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Inspired by Snowflakes: Constructing, Folding and Cutting Regular Paper Polygons to Create Art with Dihedral Symmetry

Gwen L. Fisher and Nicole Silkton
Mathematics Department
California Polytechnic State University
San Luis Obispo, CA 93407, USA
E-mail: glfisher@calpoly.edu

Abstract

The structure of natural snowflakes can be abstracted mathematically and can be reproduced in paper and fabric art. Mathematically, natural snowflakes are beautiful examples of objects with a combination of both rotation symmetry and reflection symmetry, known as dihedral symmetry. Designs with dihedral symmetry can be easily constructed from regular polygons; so we provide a brief summary of the classic methods for constructing regular polygons including Euclidean constructions (compass and straightedge), paper folding constructions, as well as a practical method that uses a protractor. Finally, we provide several examples of how artists can use regular polygons made from paper to produce art with dihedral symmetry.

1. Introduction: Snowflakes in Nature

Figure 1 shows some naturally symmetric snowflakes that were photographed by Wilson Alwyn Bentley (1865-1931). Bentley was a physicist who chose not to copyright his images of ice crystals because he wanted them to be used by the public. To make these images readily available the University of Wisconsin has published digital versions of 1,183 of Bentley's images, which can be found at http://mail.ssec.wisc.edu/snow/browsebentley.htm.

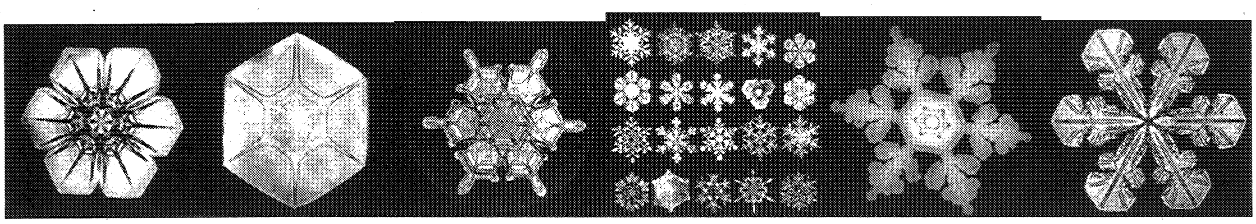

Figure 1: *Bentley's photographs of snowflakes.*

Looking at Bentley's photos, we wondered why the snowflakes all appear to have six points. At http://www.snowcrystals.com/ we learned that snowflakes form when water condenses and freezes onto a small particle of dust. The crystal quickly takes on a simple hexagonal form because the electrical bonds of water molecules cause the crystal lattice of ice to have hexagonal symmetry, at least under most atmospheric conditions. As the crystal grows, the corners of the hexagon begin to protrude, thus attracting more water. Eventually, these protrusions become the arms or branches of the snowflake. Since each branch of the snowflake is exposed to nearly identical weather conditions, the branches grow more or less identically. In short, it is because of the hexagonal structure of frozen water that most snowflakes have 6 points. In mathematical terms, we say that most snowflakes have $\mathbf{D}_6$ dihedral symmetry (such symmetry will be discussed more below). However, some natural snowflakes have $\mathbf{D}_3$ symmetry, and others have nearly $\mathbf{D}_{12}$ symmetry. Figures 2 and 3 show photographs of such snowflakes, taken by physicists Patricia

Rasmussen with hardware developed by Kenneth G. Libbrecht from California Institute of Technology. In an email from Libbrecht, he writes, "Snowflakes never grow as true 12-pointed crystals. The 12-branched snowflakes are each composed of two 6-branched crystals connected at the center."

