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On the Perception of Tone

From Music Theory to Composition in the 21st Century

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Abstract

Developments in music theory, musical acoustics, and psychoacoustics over the last 15 to 20 years have resulted in a single structural basis that provides common ground for analysis of musical systems that include non-standard/non-octave systems, microtonal systems, and systems that use unusual tunings common in non-western music. These developments make it likely that a confluence of understanding in musical acoustics, psychoacoustics, and music synthesis is about to take place. This common structural basis will lead to generalized rules of composition, unheard of since the classic work *Treatise On Harmony* by Rameau (1722), based on generalizations of the well-known properties of our familiar 12-tone scales. Indeed, some of these generalizations have already been articulated. As participants in various aspects of this work for the past 14 years [1-7], we are in a unique position to collect, assess, and distill the important contributions to this, truly, multidisciplinary field. It is our purpose, in this paper, to articulate a 21st century approach to music composition using the latest results from mathematical music theory and musical acoustics. In our presentation we will produce audio and visual examples applying these techniques.

1. Introduction

The last quarter of the 20th century has given rise to a number of remarkable discoveries in music theory and musical acoustics [1-14] that are compatible with recent discoveries in psychoacoustics [16]. In this paper, we summarize some of these recent results and show how these results can be used to define rules of composition applicable to non-traditional scales and musical systems. For the sake of clarity, we have used our familiar 12-tone system in most of the examples cited in the text. It should be keep in mind, though, that the techniques discussed may be generalized for analysis of any non-octave/microtonal system, including non-western musical systems.

In Sections 2 and 3, we discuss methods of generating musical scales that are based on the well-known and, frequently used, continued fraction approximation. Although not specifically discussed in the text, direct connection to continued fraction analysis is given in references [1-4]. Section 4 describes a procedure for generating scale sequences, and the embedded chords structures, that have the modulation properties of the familiar cycle of fifths. In Section 5 and 6, we describe perceptual analyses and a psychoacoustic measurement that allows for the evaluation of scales and chord structures to determine if they make "musical/perceptual sense."

Section 7 describes a newly developed, *dynamical-systems-like* approach to the modern transformational theory [14] of music analysis. This approach leads to tonal hierarchies, tonal distributions, and chord progressions discovered in recent cognitive studies [16]. A brief Summary/Conclusion section follows in Section 8.

2. Best Equal-Tempered Approximations

Based on work by Clough and Myerson [8], Carey and Clampitt [9], and a host of others, we have discovered criteria by which to choose a best equal-tempered musical scale of any number of notes (chromatic cardinality) and closure interval (traditionally the octave) that best approximates a set of target (usually, *just*) intervals. We have, thereby, generalized the process of choosing equal-tempered musical scales, typically chosen for ease of modulation and transposition and, therefore, compromising the consonance of *just* intervals, to include non-standard/non-traditional scales.

We [1-3] have defined a weighted, multiple-interval, 10-point desirability function (GDF – Generalized Desirability Function) thus:

$$D_b(c,N) = 10 - 20\sum_{i=1}^{N} P_i \left| \left\{ c\log_b(R_i) + \tfrac{1}{2} \right\} - \tfrac{1}{2} \right|$$

Where $\{x\}$ is the fractional part of x; N is the number of intervals to be approximated; c the chromatic cardinality of the equal-tempered system, and the base, b, of the logarithm represents the interval for closure. The R_i's are the frequency ratios of the individual intervals to be approximated and the P_i's are the respective normalized weights of each R_i (i.e. $\sum P_i = 1$).

Shown in Figure 1, is the GDF applied, simultaneously, to the intervals of the perfect fifth, major third, and minor third with weights of 0.25, 0.25, and 0.50, respectively for octave closure.

