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Mathematical Connections
in Art, Music, and Science

Star Polygon Designs of *La Alhambra's* Wooden Ceilings

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Abstract

This paper will describe how some of the beautiful Islamic star polygon designs found in the wooden ceilings of *La Alhambra* may be reconstructed using only the geometer's tools of straightedge and compass (or the electronic counterpart, the *Geometer's Sketchpad* software). The idealized, computer-generated patterns will be compared with photographs of the original ceilings. The crystallographic symmetry group classification for each ceiling design will also be discussed.

Introduction

Building upon and assimilating the traditions of ancient cultures – such as the Persian, Roman, Byzantine and Pharoanic Egyptian civilizations – Islamic artists, over a period of a thousand years, developed, as one of their main features, the extensive use of infinitely-repeating geometric patterns. Their fascination with the visual principles of repetition, symmetry and the continuous generation of pattern, led to the development of new geometric variations, thus taking the art form to unparalleled heights. These geometric designs were incorporated into all aspects of Islamic art and architecture, including their work with mosaics, decorative ceramics, and metal, and carvings in ivory and wood.

Wood, which was plentiful in Spain, was used by Islamic artisans and woodworkers to make balconies, doors, and window grilles, all of which were standard features in the domestic architecture of *al-Andalus*. The choice of wood (from among pine, cedar, teak, ebony, walnut, cypress, elm, box, pear, pomegranate, beech, willow, palm and tamarisk) depended on what was locally available. Timber paneling was also frequently installed in palaces and domestic buildings, where the degree of decoration and carving of the paneling depended upon the prestige, social position and wealth of the owner. Islamic artists and woodworkers also introduced the wooden ceiling, often richly carved (even in moderately sized houses) to Spain. The paneled ceilings were either carved and inlaid with strips of light and dark woods, to form an interlacing geometric pattern, or they were carved and painted. Paneled ceilings were also stepped into honeycomb cells (known as *artesonados* [1]), or made using a "knot" technique to produce an interlacing effect.

Islamic woodworkers (carpenters, joiners, and cabinet-makers) were highly respected for their expertise, and a Moorish tradition of woodworking skills has been handed down over the centuries in Spain. In fact, local municipal authorities issued rules by which the construction of timber roof structures incorporating the decorative ceilings were to be governed. In 1619, Diego Lopez de Arenas, a carpentry surveyor of Seville, wrote a publication specifying the type of structure, the dimensions of the wood members to be used, the framework and the decorative patterns of the ceiling. A series of drawings accompanying the rules also showed the layout and the geometric principles involved in the setting-out of the work [1].

Although wood is perishable, a wide variety of carved and joined woodwork has survived through the centuries in Spain, notably at *La Mezquita* of Cordoba and *La Alhambra* of Granada. In fact, the entire complex of buildings at *La Alhambra* only survived because Ferdinand and Isabella declared it a *Casa Real*, or royal residence, in 1492, and took it upon themselves to restore it. Most of the original woodwork surviving in Spain was restored in the 15^{th} and 16^{th} centuries by artists who were direct descendants of those artisans who had been creating it during the 13^{th} to 15^{th} century [2]. In most instances, it is hard to tell the difference between the work of the Moors and that of the Christian. (Muslims who converted to Christianity after 1492 were known as Mudejars or Moriscos).

Geometric Islamic designs may be recreated by constructing a framework of identical *repeat units*, or *motifs*, that recur regularly to form a geometrical grid or a regular division of the plane. The motif can be produced using only a compass to draw circles and a straight edge to draw line segments. The radius of a circle serves as a basic linear unit, from which equilateral triangles, squares, pentagons, hexagons, octagons, star-shaped and other polygons may be constructed within that circle. The plane may then be filled with the circles containing the design, to form an infinitely-repeating geometric tiling. The number and type of geometric designs that may be created in this manner are unlimited.

Mathematically, however, each design may only be classified as belonging to one of a finite number of possible symmetry groups, based on the transformational symmetries it permits. These symmetries include *rotations* about a point (called a center of rotation) through a given angle, *translations* in a given direction through a given distance, mirror *reflections* in a line (called a mirror line or mirror of reflection), and *glide reflections* which combine translations through a given distance and parallel to a line, and then a reflection in the line. To tile a flat plane, there are only 17 different symmetry groups (called crystallographic symmetry groups) possible, involving the various combinations of rotations, translations, reflections, and glide reflections. The crystallographic restriction only allows one-fold (the identity transformation), two-fold, three-fold, four-fold or six-fold rotations [3].