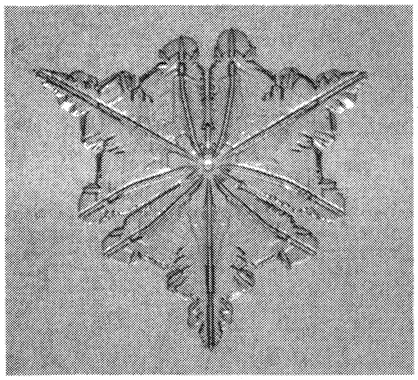

Figure 2: *A natural 3-pointed snowflake.*

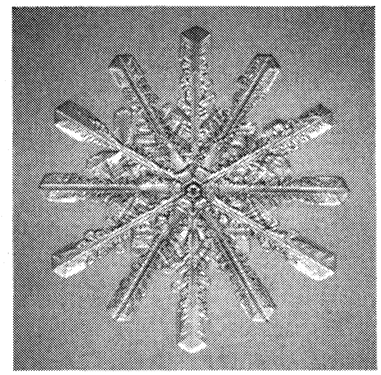

Figure 3: *A natural 12-pointed snowflake.*

In this paper, we discuss what are commonly referred to as *paper snowflakes*, which are inspired by natural snowflakes, but can have any number of sides. For convenience, throughout this paper, we use the term *snowflake* as a general term to describe any design with dihedral symmetry. While it is possible to make snowflakes with any number of sides with paper or fabric, this is not the case in nature. In particular, there are no four-, five- or eight-sided snowflakes made from water crystals. However, Libbrecht and others have conjectured that molecules other than water could produce snowflakes with symmetries not found in ice crystals. In particular, cold planets could experience CO_2 snow, which would have 4-fold symmetry.

2. Definitions

To talk about the mathematical structure of all types of snowflakes, we begin with some definitions.

2.1 Polygonal Curve. A polygonal curve is the union of line segments that meet end-to-end with no more than two line segments meeting at a single point.

2.2 Regular Polygon. A regular polygon is a simple, closed, polygonal curve in the plane in which all of the line segments are the same length and the interior angles all have the same measure.

2.3 Constructible. A figure is constructible if and only if it can be drawn with a pencil, straightedge and compass.

2.4 Dihedral Symmetry. The symmetries of snowflakes (or regular polygons) are described mathematically by dihedral groups. The dihedral symmetries include both reflections and rotations. Intuitively, reflection symmetry is when an object can be folded onto itself and matches perfectly. The line of the fold is called the line of symmetry. Thus, when a symmetric object is flipped over a line of reflection, the object will appear unchanged. Consequently, the number of sides of a regular polygon is equal to its number of lines of reflection symmetry. Regular polygons also have rotational symmetry, which is where an object can be rotated about a point by a fixed angle and the object appears unchanged to the viewer. For example, if we were to rotate and equilateral triangle 60°, it would appear unchanged (all the lines would be in the same places as before). We say that an equilateral triangle has 60°-rotational symmetry. In general, a regular n-gon has $360°/n$ degree rotational symmetry.

In definitional terms, the set of clockwise rotations r_k around a fixed center point C is a cyclic group of order n, where the $r_k = 360k/n$ are measured in degrees and $0 \leq k < n$. A dihedral group D_n contains the n elements of the cyclic group of rotations r_k together with n reflections through the n lines in the plane, all of which intersect at C, and the angles formed by the intersecting lines are the r_k. All dihedral groups have the property that a reflection followed by a rotation is equivalent to a rotation of the same size but in the opposite direction followed by the same reflection. An object that is unchanged under the set of elements of a dihedral group is said to have dihedral symmetry.

2.5 Fundamental Domain. A fundamental domain of a pattern with dihedral symmetry is a smallest piece of the pattern that can be used to build the pattern through reflections and rotations.

2.6 (Paper) Snowflake. A (paper) snowflake is any design that can be made from folding and possibly cutting a piece of paper so that the result has dihedral symmetry.

3. Building Regular Polygons

Starting with a sheet of paper shaped as a regular polygon, anyone can easily create a paper snowflake that has the same symmetries as that regular polygon. Because of this fact, a few comments on the constructibility of regular polygons are due. In this section, we consider three methods for building regular polygons on square paper. First, one may draw a regular polygon using several common tools, including a pencil, straightedge, protractor, and compass. For regular polygons other than the triangle, square, hexagon, octagon, and possibly the dodecagon, this is arguably the easiest and most accurate method, and consequently, it was used for the figures in this paper. Second, one may limit the use of tools to the classic Euclidean set: pencil, straightedge, and compass. Third, one may choose to use only paper and scissors to build a polygon. Backgrounds for these methods are discussed next.

3.1 Drawing Regular Polygons with a Pencil, Straightedge, Protractor, and Compass. Every regular polygon can be easily drawn, at least in theory, with a pencil, straightedge, protractor and compass. In practice, the accuracy of this method is often superior to that of the other methods described below. Needless to say, even better accuracy can be achieved by using a computer.