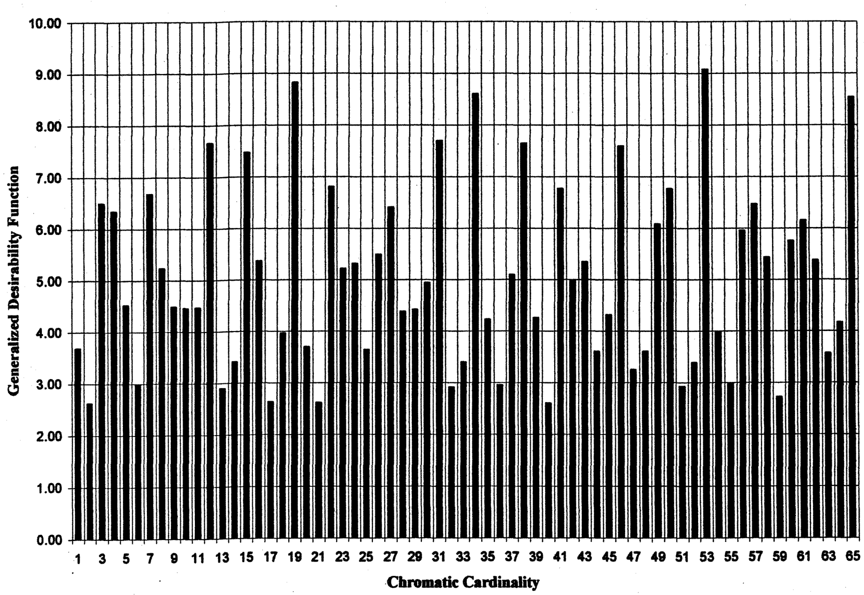


Figure 1. GDF for Pure Fifth (0.25), Major Third (0.25), and Minor Third (0.50)

Note, with a weighting of 0.50 for the minor third, 12 equal-tempered notes to the octave is no longer the best choice. For relatively few notes to the octave, 19 equal-tempered notes is a better choice. Historically, 19-tone music has been explored by various musicians and composers. A similar calculation of the GDF for equal weighting (P's = 1/3) shows that the best relatively small chromatic cardinality is 12 notes to the octave, our usual equal-tempered system.

Using our GDF, a musician could choose a particular equal-tempered scale (and closure interval) depending on his or her compositional or musical requirements; the relative importance of selected intervals for mood, for example. With the increasing use of the computer as a musical instrument such a scheme is hardly out of the question.

3. Scales Made Up of Just Tones

Just recently [4], we have discovered an approach that generates scales which preserves the consonance of the *just* intervals and compromises the modulation and transposition properties of equal-tempered scales. Suppose, for example, we choose to generate a 12-tone scale that preserves the *just* intervals of the perfect fifth, the major third, and the octave. Beginning with the home-tone (starting point) we allow tones generated by compounding the fifth (or its inversion, the fourth) no more than twice, compounding the third (or sixth) no more than once, and require the resulting scale to repeat at the octave. This process generates 15 tones. We now select a 12-tone subset from these 15 candidate tones that most closely approximates a 12-tone equal-tempered scale. We have now inverted the usual process (of compromising the *just* intervals by choosing an equal-tempered scale) by compromising the convenient modulation/transposition property of equal-temperament to preserve the consonance of the *just* intervals.

Shown in Table 1, are the 15 candidate tones generated for the example cited above.

Ratio	Cents	Angle	Composition	Ratio	Cents	Angle	Composition
1	0	0°	0	**64/45**	610	183°	- 2F - T
16/15	112	34°	- F - T	**3/2**	702	211°	F
10/9	182	55°	- 2F + T	**8/5**	814	244°	-T
9/8	204	61°	2F	**5/3**	884	265°	- F + T
6/5	316	95°	F - T	**16/9**	996	299°	- 2F
5/4	386	116°	T	**9/5**	1018	305°	2F - T
4/3	498	149°	-F	**15/8**	1088	326°	F + T
45/32	590	177°	2F + T	**1**	0	0°	0

Table 1. 15 Candidate Tones Generated by Compounding the Fifth (twice) and the Third (once)

In Table 1, the ratios are given as fractions with the equivalent cents measurement [1200 $\log_2(R)$], along with the composition of intervals that generated that particular ratio. For example, the first ratio, 16/15, was generated by compounding the fourth (the inversion of the fifth, hence –F) and the sixth (the inversion of the third, hence –T) each, one time, and reducing the resulting ratio to the octave. The third column gives the equivalent angle measurement around the octave circle relative to 0° in the 12 o'clock position. Shown in Figure 2, are the 15 candidate tones distributed around the circle.

