

## Hyperbolic Spirals and Spiral Patterns

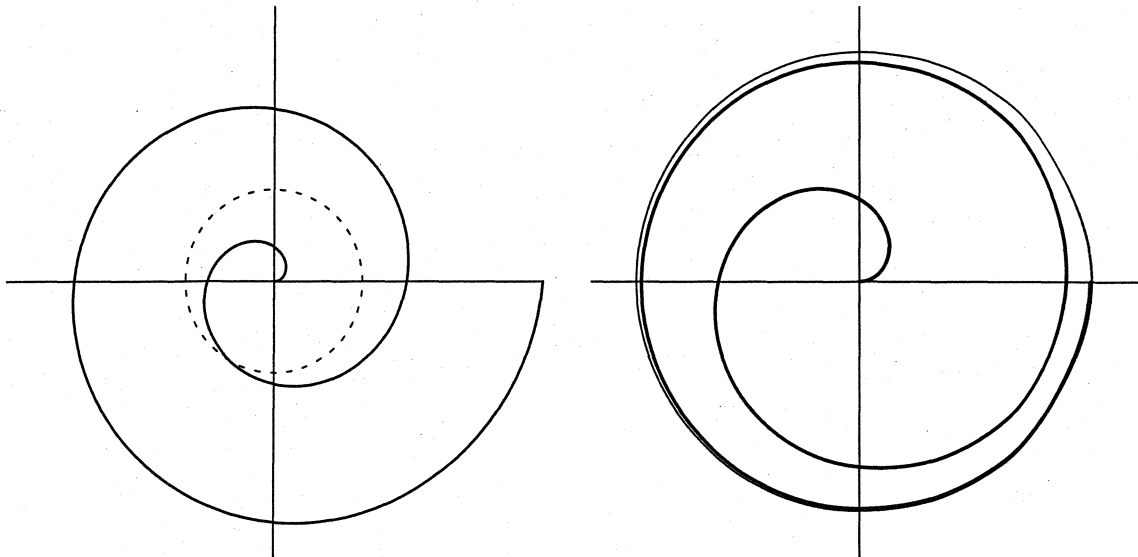
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### Abstract

Starting many centuries ago, artists in a number of cultures have created spiral patterns on flat surfaces, and even on spheres. Mathematicians have given precise definitions of spirals in the Euclidean plane and have studied them for centuries. Yet no one seems to have studied spirals in the hyperbolic plane, and until recently, no one had created hyperbolic spiral patterns. In this paper, we investigate both areas.

### 1 Introduction

This paper has three goals. The first is to develop a mathematical theory of hyperbolic spirals. This is done in Section 2 where hyperbolic spirals are compared with their Euclidean analogues. Figures 1 and 2 show an Archimedean spiral in the Euclidean plane and its analogous hyperbolic spiral.



**Figure 1:** The Archimedean spiral  $r = a\theta$ .

**Figure 2:** The hyperbolic Archimedean spiral.

The second goal is to show spiral patterns in the hyperbolic plane inspired by Celtic and Islamic Euclidean spiral patterns, which is done in Section 3. In Section 4, we combine these endeavors into our third goal by creating repeating hyperbolic patterns based on hyperbolic spirals.