To begin drawing a regular polygon, mark a point C for the center of the polygon. Use the compass to draw a circle centered at C with radius equal to that of the desired polygon. Now determine the central angle of the regular polygon by taking 360° and dividing it by the number n of sides of the polygon. Call the measure of this angle A. With the protractor centered at C, mark off n adjacent angles, each with measure A, around the circumference of the circle. Use the straightedge to draw the line segments that connect each consecutive pair of marks on the circumference of the circle. These line segments should form the sides of the desired regular polygon.

3.2 Constructible Regular Polygons. The puzzle of constructing regular polygons with only a pencil, straightedge and compass is one that has entertained people since the time of Euclid. Every year, countless high school students constructed regular polygons with 3, 4, 6, and 8 sides, but it was not until the mathematics of Carl Freidrich Gauss (1777-1855) that people determined exactly which regular polygons are constructible [1]. In 1837, Wantzel completed Gauss' proof that a regular polygon of n sides ($n \geq 3$) may be constructed if and only if

$$n = 2^k p_1 \ldots p_t$$

where $k \geq 0$ and all p, if any, are distinct odd primes, each of which has the form $p = 2^{2^h} + 1$, where h is a whole number (see Nathan Jacobson's *Basic Algebra I* for a complete proof) [2]. These primes p are called Fermat Primes, named after Pierre de Fermat (1601-1665), who conjectured (wrongly) that any integer of the form $2^{2^h} + 1$ is a prime. In fact, Leonhard Euler (1707-1783) was the first to show that $2^{2^5} + 1$ is factorable. Furthermore, p is composite for $5 \leq h \leq 19$ and at least 45 other values of h. Empirical evidence suggests that the number of Fermat primes is finite, and it is possible that the set {3, 5, 17, 257, 65537} corresponding to $h = 0, 1, 2, 3, 4$ is the complete set of Fermat primes. Using the equation above for n, we see that a compass and straightedge can be used to make a regular polygon with a number of sides equal to 3, 4, 5, 6, 8, 10, 12, 15, 16, 17, 20, 24, etc. The regular heptagon (7 sides) and nonagon (9 sides) are noticeably absent from this list.

3.3 Folding Regular Polygons from Paper. The same regular polygons that are easy to construct are also easy to fold, and the folding sequences are contained in many sources including origami books and elementary mathematics textbooks. Moreover, because the folding axioms encompass the construction axioms, all of the constructible polygons are also foldable. A particularly elegant treatment of the subject is given by Kunihiko Kasahar in his book *Amazing Origami* in which Kasahar provides detailed descriptions and pictures for folding exact regular polygons with 3, 4, 5, 6, 8, and 12 sides [3]. Although the heptagon and nonagon are missing from Kasahar's book, Robert Geretschläger has shown how to fold a regular heptagon and a regular nonagon, as well as a trisection of any angle [4]. Furthermore, Andrew M. Gleason has shown that the angle trisection allows for the construction of many more regular polygons. In particular, a regular polygon of n sides ($n \geq 3$) can be constructed with a ruler, straightedge and angle trisector (and consequently by paper folding) if

$$n = 2^k 3^l p_1 \ldots p_t$$

where $k, l \geq 0$ and all p, if any, are distinct primes (> 3), each of the form $2^s 3^t + 1$. Specifically, a regular polygon can be folded with number of sides equal to any integer between 3 and 21, except 11. Fortunately, Gleason points out that if one wishes to construct a regular 11-gon with a straightedge and compass, an angle quinsector (which cuts an angle into 5 equal angles) will assist in this construction [1].

4. Folding and Cutting Paper Snowflakes

All one needs to cut many paper snowflakes is paper and scissors. Additional supplies for more complex snowflakes include a pencil, protractor, compass, and straightedge. Fancy scissors and hole punchers can be used for artistic effects.

The general method for creating paper snowflakes with dihedral symmetry starts with a square piece of paper. Be sure that *all folds go through the center point of the original square*. After all of the folds are set, cut designs in the folded paper with scissors and hole punchers. This shape is the fundamental domain of the snowflake. Remember not to cut away an entire folded side, since doing so will cause the snowflake to fall apart. Cutting on folds will result in holes in the snowflakes. In contrast, cutting on an edge with no folds will result in changing the outside edge of the snowflake. We now discuss possible folding sequences for paper snowflakes with various dihedral symmetries.