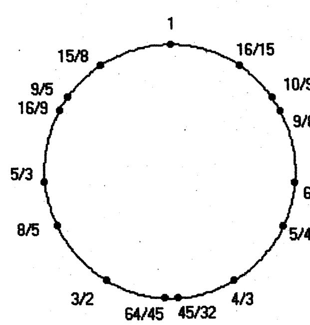


Figure 2. Candidate Tones for Pure Fifth and Major Third

Shown in Table 2, is a 12 tone subset that most closely approximates an equal-tempered scale.

Ratio	Cents	Angle	Composition	Ratio	Cents	Angle	Composition
1	0	0°	0	**45/32**	590	177°	2F + T
16/15	112	34°	- F - T	**3/2**	702	211°	F
9/8	204	61°	2F	**8/5**	814	244°	- T
6/5	316	95°	F - T	**5/3**	884	265°	- F + T
5/4	386	116°	T	**16/9**	996	299°	- 2F
4/3	498	149°	- F	**15/8**	1088	326°	F + T

Table 2. Equal-Tempered 12-Tone Subset of the 15 Candidate Tones

Figure 3 shows the resulting 12-tone scale distributed around the octave circle.

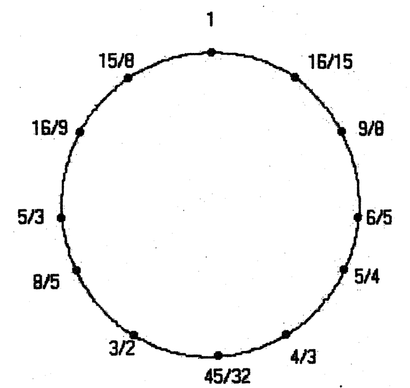


Figure 3. Equal-Tempered 12-Tone Subset of the 15 Candidate Tones

Historically, it may be plausible that when the Greeks moved beyond Pythagorean tuning, scales were chosen; first, to repeat at the octave; as the octave was the "most perfect" interval. Second, that the next most important interval be the perfect fifth; in deference to Pythagorean tuning. Third, that the scale have 12 notes; another hold over from Pythagorean tuning. And fourth, that the scale could be reasonably transposed over a reasonable number of keys to enable singing in different registers. How would one choose such a scale generated by more than one pure tone? Our process of generating candidate notes based on multiple compounding of *just* intervals and choosing a subset that approximates equal-temperament fills-the-bill. In the example above, the resulting 12-tone scale is one example of the historically important *Just Tuning System.*

4. Modulation Properties and Chord Structure

Once an n-tone scale is generated, by whatever means – for whatever reason (one such example is given above), to have any applicability to music, we need to construct a chord structure that has the modulation properties of closely related keys, the cycle of fifths in our usual 12-tone system. Figure 4 shows the cycle of fifths.

We have articulated a generalized process for constructing major and minor triads with the appropriate modulation properties from a given scale, regardless of the closure interval and the number of notes in the scale. Therefore, we now have at our disposal a prescription for generating candidate scales with chord structures that have properties that are generalizations of the historically important cycle of fifths. Next, this process is applied to our familiar 12-tone system.