## 2 A Theory of Hyperbolic Spirals

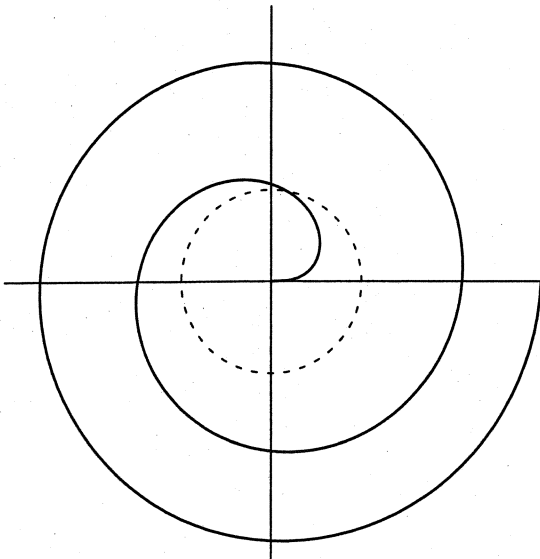
To display a hyperbolic spiral, we must first choose a (Euclidean) model of hyperbolic geometry, since as has been known for 100 years, there is no smooth distance-preserving embedding of hyperbolic geometry in Euclidean space. The *Poincaré circle model* suits our needs best because (1) it lies entirely within a *bounding circle* in the Euclidean plane (allowing the entire hyperbolic plane to be shown at once), and (2) it is *conformal* — the hyperbolic measure of an angle is equal to its Euclidean measure. In this model, the hyperbolic points are the interior points of the bounding circle and the hyperbolic lines are interior circular arcs perpendicular to the bounding circle, including diameters. Distance is measured in such a way that equal hyperbolic distances are represented by ever smaller Euclidean distances toward the bounding circle. For convenience, we assume that the bounding circle is a unit circle centered at the origin of the Euclidean plane, and that the origin of the hyperbolic plane is represented by the Euclidean origin. In fact, if  $r_H$  is the hyperbolic distance from the origin then the Euclidean distance from the origin,  $r_E$  is given by  $r_E = \tanh(r_H)$ , where  $\tanh$  is the hyperbolic tangent function:  $\tanh x = \frac{\sinh x}{\cosh x} = \frac{(e^x - e^{-x})/2}{(e^x + e^{-x})/2}$ .

Most of the common Euclidean spirals are given in polar coordinates by equations of the form:  $r = f(\theta)$ . So one natural way to define a corresponding hyperbolic spiral (in hyperbolic polar coordinates) would be:

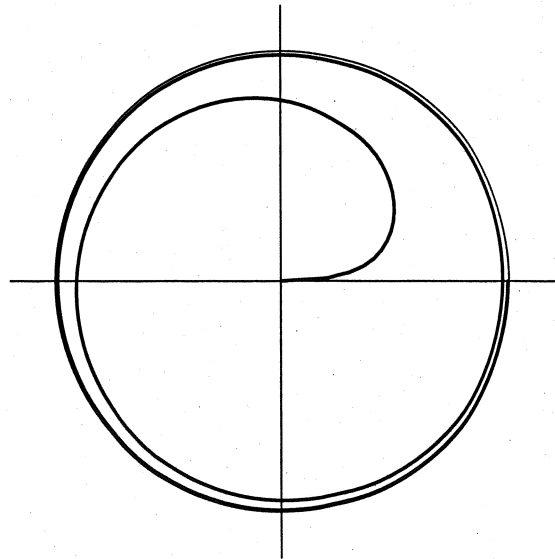
$$r_H = f(\theta),$$

which would then be represented as:  $r_E = \tanh(f(\theta))$  in the Poincaré model. One of the simplest and most widely drawn Euclidean spirals is the *Archimedean spiral* given by:  $r = a\theta$ . Two turns of this spiral are shown in Figure 1 where  $a = 3/4\pi$ , and  $0 \leq \theta \leq 4\pi$ . In our pictures of Euclidean spirals, we show the unit circle (in a dashed arc) for comparison with the corresponding hyperbolic spiral, which of course must be contained within the bounding circle. Figure 2 shows our hyperbolic version of this spiral, given by  $r_E = \tanh(a\theta)$ .

More generally, a *power spiral* is given by the equation:  $r = a\theta^n$ , where we assume  $a > 0$ . The Archimedean spiral is the special case where  $n = 1$ . If  $n = 1/2$ , we obtain another special case, the *Fermat spiral*. Figures 3 and 4 show two turns of the Fermat spiral and its hyperbolic counterpart.

