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Mathematical Connections
in Art, Music, and Science

A Search in Progress: Polyhedra from Intersecting Cylinders

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Abstract

The familiar polyhedron used as an architectural element created by the intersection of two cylinders of the same radius and their axes lying in the same plane and being perpendicular to each other was studied as early as ca 250 BCE by Archimedes [1]. Many different polyhedral forms can be created by symmetrical intersection of cylinders in the same way. A description of three unique groups of cylindrical polyhedra is described based on the symmetry axes of the five regular planar forms of polyhedra.

1. Introduction

The intersection of two cylinders of the same radius with their axes lying in a common plane has been presented as a solution to many architectural undertakings such as the ceiling vault of the Crypt under the abbey of St. Denis, 1140 [2], or the Palm house by Champs-Elysées and Jardin d'Hiver in Paris, 1854 [3].

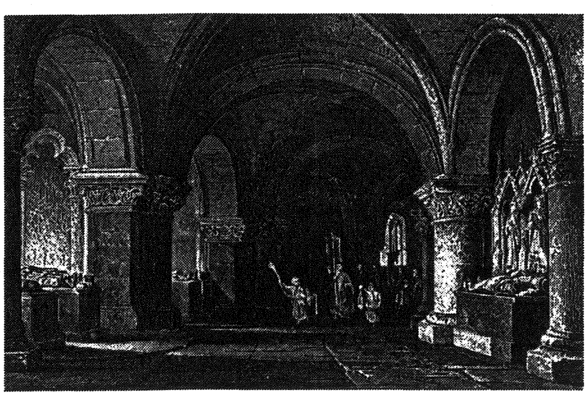

Figure 1: *abbey of St. Denis*

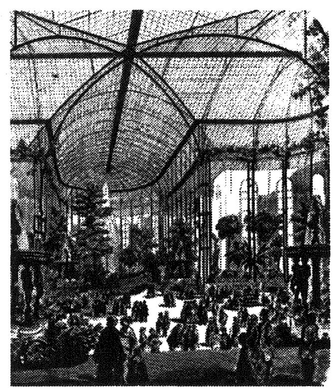

Figure 2: *the Palm house*

However, very little information appears in the literature for cylindrical polyhedra composed of cylinders having the same radius and intersecting at a common point other than those used in architecture.

In the early 1970's while teaching a freshman design course this author began exploring polyhedral forms from the intersection of cylinders based on the axes of the regular and semi-regular polyhedra. Many paper models were constructed and exhibited of the series that will be described in this paper as Group I; however, the polyhedra discovered were never published. During the same period, in correspondence with Charles E. Peck, the author of A Taxonomy of Fundamental Polyhedra and Tessellations, the studies being conducted were often discussed. Later, around the mid 1990's, he became interested in the intersection of cylinders polyhedra and began studying the Group III polyhedra. A search for any publication he may have done of his work has been to no avail. Figure 3 is the only illustration contained in the correspondence showing a few of the cylinder polyhedra he was generating. [4].

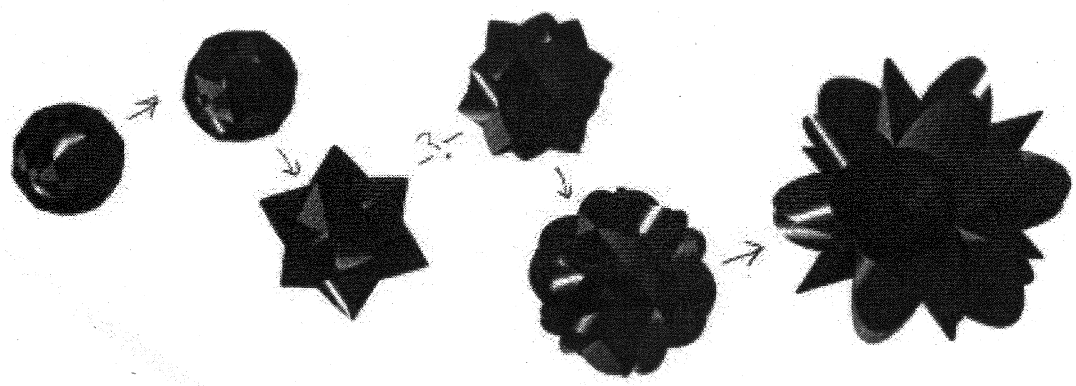

Figure 3: *Peck's Group III "Stellations of cylinders polyhedra"*

Ian O. Angell and Moreton Moore published a description of the Group I intersection of cylinders polyhedra in 1987 [5] [6]. Their article may be the first to describe this group of cylindrical polyhedra. During the writing of this article I came across Paul Bourke's web site. In December 2003 he published many illustrations of the Group I cylindrical polyhedra. He also gives the source code for creating those illustrated. [7]