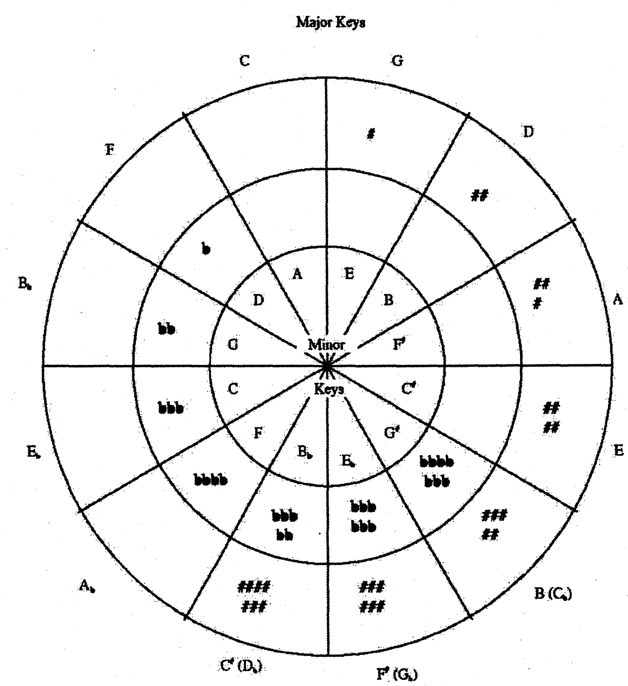


Figure 4. Cycle of Fifths

A Familiar Example: All seven note diatonic scales represented on the circle of fifths may be generated by the maximally even (ME) algorithm of Clough and Douthett [7]:

Let c and d be the chromatic and diatonic cardinalities, respectively, and let n be any fixed integer such that $0 \leq n \leq c-1$. Then, a ME set with these parameters is:

$$J^{n}_{c,d} = \left\{ \left\lfloor \frac{ck+n}{d} \right\rfloor \right\}_{k=0}^{d-1}$$

Where $\lfloor x \rfloor$ *is the smallest integer greater than or equal to x.*

For our usual 12-tone system with a 7-note diatonic scale, $c = 12$ and $d = 7$. If we let the notes C, $C^{\#}/D_b$, D, $D^{\#}/E_b$, etc. be represented by the numbers 0, 1, 2, 3, etc. this algorithm generates all the scales of the cycle of fifths:

$$J^{5}_{12,7} = \left\{\left\lfloor \tfrac{12k+5}{7} \right\rfloor\right\}_{k=0}^{6} = \{0,2,4,5,7,9,11\}.$$

Note that the, so-called, mode index, n, is equal to 5 for the C major scale. As the index increases incrementally (mod 12) the associated scales rotate clockwise around the cycle of fifths.

Progression through a portion of the cycle of fifths is shown in Figure 5.

We may construct the chord structure for our usual 12-tone system in the following way. The intervals of the fourth (inversion of the fifth) with a frequency ratio of 4/3, the major third with a frequency ratio of 5/4, and the minor third with a frequency ratio of 6/5 are of equal importance. The sequence of these ratios is 3:4:5:6 and the chromatic lengths of these intervals, in our 12-tone system, are 5 half-steps, 4 half-steps, and 3 half-steps, respectively. [Note, $4/3 \simeq 2^{5/12}$, $5/4 \simeq 2^{4/12}$, and $6/5 \simeq 2^{3/12}$.] By arranging the sequence of step intervals from largest to smallest (with rotations) we can generate all the embedded major triads in the scale. Arranging the step-interval sequence from smallest to largest (with rotations) generates all the embedded minor triads.

For example, a rotation of the major triad sequence (4, 3, 5) generates a chord using the {0, 4, 7} of the scale generated above. This is just the C major triad. All of the embedded major/minor triads of the scale can be generated in this way. Therefore, using the chromatic lengths of the frequency ratios, 3:4:5:6 in this case, which are best approximated by our usual 12-tone equal-tempered system, we have determined the chord structure of the system.