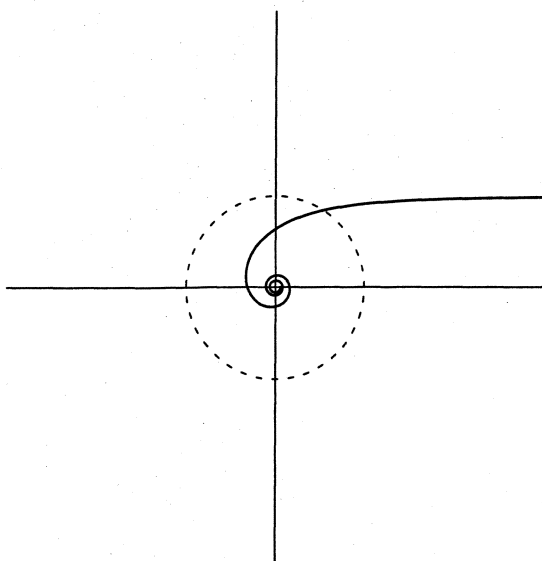


**Figure 3:** The Fermat spiral  $r = a\sqrt{\theta}$ .

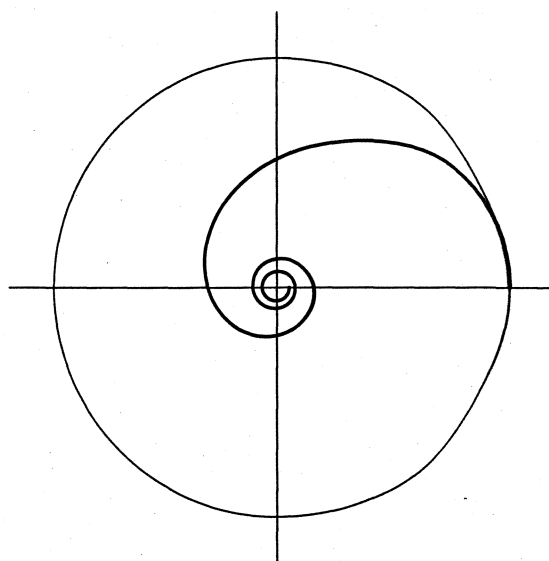


**Figure 4:** The hyperbolic Fermat spiral.

If  $n = -1$ , we obtain the *reciprocal spiral*. It is also often called the *hyperbolic spiral* since  $r = a/\theta$  is the equation of a hyperbola (if  $r$  and  $\theta$  are interpreted as Cartesian coordinates), but we wish to avoid confusion with the other use of the word “hyperbolic” in this paper, so we use the term “reciprocal” instead. Figures 5 and 6 show three turns of the reciprocal spiral and its hyperbolic counterpart. Note that the reciprocal spiral (but not its hyperbolic counterpart) has a horizontal asymptote  $y = a$ .



**Figure 5:** The reciprocal spiral  $r = a/\theta$ .

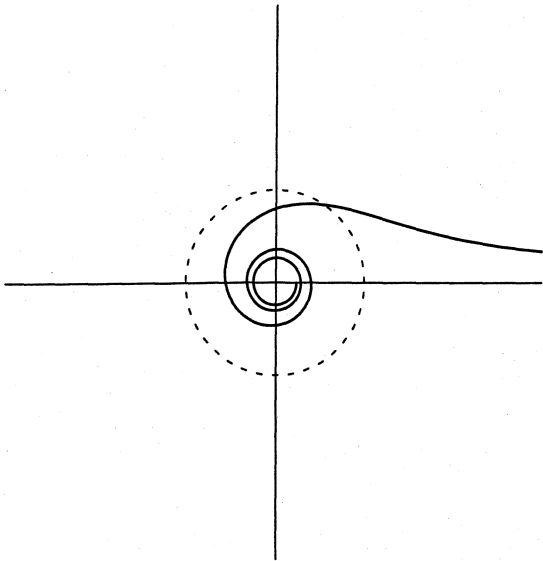


**Figure 6:** The hyperbolic version of the reciprocal spiral.

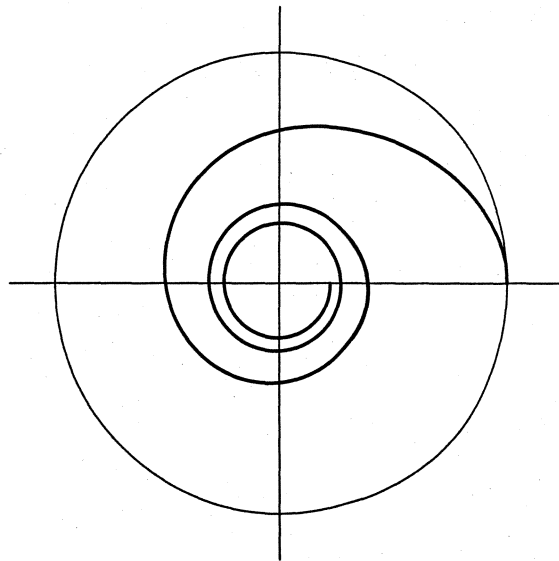
If  $n = -1/2$ , we obtain the *lituus curve*,  $r = a/\sqrt{\theta}$ . Figures 7 and 8 show three turns of the lituus and its hyperbolic counterpart. The lituus is not technically a spiral since its curvature is not a (strictly) increasing or decreasing function of its arc length, as can be seen on the right of Figure 7 by the inflection in the “tail”, which is asymptotic to the  $x$ -axis. The inflection point occurs where  $r = \sqrt{2}$  and  $\theta = 1/2$  radian (about  $28.65^\circ$ ). Similarly, the circle  $r = a\theta^0 = a$  ( $n = 0$ ) is not technically a spiral either, since its curvature is constant.