2. Polyhedra by Intersection of Cylinders

2.1. Basic polyhedra axes. The five regular polyhedra were chosen to demonstrate the three groups of polyhedra that may be formed from the intersection of cylinders. Each polyhedron has three sets of axes that may be used separately or in combination to find the polyhedral forms sought. The three sets of axes are those passing through the center of the polyhedron and through the opposite edge, opposite face, or opposite vertex. Due to the duality of symmetry only three series of polyhedra in each of the three groups will be illustrated. The tetrahedron is self dual, the octahedron is the dual of the hexahedron, and the icosahedron is the dual of the dodecahedron.

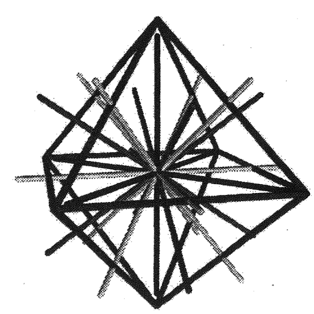

Figure 4: *The three axis sets*

For the hexahedron there are three axes in the face set, four axes in the vertex set and six axes in the edge set. The octahedron has four axes in the face set, three axes in the vertex set and six axes in the edge set. Note the duality. The dodecahedron has six axes in the face set, ten axes in the vertex set and fifteen axes in the edge set, while its dual, the icosahedron, has ten axes in the face set, six axes in the vertex set and fifteen axes in the edge set. The tetrahedron is unique from the other polyhedra in that it is non-centro-symmetric and for most cases will develop the same polyhedra by intersection of cylinders as the hexahedron and octahedron. The tetrahedron has four central axes in the face set, four central axes in the vertex set, or four opposite axes in the face/vertex set and three axes in the opposite edge set.

2.2. Definitions. Two public domain software packages were used to find the polyhedra sought: POV™-RAY and SpringDance™ [8], [9].

Intersection: two or more shapes are combined to make a new shape that consists of the volume common to both shapes. [8]
Union: two or more shapes are added together. [8]

Three unique groups of cylinder polyhedra are generated:

Group I: generated by the intersection of cylinders thus defining a common volume to all cylinders
Group II: generated by the intersection of unique sets of cylinders and adding sets together thus generating a compound polyhedra.
Group III: generated by adding sets of cylinders together and then intersecting sets thus generating Stellation types of polyhedra.

Figure 5 is a crossed eye stereo view of the polyhedron found by the intersection of the face/vertex axes of a tetrahedron.

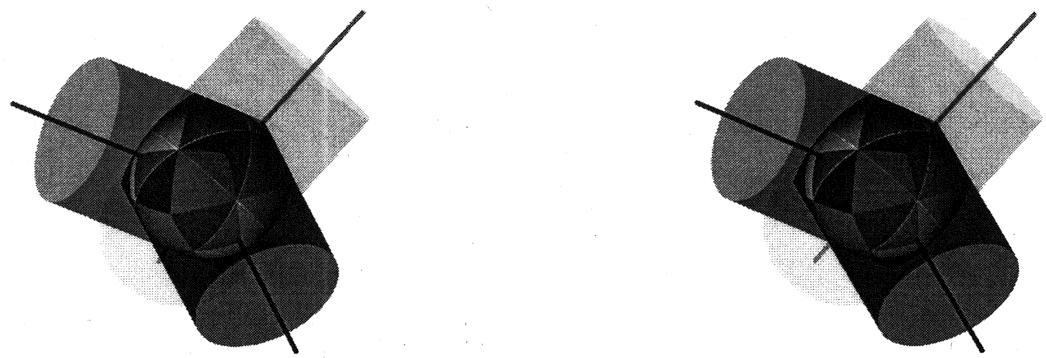

Figure 5: *intersection of cylinder axes of a tetrahedron*

For each group the axes sets for each base polyhedron may be treated as a single set, or combination of sets. Those illustrated in this paper will be:

Edge set	{E}	Face set	{F}	Vertex set	{V}
Edge/Face/Vertex set	{EFV}	Face/Vertex set	{FV}		
Edge/Face set	{EF}				
Edge/Vertex set	{EV}				

For each cylindrical polyhedron illustrated; a notation will be given to identify the group name, the base polyhedron, set combination and number of cylinders in the axes set. H_{GI}_EFV_C_6, C_3, C_4 refers to a base polyhedron as the hexahedron (H), Group I (GI) cylindrical polyhedron created form a combination of edge, face, vertex (HFV) sets having a six cylinder set (C_6), three cylinder set (C_3) and a four cylinder set (C_4). In place of H; I, O, D, or T may be used to refer to the icosahedron (I), octahedron (O), dodecahedron (D), or tetrahedron (T). In place of GI associated with the base polyhedron letter; GII or GIII may be used to refer to Group I (GI), Group II (GII), or Group III (GIII).

In order to give some comparison between each group, the three groups' illustrations will be arranged in columns in the same order of set combinations. Only the cylindrical polyhedron derived from the hexahedron and icosahedron set combinations will be illustrated in this paper.

2.3. Group I (GI): Cylindrical polyhedra by intersections.

The volume common to the intersection of cylinders common to the edge axes, face axes, and or vertex axes may be found by:

Intersection_of_Cylinders Polyhedra = intersection { E_Axis_1.........E_Axis_n,, F_Axis_1.........F_Axis_n , V_Axis_1........V_Axis_n }

Where: the radius of all axes are the same.

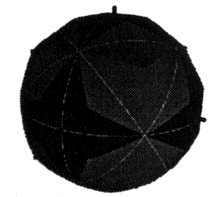

Figure 6: $T_{GI}\_E\_F/V\_C_7$

2.4. Group II (GII): Cylindrical polyhedra by intersection of cylinder sets and union of sets.

Start by defining:

Three sets of:

Edge_Cylinders set = intersection { E_Axis_1.........E_Axis_n }
Face_Cylinders set = intersection { F_Axis_1..........F_Axis_n }
Vertex_Cylinders set = intersection { V_Axis_1.........V_Axis_n }

Where: the radius of all axes are the same.
Then: the compound is determined by:

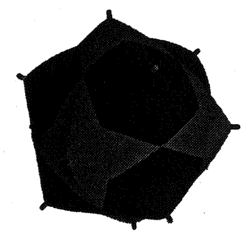

Figure 7: $T_{GII}\_E\_F/V\_C_3, C_4$

Compound Cylinder Polyhedra = union { Edge_Cylinders, and/or F_Cylinders, and/or V_Cylinders }

2.5. Group III (GIII): Cylindrical polyhedra by union of axes cylinders as sets and intersection of sets.

Start by defining:

Three sets of:

Edge_Cylinders set = union { E_Axis_1........E_Axis_n }
Face_Cylinders set = union { F_Axis_1.........F_Axis_n }
Vertex_Cylinders set = union { V_Axis_1........V_Axis_n }

Where: the radius of the axes in each set is the same and the radius of the axes between sets may not be the same.
Then: Stellation is determined by:

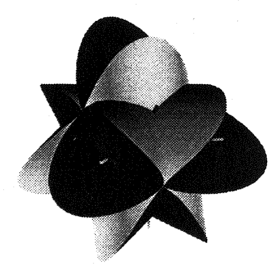

Figure 8: $T_{GIII}\_E\_V\_ C_3, C_4$

Stellation Cylindrical Polyhedra = intersection { Edge_Cylinders, and/or F_Cylinders, and/or V_Cylinders }

2.6. Mathematical operations for the axes cylinder sets. Figures 9, 10 and 11 of the hexahedron are created from each single axes cylinder set by an intersection operation.

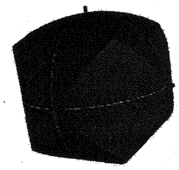

Figure 9: $H_I\_F\_C_3$

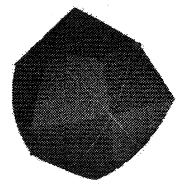

Figure 10: $H_I\_V\_C_4$

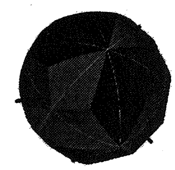

Figure 11: $H_I\_E\_C_6$

Figures 12, 13 and 14 of the hexahedron are created from each single axes set by a union operation. For the union operation there is variation in cylinder radius between sets.

