Why Do Penrose Tilings Diffract?

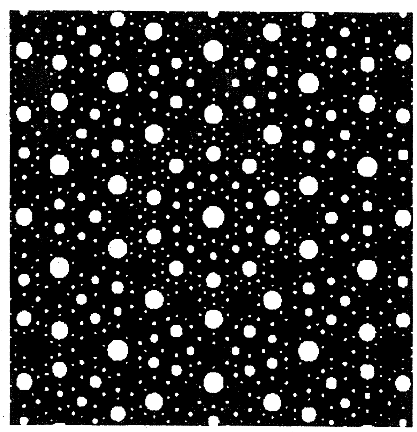
Robert V. Moody
Scientific Director, Banff International Research Station
Department of Mathematical and Statistical Sciences
University of Alberta, Edmonton, Alberta
Canada, T6G 2G1
Email: rmoody@ualberta.ca

1 Hidden symmetry

Symmetry is one of those concepts that easily bridges the underlying themes of this conference: art, architecture and mathematics. Part of the fascination of Penrose tilings is their rather quirky kind of symmetry. But if one is pressed to describe exactly what this symmetry is, say group theoretically, one is likely to come up empty handed. Penrose tilings are **aperiodic** – they have no translational symmetries. They are not built upon simple repetition of a motif. Furthermore, in general Penrose tilings (there are infinitely many different ones) do not have 5-fold symmetry, though they have a strong feeling of "five-ness" about them. So in general they have no symmetries in the conventional way.

However Penrose tilings, and all the other famous aperiodic tilings (Fibonacci, square-triangle, Robinson, etc) do have a hidden world of symmetry and it can be revealed in a very striking way: they are pure-point diffractive.

Here is a picture of the diffaction pattern of a Penrose tiling.



Diffraction Image of a Penrose Tiling

Which Penrose tiling? Actually all of them! They all have the same diffraction. Furthermore it is perfectly 5-fold symmetric – in fact, perfectly 10-fold symmetric.

The way to look at this pattern is to interpret each dot as a spike, an infinite spike, located at the centre of the dot and weighted by a quantity equal to the area of the dot. So this is like an Indian bed of nails, where the nails are varying strengths. Technically it is an pure point measure.

There are lots of articles on Penrose and other aperiodic tilings, but few of them address its remarkable diffraction – for the simple reason that it is not particularly easy to describe what it really is. The objective of this talk is to shed a little more light on this rather mysterious subject.

2 Repetition

One of the striking features of a, say, a rhombic Penrose tiling is that it is *repetitive*. Simply put, any finite patch of rhombi (no matter how large) in the pattern repeats. In fact it repeats regularly in the following way: for each radius r there is another radius R so that for any patch of tiles of lying in any ball of radius r, a translated image of it is bound to occur in *any* ball of radius R. What you see once, you see again, and you don't have to go too far to find it.

There is another sense, though, in which a Penrose tiling $\mathcal{T}$ repeats. $\mathcal{T}$ has no translational symmetry. But if t is a vector and $\mathcal{T}$ is translated by t to get $t + \mathcal{T}$ then one can ask to what extent it coincides with $\mathcal{T}$. We can measure this say by comparing the area of perfectly lined up tiles to the total area – i.e. as the density of perfectly matched up tiles.

Question: How good can we make this matching?

Answer: As good as we wish. If $0 < \epsilon < 1$ then there are t for which $\mathrm{den}\,((t + \mathcal{T}) \cap \mathcal{T}) > 1 - \epsilon$.

So Penrose tilings are aperiodic but nonetheless allow almost translations as close to perfect translations as you desire. In fact, just as for repetitivity, there is an R so that every ball of radius R has such a translation t in it. Of course R will be very large if ϵ is very small, but in a strict sense the ϵ-almost periods do appear in a regular (though aperiodic!) way. This is what is meant by **almost periodicity**.

Almost periodicity in this sense is *equivalent* to pure point diffractivity, and the diffraction pattern is just a manifestation of it.

The rest of this talk will be devoted to explaining this. There are three aspects to this:

- the way in which group theory is still underlying the structure;
- the physical nature of diffraction - why it is so relevant?;
- the way in which the diffraction encapsulates the group theory.

The first is easy to describe. For each $t \in \mathbb{R}^2$ let $\eta(t)$ denote the matching density of translation by t. For $t = 0$ it is 1 – the matching is obviously of full density. For most t it is zero – there are no tiles matched at all. But for a countable number of "good" t it is positive. Remarkably the set of all these good t actually form a group L under addition. L is a subgroup of rank 4 inside the group $\mathbb{R}^2$ of all translations of the plane. (For a normal periodic structure in the plane we would expect a rank 2 subgroup to be carrying the symmetry.) The elements of L are not perfect symmetries, only partial ones, so we attach to each its measure of goodness: $\eta : L \mapsto \mathbb{R}$. So the notion of symmetry group needs to be modified by a goodness function.

All this information can be assembled into one generating object – another bed of nails (formally the autocorrelation measure):

$$\eta := \sum \eta(t)\delta_t \,.$$

Nature does not give us a way of seeing this measure directly, but she does allow us to see its Fourier dual – and that is the diffraction. The last part of the talk briefly describes what physical diffraction does and what the resulting pictures mean.

The talk will be illustrated with various tilings and their diffraction images, a schematic of diffraction experiments, and some recent diffraction images from real quasicrystals.