Photographs of some of the beautiful Islamic star polygon designs found in the wooden ceilings of La Alhambra will be illustrated. Each of the designs will be analyzed for their symmetry elements and classified based on the transformational symmetries it permits. (All of the wooden ceiling designs discussed in this paper may be classified as belonging to either the *p*4*m* or *cmm* crystallographic groups). Lastly, the designs will be recreated in a manner described by El-Said and Parman [4] and also by the author ([5] – [8]), using only the geometer's tools of straightedge and compass (or the electronic equivalent, the *Geometer's Sketchpad* software [9]).

Ten- and Five-Pointed Stars Design in *El Mexuar*

The first pattern, a ten- and five-pointed stars design, is found in a wooden ceiling panel (see Figure 1.) in *El Mexuar* (named for the Arabic word, *mashwar*). This room, the most modified of all buildings of *La Alhambra*, originally had a cupola, which was replaced with a flat wooden roof and a Moriscos-decorated wooden ceiling in the16th century [2]. An idealized, computer-generated rendition of this design, without the interlacing, is given in Figure 2.

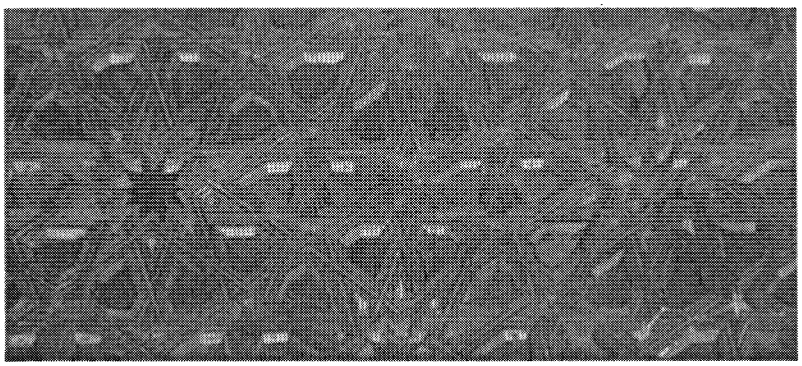

Figure 1. Detail of ceiling panel in *El Mexuar*

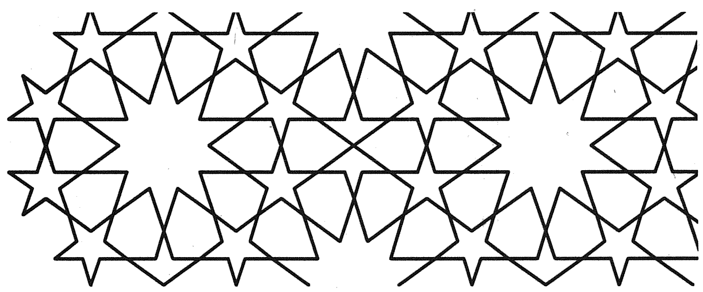

Figure 2. Computer-generated version of Figure 1., without the interlacing

This pattern consists of ten-pointed stars, each surrounded by eight five-pointed stars. All of these star polygons may be constructed from regular pentagons inscribed within a circle. The design has a crystallographic group classification of *cmm*, since it permits a two-fold rotation (about a point located in the center of the ten-pointed star) and mirror lines through the center point, where not all rotation centers are on mirror lines. In Figure 3. below, a segment of the skeletal design, showing only the star polygon motifs, indicates the position of four of the possible mirror lines.