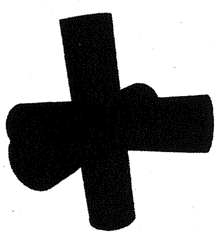

Figure 12: $H_U\_F\_C_3$

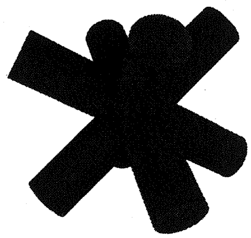

Figure 13: $H_U\_V\_C_4$

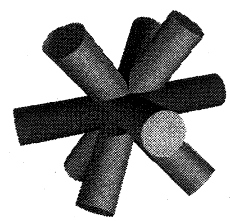

Figure 14: $H_U\_E\_C_6$

Figures 15, 16 and 17 of the icosahedron are created from each single axes cylinder set by an intersection operation.

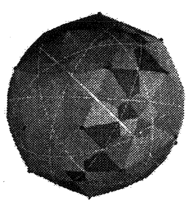

Figure 15: $I_I\_F\_C_{10}$

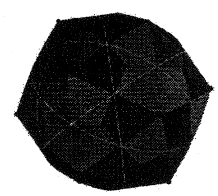

Figure 16: $I_I\_V\_C_6$

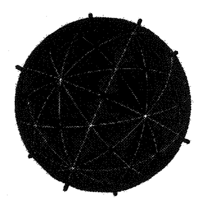

Figure 17: $I_I\_E\_C_{15}$

Figures 18, 19 and 20 of the hexahedron are created from each single axes set by a union operation. For the union operation there is variation in cylinder radius between sets.

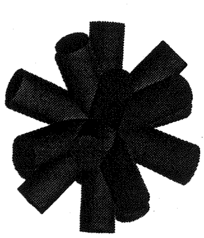

Figure 18: $I_U\_F\_C_{10}$

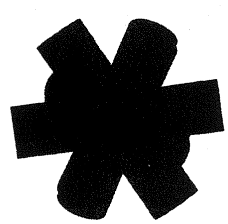

Figure 19: $I_U\_V\_C_6$

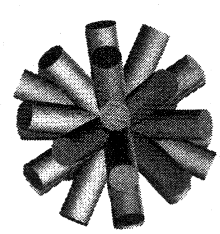

Figure 20: $I_U\_E\_C_{15}$

2.7. Hexahedron/octahedron cylindrical polyhedra. The hexahedron Groups I, II and III were chosen to be illustrated below. Group I appears in the center column of figures, Group II in the left column of figures and Group III in the right column of figures. These polyhedra are the results of the last step of mathematical operations applied within the various combinations of axes cylinder sets for its Group.