The equation of a *exponential spiral* is given by the equation:  $r = ab^\theta$ , where we assume  $a > 0$ , and  $b > 1$ . It is also often called *logarithmic spiral*. The *golden spiral* is the special case in which  $b = \tau^{2/\pi}$ , where  $\tau = \frac{1+\sqrt{5}}{2}$  is the golden section. Figures 9 and 10 show two turns of the golden spiral and its hyperbolic counterpart. It is related to the following construction. Start with a horizontal golden rectangle and construct a square on top of it, producing a bigger (vertical) golden rectangle. Construct a square on the left side of the new golden rectangle, producing a bigger (horizontal) golden rectangle. Continue successively constructing ever bigger squares on ever bigger golden rectangles on the top, left, bottom, and right sides in that sequence. A golden spiral can be passed through the diagonal corners of the squares — that is, through the bottom right and top left of the first square, which is the top right of the second square, then through the bottom left of the second square, which is the top left of the third square, etc. This construction is discussed by Coxeter [Coxeter1], and Swimmer discussed related spirals in his Bridges 2000 paper [Swimmer1].

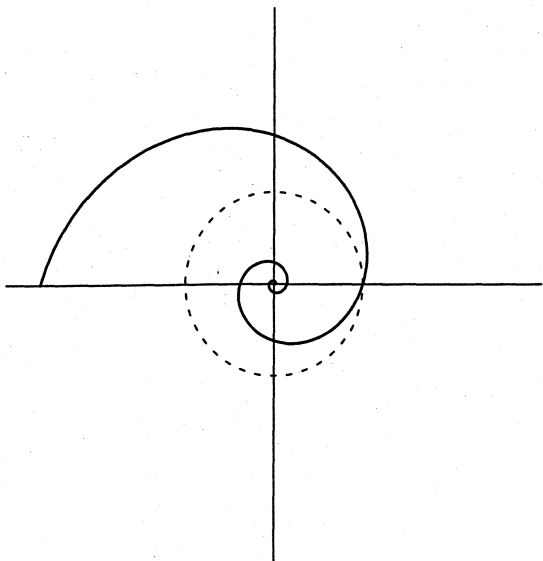
The exponential spiral is also called the *equiangular spiral* since rays from the origin cut it at the same angle  $\phi$ . In fact the equation can be rewritten as  $r = ae^{\theta \cot \phi}$  (i.e.  $b = e^{\cot \phi}$ ). As can be seen in Figure 10, our hyperbolic counterpart of the exponential spiral is not an equiangular spiral. But since the Poincaré



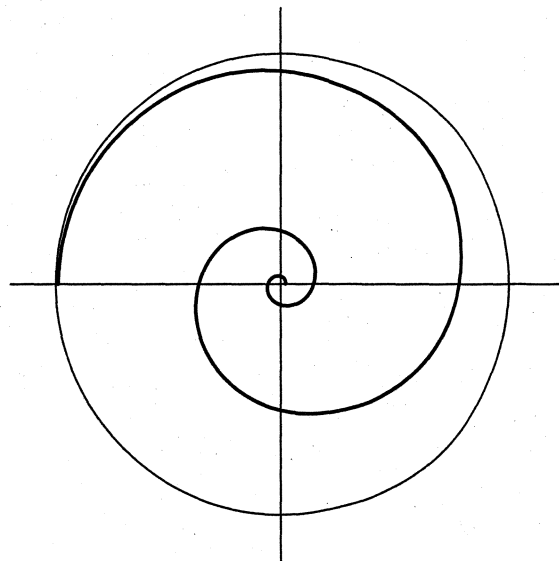
**Figure 7:** The lituus curve  $r = a/\sqrt{\theta}$ .