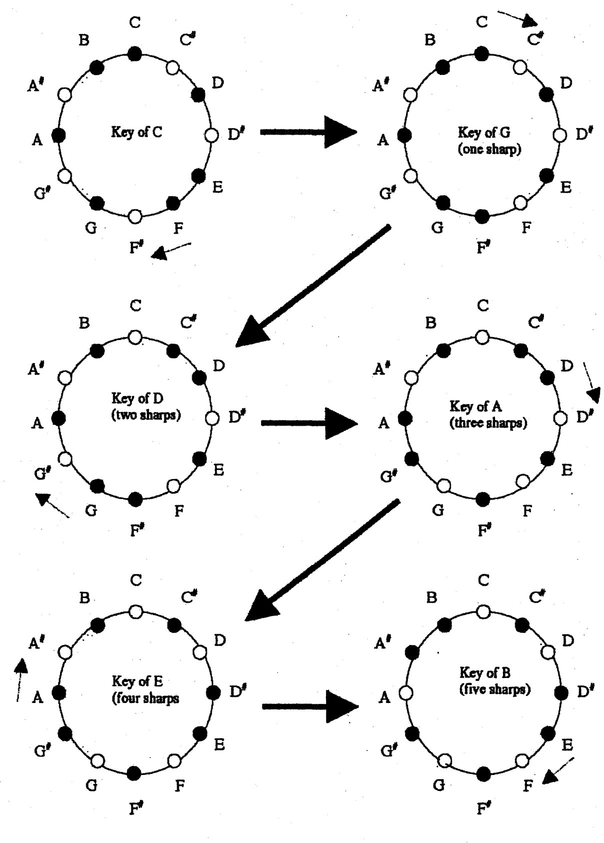


Figure 5. Progression through the Cycle of Fifths

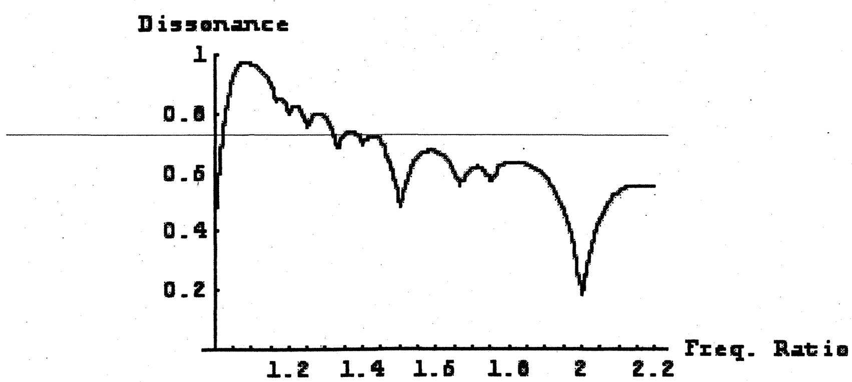
5. Consonance and Dissonance

Questions remain. Which of these candidate scales and chord structures is meaningful musically? How can we tell? And, are we relegated to waiting for the successes of hit-and-miss, trial-and-error basement computer composers and performers? Or, is there a direction defined by the above results?

We believe the latter, but much else has to be done. Because one has chosen a candidate scale, or set of scales, according to the numerics described above, there is no guarantee that the scale will be useful acoustically. In fact, the music theory analysis that leads to the candidate scales conspicuously avoids any mention of complex tones and the acoustic subtleties associated with combining notes that have complex spectra. It is just these subtleties that determine the acoustic richness of complex tones and chords. Fortunately, in the last 12 years a prescription for determining the consonance/dissonance of scales made up of complex tones has been articulated.

William Sethares [10], at the University of Wisconsin, has developed a measure of dissonance based on the "critical bandwidth" research of Plomb and Levelt [11]. One can construct complex spectra for tones that make up the candidate scales and apply Sethares' dissonance measure to the scale and chord structure. This allows us to determine whether the candidate scale, including complex tones, has the requisite structure for possible use as a musical scale. If the complex candidate scale is low in consonance we have justification for abandoning it and looking for scales that have a higher "consonance measure."

For example, Figure 6 shows Sethares' dissonance measure, for a complex acoustical spectrum, made up of the first 7 harmonics each with amplitudes 90% of the previous harmonic.



A relative maximum on this curve represents a frequency ratio (interval) which is relatively dissonant. A sharp minimum represents a consonant interval. Note, in the figure, that for this complex spectrum the octave (frequency ratio 2/1) and the perfect fifth (frequency ratio 3/2) are consonant intervals. We, therefore, would consider using a scale generated by the perfect fifth that closes at the octave – such as our usual 12-tone scale.