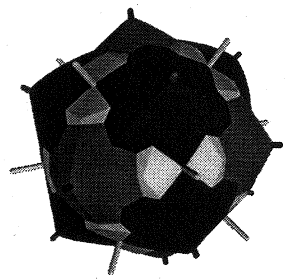

Figure 21: $H_{GII}\_EFV\_C_6, C_3, C_4$

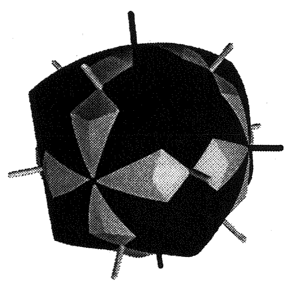

Figure 22: $H_{GII}\_EF\_C_6, C_3$

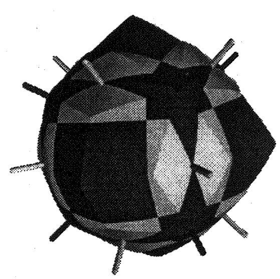

Figure 23: $H_{GII}\_EV\_C_6, C_4$

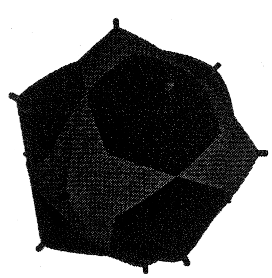

Figure 24: $H_{GII}\_FV\_C_3, C_4$

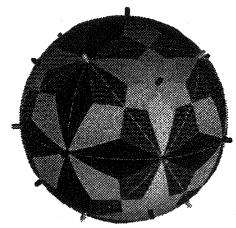

Figure 25: $H_{GI}\_EFV\_C_{13}$

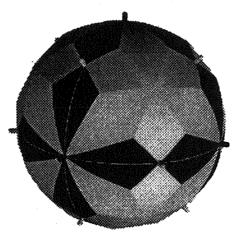

Figure 26: $H_{GI}\_EF\_C_9$

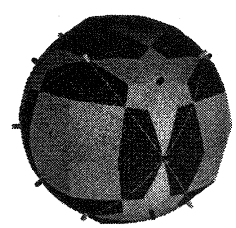

Figure 27: $H_{GI}\_EV\_C_{10}$

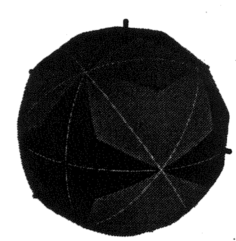

Figure 28: $H_{GI}\_FV\_C_7$

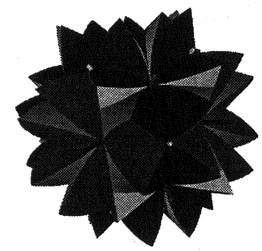

Figure 29: $H_{GIII}\_EFV\_C_6, C_3, C_4$

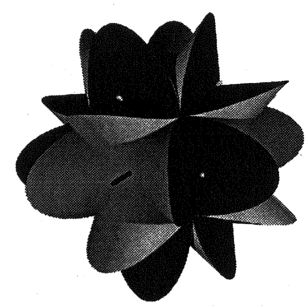

Figure 30: $H_{GI}\_EF\_C_6, C_3$

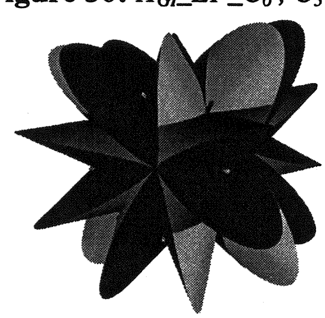

Figure 31: $H_{GI}\_EV\_C_6, C_4$

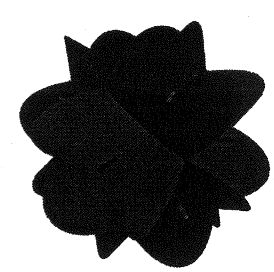

Figure 32: $H_{GI}\_FV\_C_3, C_4$

2.8. Dodecahedron/icosahedron cylindrical polyhedra. The icosahedron Groups I, II and III were chosen to be illustrated below. Group I appears in the left column of figures, Group II in the center column of figures and Group III in the right column of figures. These polyhedra are the results of the second step of mathematical operations applied within the various combinations of axes cylinder sets for its Group.