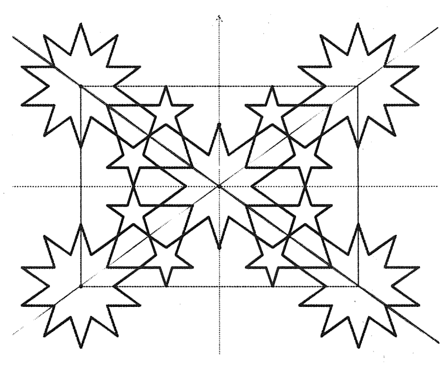

Figure 3. Skeletal drawing of Figure 1., without the interlacing

Painted, Eight-pointed Stars Design in *El Mexuar*

The second pattern, an eight-point stars design, is also found in *El Mexuar* in another Moriscos-decorated ceiling of the16th century (Figure 4.). The design, part of a painted, wooden ceiling panel, consists of three different, but related, eight-pointed star polygons embedded within each other. The smallest eight-pointed star may be formed by offsetting two squares by 45 degrees. This small star is embedded in a larger eight-pointed star, which may be created by extending the edges of the smallest star until the lines intersect; these points of intersection form the vertices of the star. The third and largest eight-pointed star is also produced by extending the edges of the second star, but to enclose the edges to form a "rose petal," extra intersecting lines in the shape of an "X" must be introduced at the point of tangency of a circle and a radius that intersects the vertices of the smallest star.

If the three concentric star shapes are thought to be positioned at the origin of a rectangular coordinate system, then the smallest star shape also appears an equal distance away, along both the positive and negative x- and y-axes. The smallest star shape is also the basis for the "comet-like" polygons. The smallest eight-pointed stars and the four "comets" are centered at the vertices of a regular octagon. An idealized, computer-generated version of this design, without the interlacing, is given in Figure 5.

Figure 4. Detail of ceiling in *El Mexuar*

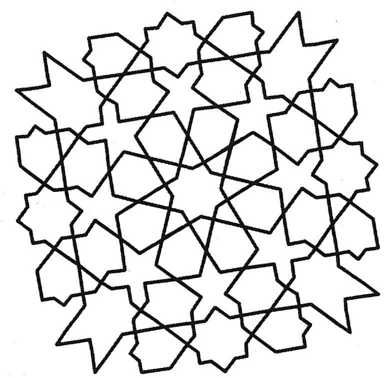

Figure 5. Computer-generated version of Figure 4. without the interlacing

The skeletal design (that is, with the interlacing disregarded), is highly symmetric, with a crystallographic group classification of $p4m$, since it permits both four-fold rotations (about a point located in the center of the eight-point star) and mirror lines through the center point in four directions (at 45 degree angles to one another). The design, showing only the star polygon motifs (along with the "comet-like" eight-pointed stars) and four of the possible mirror lines, is below, in Figure 6.

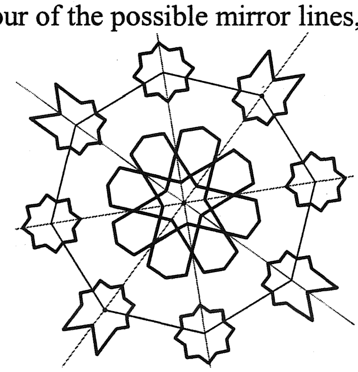

Figure 6. Skeletal drawing of Figure 4., without the interlacing

Twelve- and Eight-Pointed Stars Design in *la Sala de la Barca*

The third pattern, a twelve- and eight-pointed stars design, is part of a gilded, pine ceiling in la *Sala de la Barca*, an anteroom to the main throne room, *Salon de Embajadores* (Hall of Ambassadors) in the Comares Palace. The room's name, the "Hall of the Blessing," may derive from the Arabic word *al-baraka* ("blessing" or "greeting"). Or, alternately, the room may be known as the "Hall of the Boat" because the ceiling looks like the hull of an upturned row boat (*barca* in Spanish); each end of the ceiling is in the form of a semi hemisphere. The original ceiling, almost half of which was destroyed by fire in 1890, was restored in 1965 (see Figure 10.), based on drawings, photographs and the surviving ceiling fragments [2]. An idealized rendition of this design, without the interlacing, is given in Figure 11.