Figure 6. Dissonance Function of Sethares

This still does not tell us whether the chord structure for the candidate scale makes sense musically. Happily, we have recourse to research done in the last 20 years to assess the "perceptibility" of chords and determine whether these chords possess the richness of traditional harmonies.

6. Musical Sense

To evaluate the musical perceptibility, so-called *intonation sensitivity* measurements can be done. On the basis of this perceptual measure we may pick scales and determine the musical relevance of the candidate chord structure. The "*intonation sensitivity*" measure developed by Roberts and Mathews [12] may be applied to the chords of the candidate scales. Operationally this means that audio files of candidate chords are synthesized and subjects are asked which chords they prefer. The original work on intonation sensitivity showed a difference in response of musically trained and untrained individuals. It might be of benefit to a modern composer/musician to know what scales and chords structures appeal to which group; i.e. whether one wants to "appeal-to-the-masses", be a "musicians-musician", or both.

7. Chord Progression

At this point, one could turn the candidate scales over to a modern, microtonal composer to use as he or she sees fit to develop the rules of composition. An alternative would be to articulate the "rules of composition" appropriate to the given scale and chord structure. Recent results in music theory (Cohn [13], Clough et al. [6], Douthett and Steinbach [5], and Lewin [14]) are available to construct a "musical space" and the network of transformations among chords ("rules of composition") appropriate for the scale. Composition would then take place in this musical space according to these rules of composition.

One approach to this, so-called, "transformational" theory has been articulated by one of us (JD). Techniques similar to those used in the study of *dynamical systems* have been adapted to study transformational theory. Although dynamical systems are probably best known today for the *fractals* they sometime generate, fractals are only a part of this field of study. As Strogatz [15] puts it in his text on nonlinear dynamics and chaos, "[dynamics] . . . is the subject that deals with change, with systems that evolve in time. Whether the system in question settles down to equilibrium, keeps repeating in cycles, or does something more complicated, it is dynamics we use to analyze the behavior." In the *dynamical systems* approach to transformational theory *periodic orbits* represent cycles of scales and chords and form a basis for the "rules of composition."

As the technical details of this approach are outside the scope of this article, we will appeal to a specific application of the theory, with the aid of the constructions shown in Figures 7 and 8. To begin construction of a dynamical system, consider two concentric circles of different radii. The outside circle has 12 holes (numbered 0 through 11), equally spaced about its circumference. The inside circle, called the *beacon*, has 7 *lamps*, equally spaced about its circumference (numbered 0 through 6). Each *lamp* transmits a beam in the radial direction. Two rules apply when the beam hits the outside circle:

1. If the beam hits a hole on the circumference of the outside circle, the beam travels through the hole.
2. If the beam hits the inside wall of the circumference, the beam moves counterclockwise on the circumference of the outside circle and travels through the first hole it encounters.

In this way, the outside circle acts as a type of *filter*, slightly modifying the paths of the beams. With the configuration in Figure 7a (7 beams, 12 holes), the set of *beam numbers* (numbers corresponding to the holes that the beams pass through on the outside circle) is {0, 1, 3, 5, 6, 8, 10}, the D_b Major scale. As the beam circle is slowly rotated clockwise, the beam numbers stay the same until the beacon has passed through an angle of 4 and 2/7 degrees, or 1/7 x 1/12 of a revolution. At this point, beam 4 hits hole 7, changing the set of beam numbers to {0, 1, 3, 5, 7, 8, 10}, the A_b Major scale (Figure 7b).