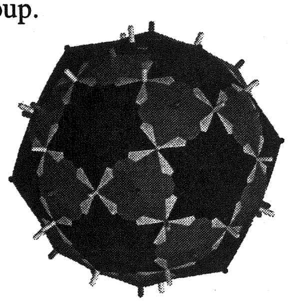

Figure 33: $I_{GII}\_EFV\_C_{15}, C_{10}, C_6$

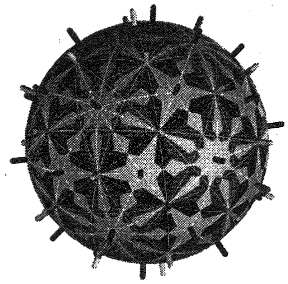

Figure 37: $I_{GI}\_EFV\_C_{31}$

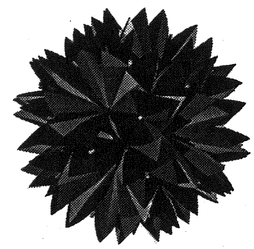

Figure 41: $I_{GI}\_EFV\_C_{15}, C_{10}, C_6$

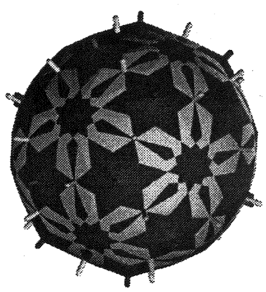

Figure 34: $I_{GII}\_EF\_C_{15}, C_{10}$

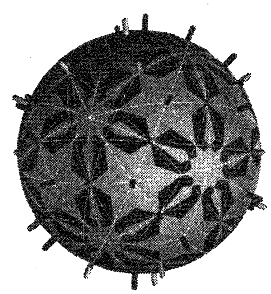

Figure 38: $I_{GI}\_EF\_C_{25}$

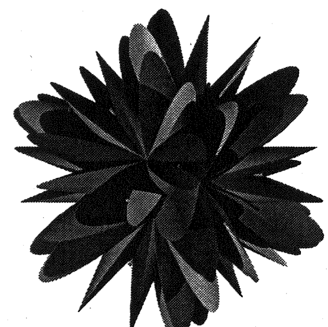

Figure 42: $I_{GI}\_EF\_ C_{15}, C_{10}$

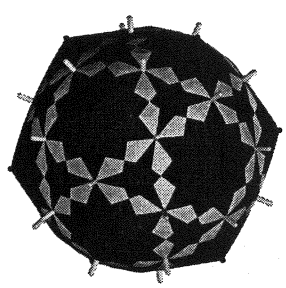

Figure 35: $I_{GII}\_EV\_C_{15}, C_6$

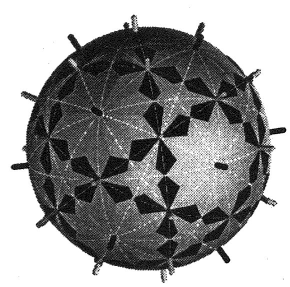

Figure 39: $I_{GI}\_EV\_C_{21}$

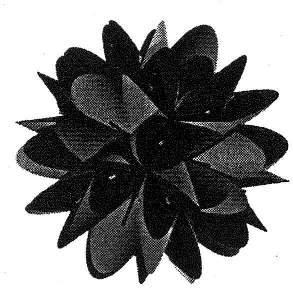

Figure 43: $I_{GI}\_EV\_C_{15}, C_6$

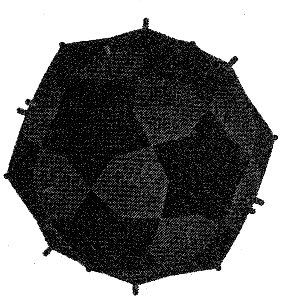

Figure 36: $I_{GII}\_FV\_C_{10}, C_6$

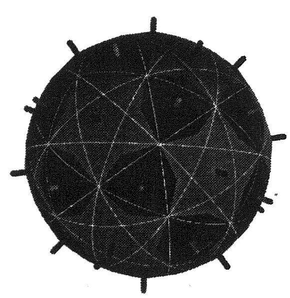

Figure 40: $I_{GI}\_FV\_C_{16}$

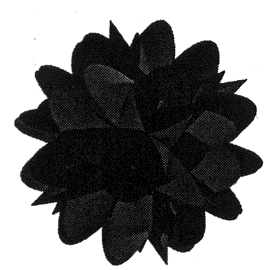

Figure 44: $I_{GI}\_FV\_C_{10}, C_6$

3. Summary

Three groups of cylindrical polyhedra can be constructed from the intersection of cylinders passing through the center of a planar polyhedron. Group I- intersection of cylinders polyhedra, may be though of as the volume common to the intersection of all cylinders passing through a common point. Group II-compound cylindrical polyhedra, may be considered as similar to the compound polyhedra found in the planar class of polyhedra. Group III-stellation of cylindrical polyhedra, has the appearance of the planar stellated polyhedra. However, the rules of Stellation for these forms have not been completely defined at this time. The cylinder axes in Group III described in this paper all pass through a common center point. However, this may not hold true for other polyhedra within the group. We have already noted that the radius for the Group III may not be the same throughout the sets. In Groups I and II the cylinder axes radii remain the same through the construction process. It may be interesting to explore cases where their radii are allowed to change.

Further search into these fascinating forms is still on going. A catalogue is in progress and a search for those forms based on the thirteen semi-regular and higher ordered forms is being pursued.

I would like to thank the referees for their valuable suggestions for presenting this paper.

References

[1] Archmedes, *The Method Treating of Mechanical Problems: a letter to Eratosthenes*, Great Books of the Western World, Encyclopædia Britannica, Inc., Chicago, IL, 1952, Vol. 11, pp. 569-592.
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