4.1 Paper Snowflakes with 2^n Lines of Symmetry. The 2^n-pointed snowflakes can all be folded by starting with a square sheet of paper. To fold and cut a snowflake with $2^0 = 1$ line of symmetry, simply fold the square in half and then make cuts. The unfolded paper design will have $\mathbf{D}_1$ symmetry, which means that its rotational symmetry is the trivial one of 360°.

We can fold and cut a snowflake with $2^1 = 2$ lines of symmetry. Fold the square in half to make a rectangle. Fold it in half again to make a square. Now make cuts to create the snowflake. Open the snowflake. This snowflake has $\mathbf{D}_2$ symmetry, including 180° rotational symmetry. An alternate folding sequence is to first fold the square in half to make a triangle and then fold it in half again to make another triangle. Then cut and open.

We can fold and cut a snowflake with $2^2 = 4$ lines of symmetry. Fold the square in half three times to give a triangle, always making sure that each fold goes through the center of the original square of paper. Now make cuts to create the snowflake. Open the snowflake. This snowflake has $\mathbf{D}_4$ symmetry, including 90° rotational symmetry.

In general, we can fold and cut a snowflake with 2^n lines of symmetry by folding a square sheet of paper in half n times with all folds going through the center of the original square of paper. For eight sides or more, even out the edge with no folds by cutting through all layers near that edge. Now make cuts to create the snowflake. Open the snowflake. This snowflake has $\mathbf{D}_{2^n}$ including $360°/2^n$ rotational symmetry.

4.2 Paper Snowflakes with Any Number of Lines of Symmetry. The method described in this subsection will work for a paper snowflake with any number of points. The only restrictions are in the thickness of the paper, the sharpness of the scissors, and the agility of the paper cutter.

Start with a regular polygon of n sides where n is equal to the desired number of points on the finished snowflake. Figure 4 shows the construction of a snowflake with $\mathbf{D}_7$ symmetry, and Figure 5 shows a set of original paper snowflakes with various dihedral symmetries. The steps shown in Figure 4 are as follows.

1. Drawing of a circle;
2. Marking 7 points on the circle with the angle 360°/7, which is about 51.4°;
3. Drawing of a regular heptagon (7 sides) inscribed in the circle;
4. A cut heptagon, with all lines of symmetry pre-folded;
5. A heptagon folded into a triangle with one angle equal to 360°/14, ready to be cut;
6. A cut, folded heptagon snowflake (this is the fundamental domain of the snowflake); and
7. The unfolded snowflake.