**Figure 8:** The hyperbolic version of the lituus curve.



**Figure 9:** The golden spiral  $r = \tau^{2\theta/\pi}$ .



**Figure 10:** The hyperbolic version of the golden spiral.

model is conformal, the Euclidean equation of a hyperbolic equiangular spiral must itself just be the equation of a Euclidean equiangular spiral. Hence the equation of a hyperbolic equiangular spiral in hyperbolic polar coordinates is just  $r_H = \tanh^{-1}(ab^\theta)$ . So to see a hyperbolic equiangular spiral, one can just look at that part of the spiral in the interior of the unit circle in Figure 9. Note that unlike the Euclidean case,  $\theta$  cannot increase indefinitely — there is a limiting angle, given by  $\theta_0 = -\log_b a$  (from solving  $ab^{\theta_0} = 1$ ).

Finally, we mention that spherical equiangular spirals are called *loxodromes* or *rhumb lines*. A loxodrome is the path one would travel on a sphere by keeping a constant heading, i.e. a path that makes a constant angle with lines of latitude (or with lines of longitude). A loxodrome is the stereographic projection of a Euclidean equiangular spiral onto the sphere.

This finishes our discussion of the mathematics of hyperbolic spirals. In the next section, we look at spiral patterns from two cultures and their hyperbolic counterparts.

### 3 Celtic and Islamic Spiral Patterns

The art of many cultures has been enriched by spiral patterns. We will examine two such patterns, one from Celtic art and one from Islamic art, and show hyperbolic versions of each pattern. It seemed to have been common in Celtic art to form patterns from  $k$  pieces of a spiral like Fermat's spiral arranged around a center to form a circular "medallion" (popular values of  $k$  were 2, 3, and 4). Then copies of the medallion were arranged in a (possibly) repeating Euclidean pattern with their spiral pieces connected. Figure 11 shows such a pattern with  $k = 3$ , and 3-fold rotation centers between the medallions. Figure 12 shows a related hyperbolic pattern based on the Fermat spiral, with  $k = 3$  and 4-fold rotation centers between the medallions.

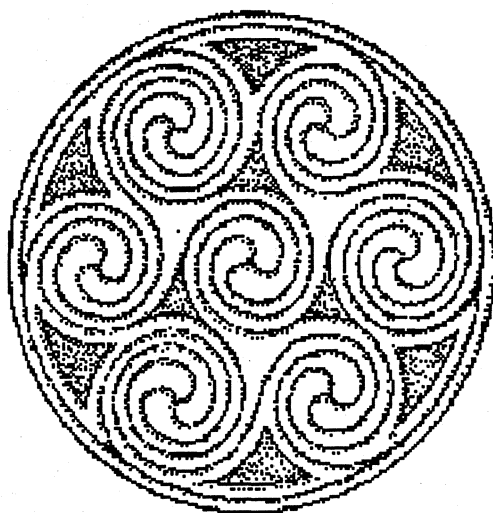


Figure 11: A Celtic spiral pattern.

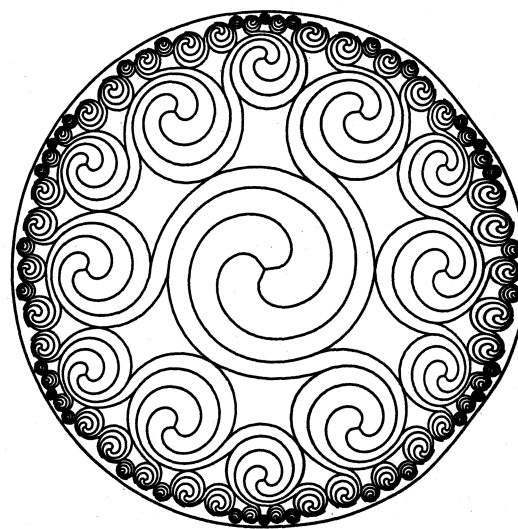


Figure 12: A hyperbolic version of Figure 11.

Figure 13 shows an Islamic spiral pattern from the Alhambra. It is a Euclidean tessellation by angular pinwheel medallions with 6-fold chiral symmetry. Figure 14 shows a hyperbolic tessellation by similar angular pinwheels with 7-fold rotation centers. Figures 13 and 14 were shown in Bridges 2001 [Dunham5]. The art of several cultures includes spiral patterns made up of "medallions" of spiral pieces, which are then linked together to make a repeating Euclidean pattern. We have shown simple examples from two cultures.