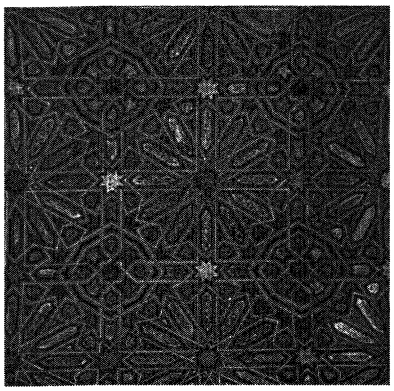

Figure 10. Detail of the ceiling in la Sala de la Barca

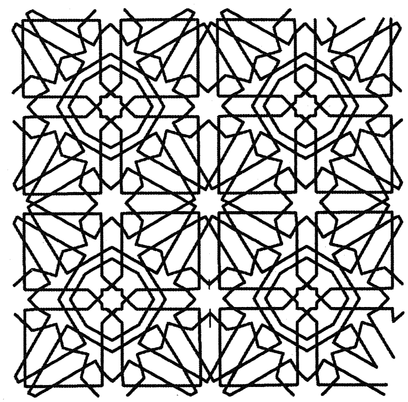

Figure 11. Computer-generated version of Figure 10., without the interlacing

The center of this pattern also has three stars embedded within each other, except in this case, the stars have 12 points. The smallest star in the center is formed from regular hexagons offset from each other by 30 degrees and inscribed within a circle. The next larger star is produced by extending the edges of the smallest star. The largest star, however, is created from lines used to create the small eight-pointed star within the double octagon shapes. A second, "right-angled" eight-pointed star is also produced from these lines. If the three concentric star shapes are thought to be positioned at the origin of a rectangular coordinate system, then the right-angled eight pointed star shape appears an equal distance away, along both the positive and negative x- and y-axes.

As was the case for the second design, the pattern is highly symmetric, with a crystallographic group classification of *p4m*, since it also permits both four-fold rotations (about a point located in the center of the eight-point star) and mirror lines through the center point in four directions (at 45 degree angles to one another). The design, showing only four possible mirror lines and the star polygon motifs, is below, in Figure 12.

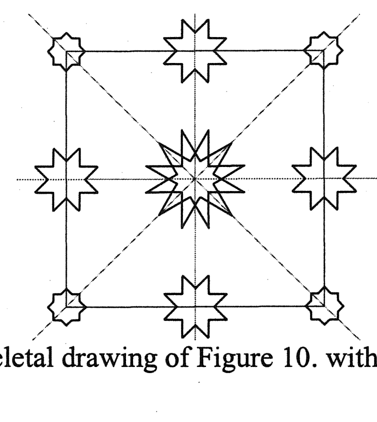

Figure 12. Skeletal drawing of Figure 10. without the interlacing

Sixteen- and Eight-Pointed Stars Design in the *Patio de los Arrayanes*

A fourth pattern is a gilded sixteen- and eight-pointed stars design found in the wooden ceiling of the northern portico of the *Patio de los Arrayanes* (Court of the Myrtles), also known as *Court de la Alberca* (Court of the Pool). The photograph of a segment of the ceiling (Figure 13.) and an idealized rendition of this design (Figure 14.), without the interlacing, are given below.

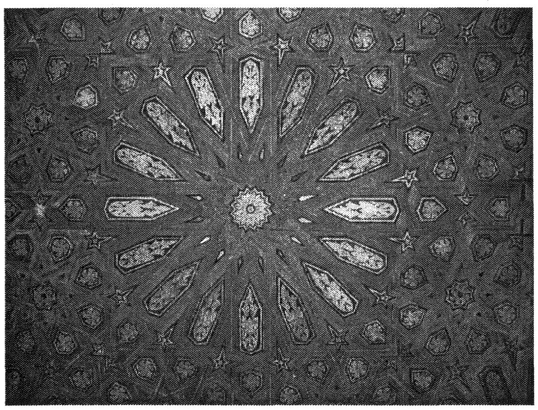

Figure 13. Detail of ceiling of the *Patio de los Arrayanes*

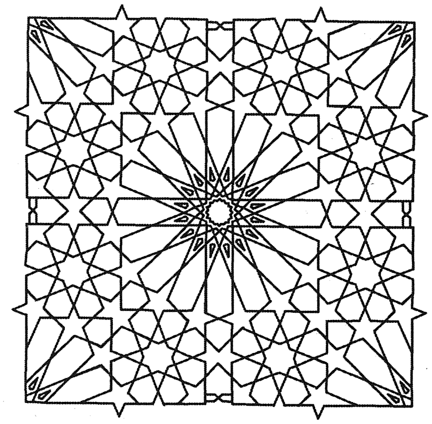

Figure 14. Computer-generated version of Figure 13., without the interlacing

This pattern also has five, 16-pointed stars embedded within each other. The smallest star in the center is formed from squares offset from each other by 22.5 degrees and inscribed within a circle. The four smallest stars are all produced by extending the edges of the next smallest star. The largest star is created in a similar manner to the previous ones with "rose petal" points. These 16-pointed stars are surrounded by eight, eight-pointed stars which consist of three embedded stars as described in the second wooden ceiling example.