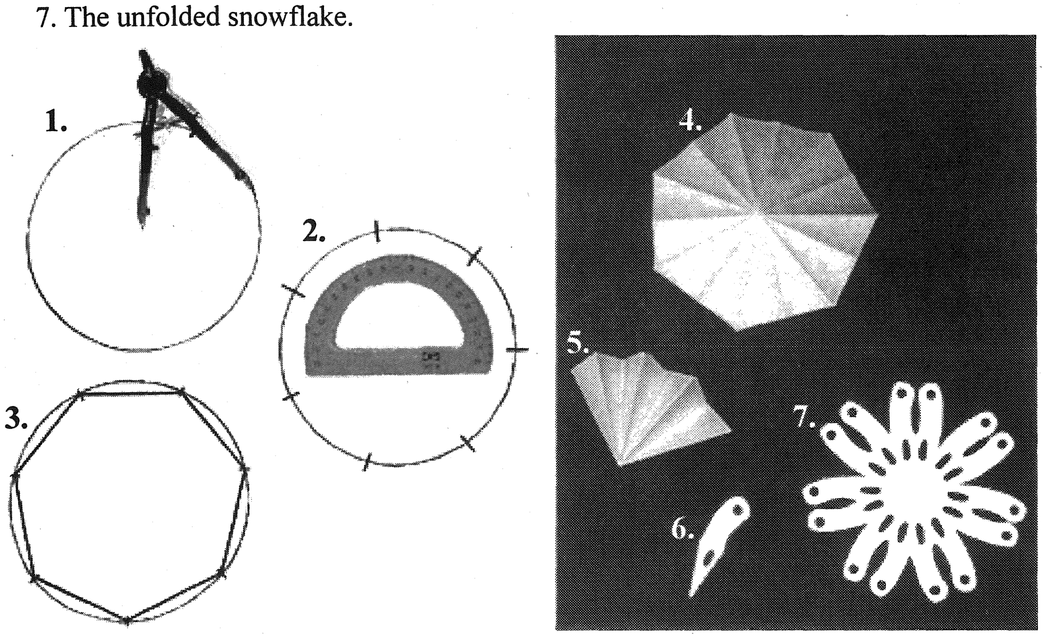

Figure 4: *Making a cut paper snowflake with* $\mathbf{D}_7$ *symmetry.*

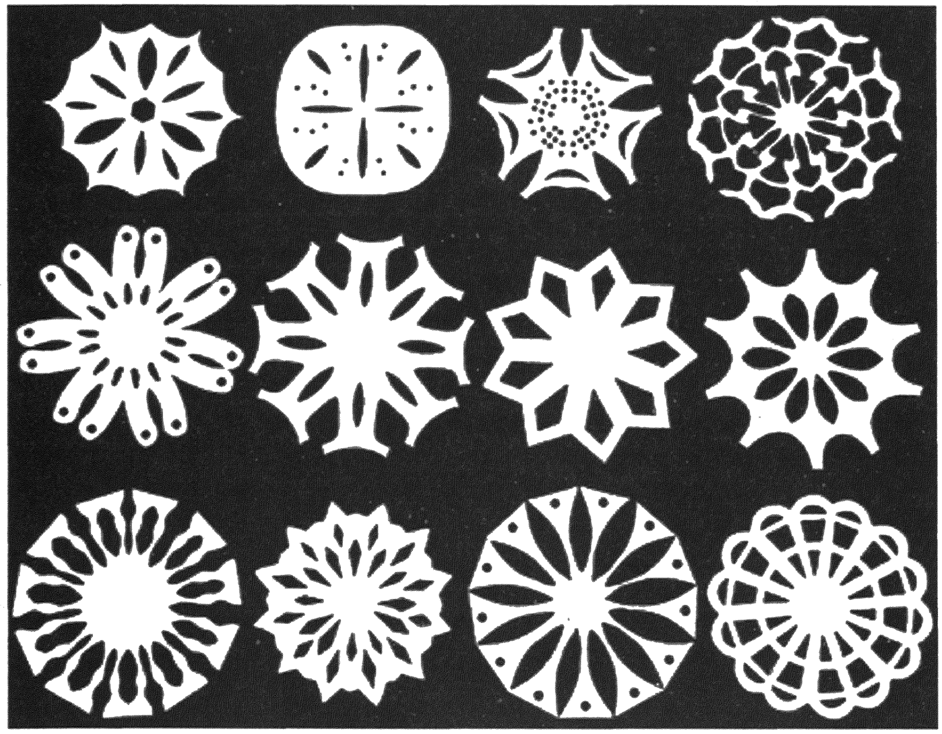

Figure 5: *Cut paper snowflakes.*

Practically speaking, the construction of paper snowflakes with high orders of symmetry can be tricky. Looking at Figure 5, one may notice that the snowflakes with D_{11} and D_{13} symmetry (bottom row, first and third) are not as symmetric as the snowflakes with lower orders of symmetry. This is due to the nature of construction of these snowflakes. To create snowflakes like these, the paper needs to be folded 11 or 13 times, respectively. However, a paper with this many folds quickly becomes unwieldy. The folded edges of the paper no longer line up with one another; so, when the fundamental domain is cut, it looks different on the different layers of the folded paper. When the paper is unfolded, the cuts appear in different places in each of the branches, thereby ruining the perfect symmetry of these snowflakes. Even though the D_{12} and D_{14} snowflakes (bottom row, second and fourth) also have high orders of symmetry, they are not as difficult to construct because we take advantage of 12 and 14 being even, by cutting the fundamental domain into a symmetric design. This allows us to fold the paper only half of the number of times as we would have to do otherwise.

