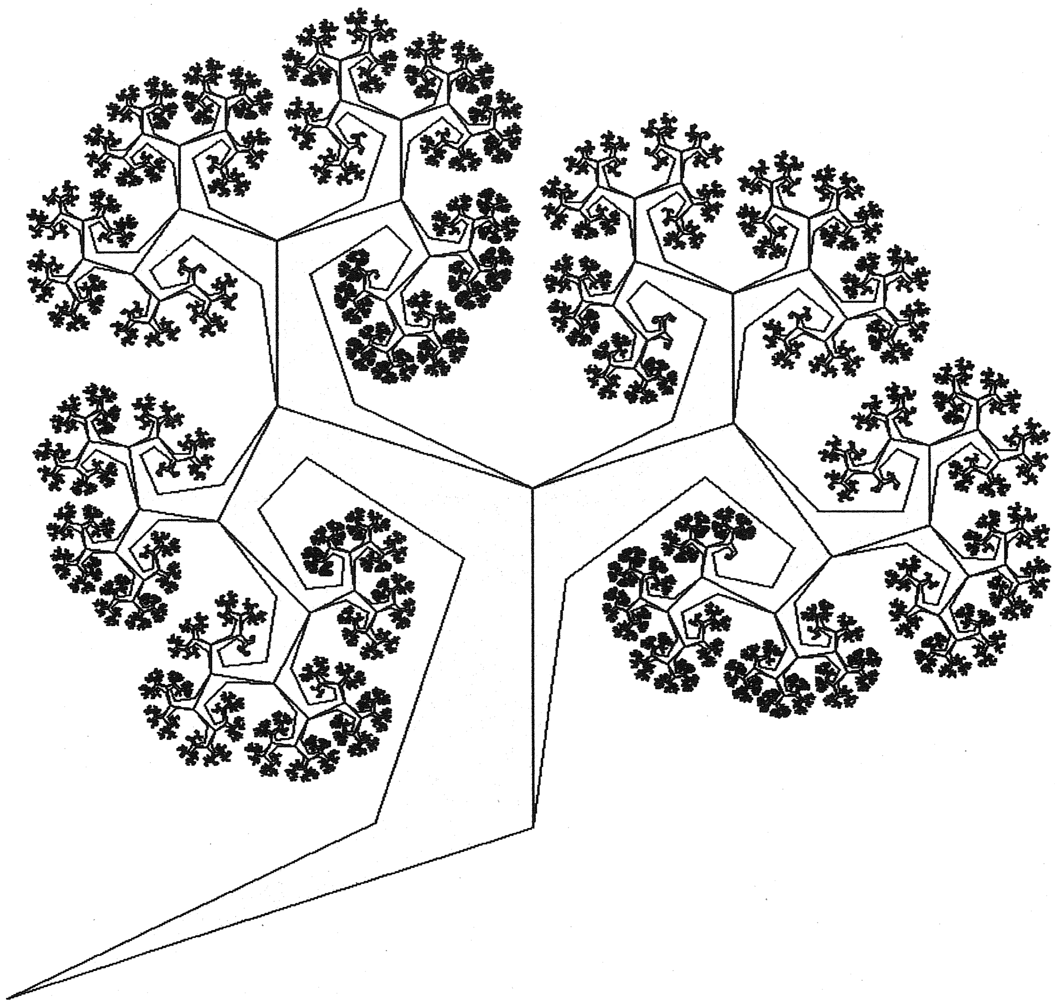

Figure 9: *Pseudo-tiling based on the right prototile of Figure 7.*

References

[1] See, for example, the Infinite Fractal Loop: http://www.fractalus.com/ifl/.
[2] Robert W. Fathauer, *Self-similar Tilings Based on Prototiles Constructed from Segments of Regular Polygons*, in Proceedings of the 2000 Bridges Conference, edited by Reza Sarhangi, pp. 285-292, 2000.
[3] Robert W. Fathauer, *Fractal tilings based on kite- and dart-shaped prototiles*, Computers & Graphics, Vol. 25, pp. 323-331, 2001.
[4] Robert W. Fathauer, *Fractal tilings based on v-shaped prototiles*, Computers & Graphics, Vol. 26, pp. 635-643, 2002.
[5] John Sharp, *Beyond the Golden Section – the Golden tip of the iceberg*, in Proceedings of the 2000 Bridges Conference, edited by Reza Sarhangi, pp. 87-98, 2000.
[6] Heinz-Otto Peitgen, Hartmut Jürgens, and Dietmar Saupe, Fractals for the Classroom, Part One (Springer-Verlag, New York, 1992).
[7] For descriptions of different types of spirals, see for example the Geometry Junkyard: http://www.ics.uci.edu/~eppstein/junkyard/spiral.html.