As was the case for all but the first design, the pattern is highly symmetric, with a crystallographic group classification of *p*4*m*, since it also permits both four-fold rotations (about a point located in the center of the eight-point star) and mirror lines through the center point in four directions (at 45 degree angles to one another). The design, showing only the star polygon motifs and four of the possible mirror lines, is below, in Figure 15.

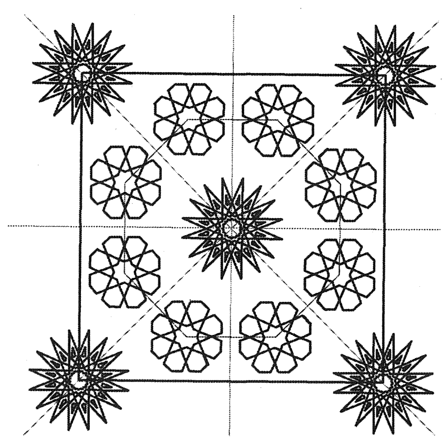

Figure 15. Skeletal drawing of Figure 13., without the interlacing

Twelve- and Eight-Pointed Stars Design

The last pattern is another twelve- and eight-pointed stars design found in a painted wooden ceiling. The photograph (Figure 16.) and an idealized rendition of the design (Figure 17.), without the interlacing, are given below. This pattern also has twelve-pointed stars embedded within a cruciform shape, and eight-pointed stars (formed from two offset squares) within an octagonal shape. If the twelve-pointed star shape is thought to be positioned at the origin of a rectangular coordinate system, then the eight-pointed star shape appears an equal distance away, along both the positive and negative *x*- and *y*-axes.

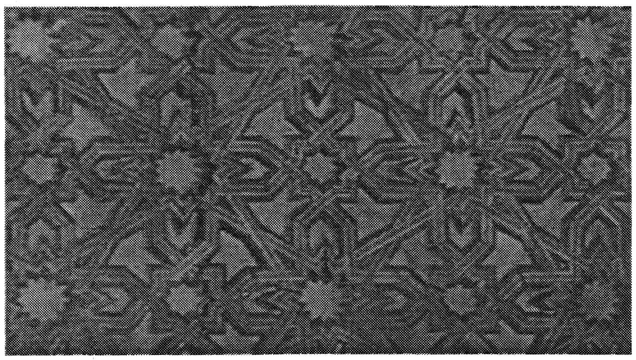

Figure 16. Detail of 12- and 8-pointed star wooden ceiling

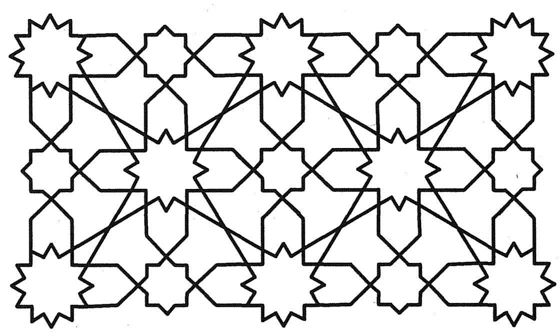

Figure 17. Computer-generated version of Figure 16., without the interlacing

As was the case for all but the first design, this last pattern is highly symmetric, with a crystallographic group classification of *p*4*m*, since it also permits both four-fold rotations (about a point located in the center of the eight-point star) and mirror lines through the center point in four directions (at 45 degree angles to one another). The design, showing only the star polygon motifs and four of the possible mirror lines, is in Figure 18 below.