5. Snowflakes in Origami

Some artists have made paper snowflake designs without the use of scissors. In his book *Extreme Origami* Kunihiko Kasahara provides steps for folding paper snowflakes that do not require any cutting [5]. Kasahara's method was inspired by Friedrich Wilhelm August Froebel (1782-1852), the German originator of the first kindergartens. Froebel used basic forms to fold snowflakes with $\mathbf{D}_3$, $\mathbf{D}_4$, and $\mathbf{D}_6$ symmetry. Figure 6 shows examples of folded paper snowflakes with 3, 4, 5, 6, 7, 8, and 9 sides. Before folding any paper, we cut each sheet into the regular polygon to match the final shape of the paper snowflake. All of these snowflakes use at least two sheets of paper, which we nested after each sheet was folded individually. After having completed the folds in Figure 6, we questioned whether or not we could fold a set of paper snowflakes using exactly the same folding sequence on each polygon, only varying as needed for the number of sides in the polygon. Figure 7 shows the results of this task.

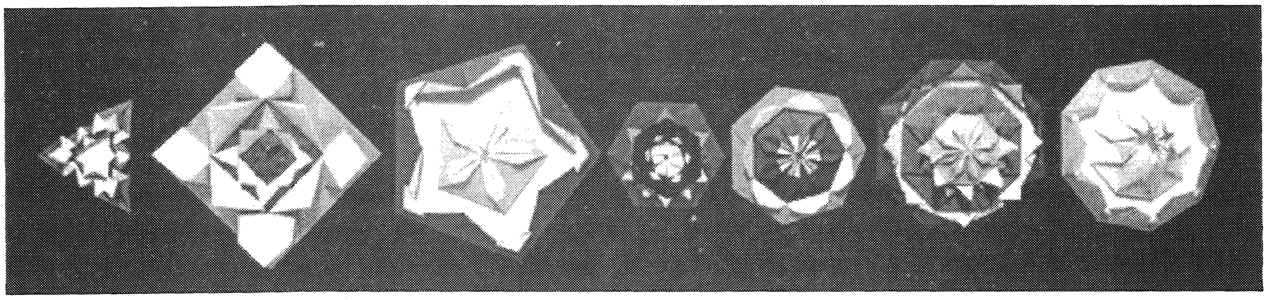

Figure 6: *Folded paper snowflakes.*

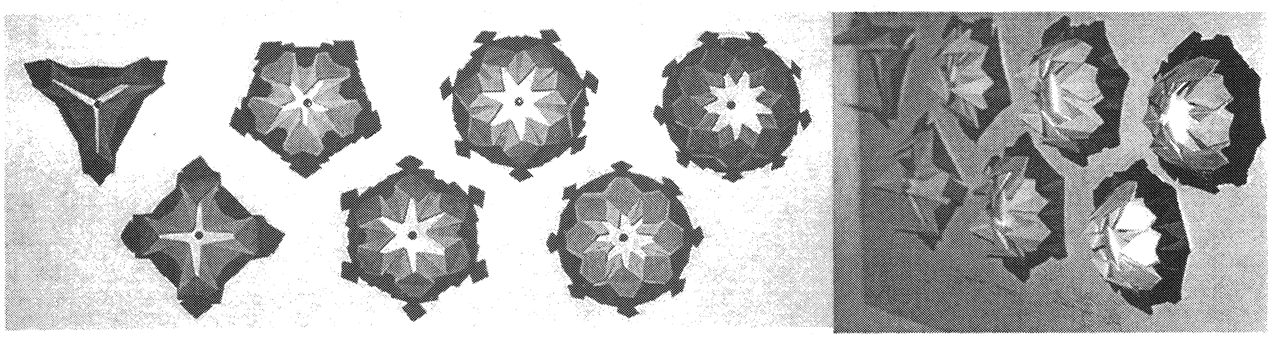

Figure 7: *Matched folded paper snowflakes from two different angles.*

Each of the paper snowflakes in Figure 7 uses two sheets of paper, cut as congruent regular polygons. All of these regular polygons were inscribed in circles with equal radii. This allows one to see how the number of sides of the polygon affects the appearance of the completed snowflake. After having folded the matched set of paper snowflakes, we decided to make a set of three nested folds from regular 13-gons. Two preparatory folds are shown in Figure 8: The fold on the left shows all of the lines of reflection symmetry, plus creases created by folding each vertex to the center, and the fold on the right of Figure 8 includes all of these folds plus those created by folding each edge to the center. The crease patterns in both cases admit $\mathbf{D}_{13}$ symmetry. Similar folds can be set on any regular paper polygon to produce dihedral symmetry of other orders. The completed $\mathbf{D}_{13}$ paper snowflake is shown in Figure 9.