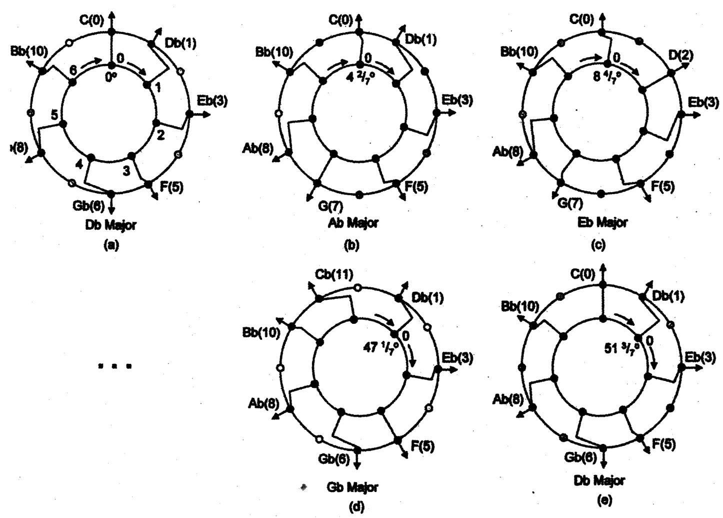


Figure 7. The 7 through 12 Diatonic System

Continued rotation of the beam circle generates the scales of the cycle of fifths discussed above (see the remainder of Figure 7). This construction is an alternative approach to generating the cycle of fifths.

Next, consider a beacon rotation that yields the C Major scale (Figure 8a); change the lamps to holes on the beacon, and add a new beacon with 3 lamps inside the old beacon. This 3 through 7 through 12 system has two filters, and the set of beam numbers on the outside circle is (0, 4, 7}, the C chord (Figure 8b). When the beacon is rotated clockwise, the next triad (chord) encountered is Am (Figure 8c). This is followed by F, d, B dim, G, and e; and the cycle begins anew. These are, of course, all the embedded triads in the C Major scale. Variations of this construction, interchanging beacon and hole circles, etc. lead to well-known chord progressions studied by music theorists [6, 7, 13].

Application of this dynamical systems generalization of transformational theory to our usual 12-tone system reproduces tonal hierarchies and tonal distributions, uncovered by the cognitive studies of Krumhansl [16], that are perceptually recognized as representative of "good phrasing" (i.e. a fundamental of good composition). It appears that the recent discoveries of music theory and musical acoustics are compatible with recent discoveries in psychoacoustics. At this point one could (should?) perform cognitive studies on "test" musical pieces that use the rules of composition, articulated above, to determine if they are, indeed, perceived as "good compositions."

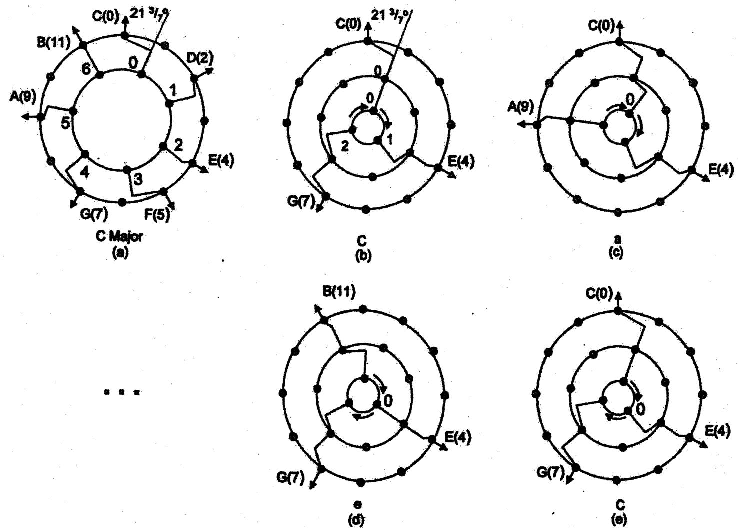


Figure 8. Embedded Triads in the C Major Scale

8. Summary/Conclusion

We have articulated an approach to musical composition that leads to "rules of composition" based on the latest results in music theory and musical acoustics that is compatible with recent discoveries in psychoacoustics. The way is now clear. Generate a "best" n-tone candidate scale with a chosen interval for closure, embellish the notes with a complex spectra, test the scale for consonance/dissonance, construct the chord structure of the scale and test the intonation sensitivity of chords, generate the musical space and rules of composition for the scale, and compose and test the compositions for their perceived "goodness."

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