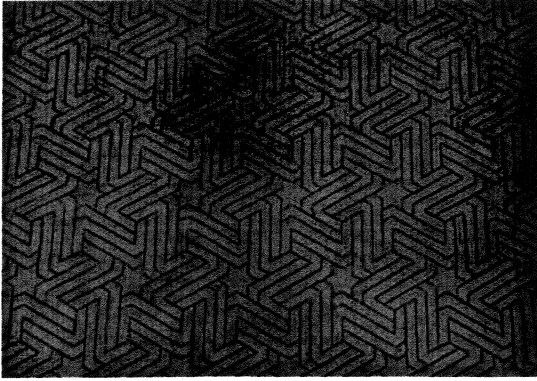


Figure 13: An Islamic spiral pattern.

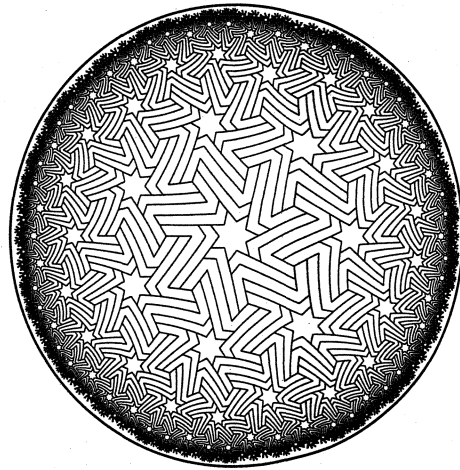


Figure 14: A hyperbolic version of Figure 13.

Spiral patterns are often colored and elaborated, almost to the point of obscuring the original spirals. In the next section, we will look at some simple hyperbolic spiral patterns.

#### 4 Hyperbolic Patterns of Spirals

The patterns in Figures 11 and 12 are not “pure” spiral patterns since the spirals branch at one point, forming two arms that connect to two adjacent spirals. Figure 15 below shows a spiral pattern with no such branching. This pattern is based on the regular tessellation  $\{4,5\}$ . Figure 16 shows the tessellation  $\{4,5\}$  (in dark

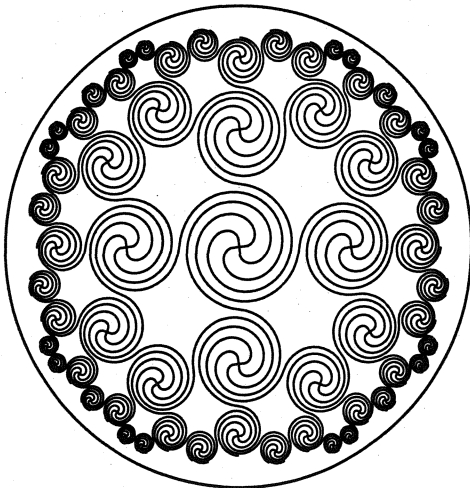


Figure 15: A spiral pattern based on the Fermat spiral and the tessellation  $\{4,5\}$ .

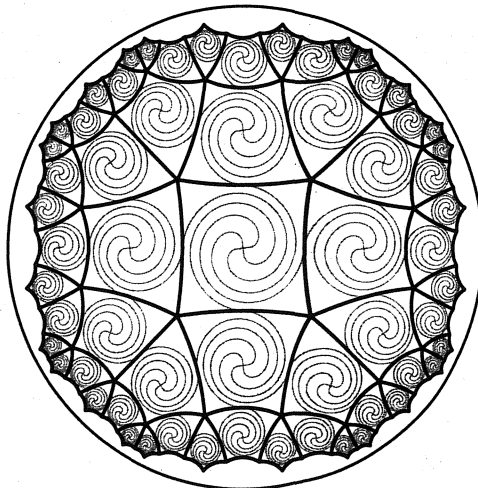
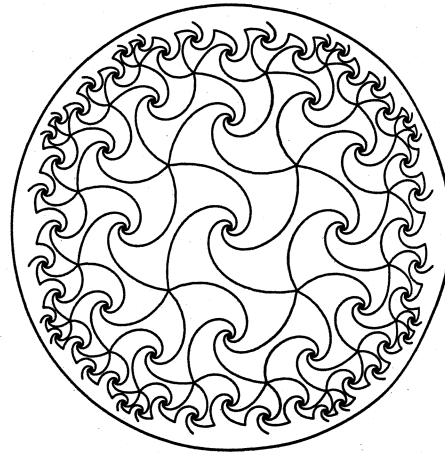
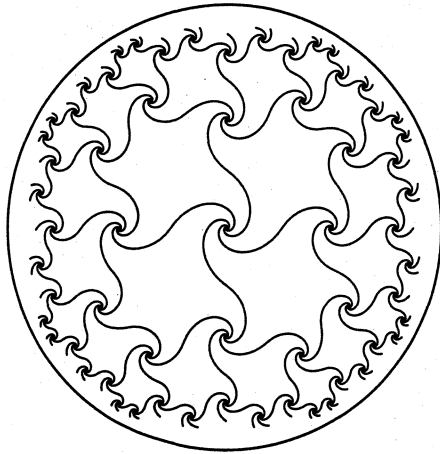