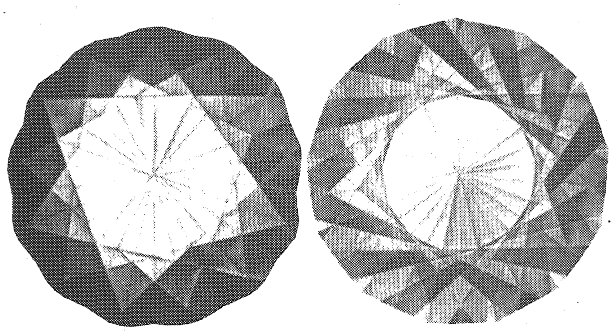

Figure 8: *Preparatory folds for the $\mathbf{D}_{13}$ folded snowflake.*

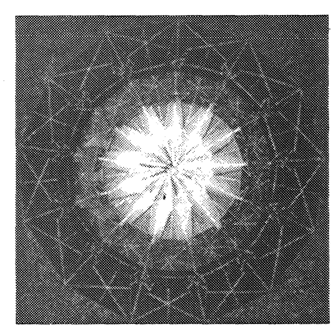

Figure 9: *$\mathbf{D}_{13}$ folded snowflake.*

6. Dihedral Symmetry in Hawaiian Quilt Design

To provide a wider context for the application of the art of paper snowflakes, we note one traditional art form that displays such designs: Hawaiian quilts. The Hawaiian quilt design was a clearly identifiable style as early as the 1910s. The method for construction of Hawaiian quilts uses, in part, the same techniques for constructing paper snowflakes: folding the fabric (or a paper pattern) into quarter or eighths before cutting. Hawaiian quilts regularly display dihedral symmetry, almost exclusively using $\mathbf{D}_2$ or $\mathbf{D}_4$ symmetry. Nancy Lee Chong designed the quilt patterns shown in Figure 10, which were found at http://www.quilthawaiian.com/. As is typical of Hawaiian quilt patterns, each quilt in Figure 10 uses only two colors of fabric, one for the background and one for the appliquéd "snowflake," which, as one might expect, does not represent snow. Instead, Chong's designs represent the natural flora of Hawaii, a traditional feature of Hawaiian quilts. Notice how the six design elements from the left quilt are the same six design elements in the right quilt. One can reasonably argue that Chong has maintained the essence of the fundamental domain in each case, but has changed its relative location or orientation in the block.

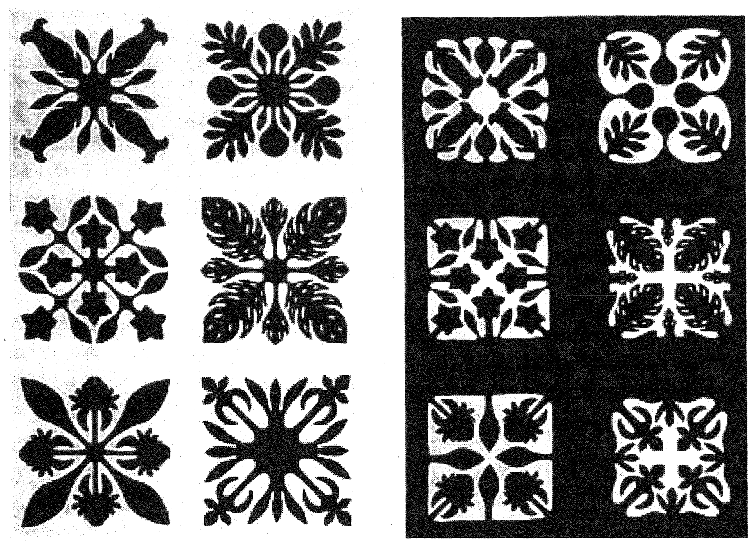

Figure 10: *Nancy Lee Chong's Hawaiian quilt designs.*

To conclude, natural snowflakes can be abstracted by using the mathematical construct of the dihedral group, usually, although not exclusively, $\mathbf{D}_6$. To make paper snowflake designs with dihedral symmetries of any order, one can start with a paper cut as a regular polygon. The number of sides of the regular polygon determines which tools are needed to build the polygon. Finally, we showed how any regular paper polygon can be folded and cut to create a design with dihedral symmetry.

References

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