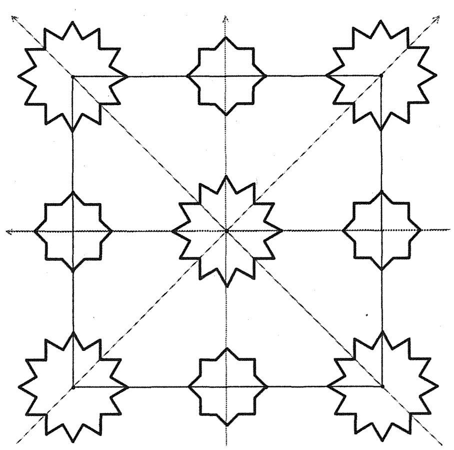

Figure 18. Skeletal drawing of Figure 16., without the interlacing

Discussion

All but the first of the wooden ceiling designs are highly symmetric (if one disregards the interlacing) with four-fold rotational symmetry about a point and four mirrors at 45 degrees to one another through that point. As a result, they are all classified as belonging to the *p*4*m* symmetry group, despite the fact that the designs look quite varied and contain star polygons with differing number of points ($n = 8, 12, 16$). The first example, classified as *cmm*, has 2-fold symmetry with centers of rotation not all on mirror lines and consists of star polygons with five and ten points. Since both *cmm* and *p*4*m* designs are very common to Islamic art, with *p*4*m* being one of the two predominating patterns [10], it is not surprising that most of the wooden ceiling designs discussed in this paper may be classified as *p*4*m*.

Patterns consisting of stars with the same number of points (or integer multiples of each other, if there was more than one type of star), were the most straight forward to reconstruct. For example, the second design consisted only of four different types of eight-pointed stars, the sides of which could all be constructed from line segments arising from a square grid inscribed in a circle. Likewise, the fourth design, consisting of eight- and 16-pointed stars, could also be recreated from a square grid. In a similar manner, the first design, containing star polygons with five and ten points, could be created from line segments arising from a regular pentagonal grid inscribed in a circle. For these three designs, one basic polygon inscribed within a circle could form the basis for the entire pattern. This does not appear to be the case for the third and last designs involving eight- and 12-pointed stars. For these designs, both hexagonal and square grids are needed for the patterns, and the interface between the 12-pointed and the eight-pointed stars is not always obvious. That is to say that some of the sides of the star polygons came from line segments arising from the square grid and some from the hexagon grid. As a result, the patterns consisting of multiple star polygons, where the number of points of one star is not an integer multiple of the number of points of another, are the most challenging and also the most interesting to decipher.

By creating the star polygon designs of *La Alhambra's* wooden ceilings, the Muslim rulers may have intended to create and surround themselves with an earthly paradise [1]. It is hoped that the designs presented in this paper offer the reader (and viewer) a stunning visual impact and appeal. These intriguing designs also issue a challenge to us to discover their underlying mathematical structures. Being able to find the hidden geometric structure and the relationships between the design elements was one of the author's goals. A second objective was to inspire the reader to look for the mathematics underlying the many bewitching patterns found in the lavish decoration of the wooden ceilings of *La Alhambra*.

References

[1] M. Danby. *Moorish Style*. Phaidon, 1999.
[2] J. B. Lopez and P. G. Andrew. *Official Guide: The Alhambra and Generalife*. Patronato de la Alhambra y Generalife, 1999.
[3] B. Grunbaum and G. C. Shepard. *Tilings and Patterns*. W. H. Freeman, 1987.
[4] I. El-Said and A. Parman. *Geometric Concepts in Islamic Art*. Dale Seymour Publications, 1976.
[5] B. L. Bodner, *An Unusual Nine-Pointed Star Polygon Design of La Alhambra,* Proceedings of ISAMA, 2004.
[6] B. L. Bodner, *Constructing and Classifying Designs of al-Andalus,* Proceedings of ISAMA-Bridges, 2003.
[7] B. L. Bodner, *Mathematics of Islamic Art,* 45-minute invited talk and 1-hour invited workshop at the Mathematical Association of America (MAA), New Jersey Section, 2002.
[8] B. L. Bodner, *Islamic Art,* 2-hour special invited session at the MAA Mathfest meeting, 2001.
[9] The *Geometer's Sketchpad* (version 4) software, distributed by Key Curriculum Press, 2001.
[10] S. J. Abas and A. S. Salman, *Symmetries of Islamic Geometrical Patterns*. World Scientific, 1998.