Figure 16: The relation between the tessellation  $\{4,5\}$  and the pattern of Figure 15.

lines) overlaying the pattern of Figure 15 (in light lines). In general, the symbol  $\{p, q\}$  denotes the *regular tessellation* by regular  $p$ -sided polygons with  $q$  of them meeting at each vertex. Figures 12 and 14 are

based on the tessellations  $\{6,4\}$  and  $\{7,3\}$  respectively. Regular tessellations are discussed in more detail in [Dunham4].

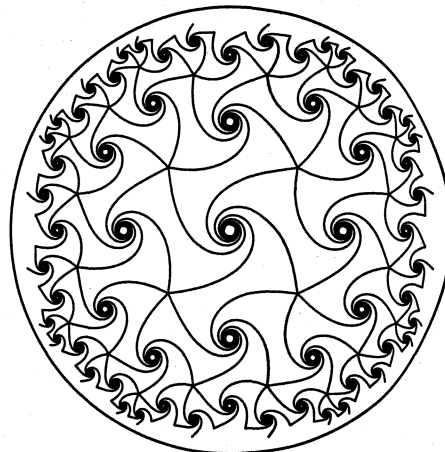
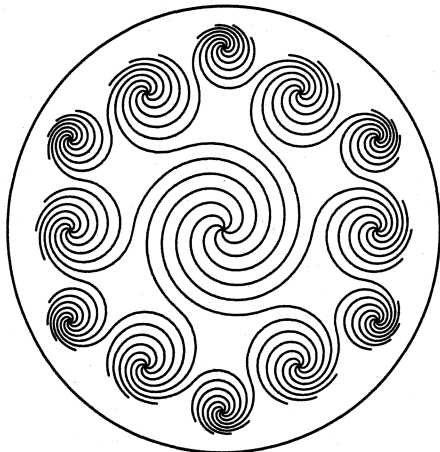
Figure 17 shows an exponential spiral pattern with the spiral pieces connected at the midpoints of the edges of the  $\{4,5\}$  tessellation. Figure 18 shows a pattern based on the spiral of Figure 17 which has been slightly contracted so that the spiral pieces meet at vertices of the  $\{4,5\}$  tessellation.



**Figure 17:** An exponential spiral pattern based on the tessellation  $\{4,5\}$ .

**Figure 18:** An exponential spiral pattern with spiral pieces meeting at vertices of the tessellation  $\{4,5\}$ .

We finish by showing spiral patterns based on the Archimedean spiral and the reciprocal spiral. Figure 19 shows an Archimedean spiral pattern based on the  $\{6,4\}$  tessellation. Figure 20 shows a reciprocal spiral pattern based on the  $\{4,5\}$  tessellation with spiral pieces meeting at tessellation vertices.



**Figure 19:** An Archimedean spiral pattern based on the tessellation  $\{6,4\}$ .

**Figure 20:** A reciprocal spiral pattern with spiral pieces meeting at vertices of the tessellation  $\{4,5\}$ .

## 5 Conclusions and Future Work

We have surveyed some of the natural hyperbolic spirals, and have used pieces of them to construct repeating hyperbolic patterns. It would seem possible to construct hyperbolic versions of Krawczyk's spirolaterals [Krawczyk1, Krawczyk2]. It would also seem possible to construct hyperbolic spiral tilings analogous to those of Gailiunas [Gailiunas1]. Although Euclidean spirals have been studied extensively, there is probably more work to be done on spherical spirals too. In fact, given the large literature on the subject of spirals, there would seem to be numerous directions for further research.

### References

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