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Petrie Polygons

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Abstract

A Petrie polygon is a closed series of edges on a polyhedron (see Coxeter[1] for a more detailed treatment). It is generally taken to mean an equatorial polygon (usually skew), for example the regular dodecahedron has six skew decagons with vertices that alternate across the equatorial planes parallel to its faces.

A sequence of regular frameworks can be generated by moving the vertices of Petrie polygons anywhere between the equatorial planes and the ends of the axes perpendicular to the planes. The sequence passes through a position where the skew polygons define the edges of the original polyhedron, and the convex hull of the framework corresponds to the polyhedron.. The full sequence defines a series of polyhedra (which are isogonal if the original polyhedron is regular). Other isogonal polyhedra can be generated, for example compounds of the antiprisms defined by each skew polygon.

The vertices of moving frameworks generated from Platonic polyhedra define a conjugate framework which passes through an identical sequence, and also other frameworks that match the sequences generated by the dual Platonic polyhedron.

The Regular Tetrahedron

The regular tetrahedron can be considered to be an antiprism with line-segments as the top and bottom faces. Omit these line-segments, which can be done in three different ways, and skew quadrilaterals are left which are the Petrie polygons of the tetrahedron (figure 1). If polyhedra are allowed to have non-planar faces, a possibility touched on by Grünbaum[2], then they form the three faces of a polyhedron with six edges and four vertices, since they meet in pairs at the edges, which is easily verified by counting: 3 $\times$ 4 = 12 quadrilateral edges, 6 edges of the tetrahedron, so each tetrahedral edge has two quadrilaterals.

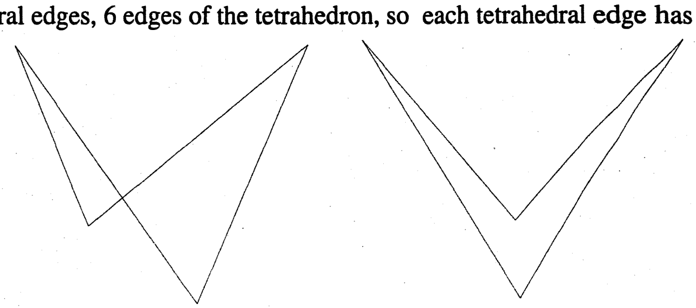


Figure 1. Two views of a skew quadrilateral derived from a regular tetrahedron.

Think of each quadrilateral as being composed of rigid rods, hinged at their ends, and moving so that symmetry is preserved, from a single line-segment when vertices correspond in pairs (figure 2), through the regular tetrahedron (figure 4), a square (figure 6), a regular tetrahedron oriented the opposite way (figure 7), then finally another line-segment. (Alternative stereo pair images for some of the more complicated figures have been provided as an appendix.)

It is quite easy to show that during this motion each vertex sweeps out a semi-ellipse. Consider a pair of vertices joined by an edge. Each vertex moves in a plane. Suppose that the planes intersect along the y axis, each makes an angle of θ to the xy plane, and that one vertex is at a distane of a from the y axis and b from the xz plane (a and b are its coordinates in its own plane). Its coordinates are ($a\,Cos\theta$, b, $a\,Sin\theta$). The other vertex will be at ($a\,Cos\theta$, $-b$, $-a\,Sin\theta$).

$$(2b)^2 + (2a\,Sin\theta)^2 = l^2$$

(where l is the length of the edge), which is the equation of an ellipse in a, b coordinates.

The vertices and edges can be chosen in different ways to define the faces of a polyhedron. One possibility is based on the regular cuboctahedron (figure 8) defined as the convex hull when the three quadrilaterals are expanded fully to squares (mutually at right angles). As they move together the squares become rectangles (figure 9) , and eventually collapse to line-segments, forming the regular tetrahedron (figure 10). The vertices pass through each other , and rectangles start to form again, intersecting inside the convex hull (figure 11), eventually becoming squares when the hinged quadrilaterals have collapsed into line-segments. The vertices define a regular octahedron, and a tetrahemihexahedron (figure 12) is produced. The squares pass through each other and the sequence reverses (in the opposite orientation).

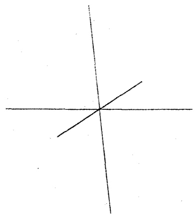
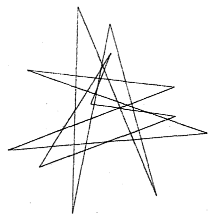
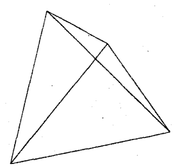




Figure 2 **Figure 3** **Figure 4**

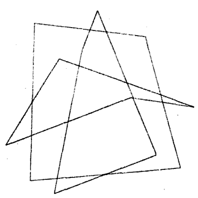
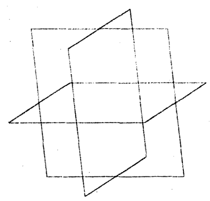
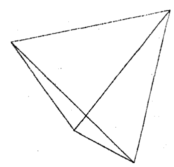




Figure 5 **Figure 6** **Figure 7**

Figures 2 – 7. Stages during the motion of three skew quadrilaterals.

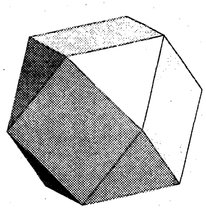
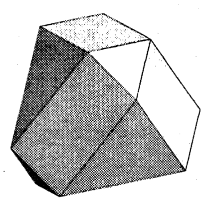
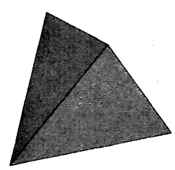
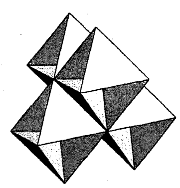
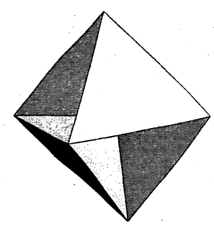






Figure 8 **Figure 9** **Figure 10** **Figure 11** **Figure 12**

Figures 8 – 12. One possible series of polyhedra generated by three skew quadrilaterals.

Another possibility is the compound of three tetrahedra defined in an obvious way by the quadrilaterals . They coincide when they are regular, collapse to squares when the quadrilaterals are fully expanded, and to line-segments when the quadrilaterals have collapsed to line-segments.

The Regular Octahedron/Cube

The regular octahedron is a triangular antiprism, so its Petrie polygon is a skew hexagon in a very obvious way. The hexagon can be chosen in four ways, by omitting opposite pairs of the eight faces, and counting edges verifies that the four hexagons comprise a polyhedron with four non-planar faces. Treating the hexagons as hinged rods leads to a framework that moves from four line-segments oriented along the threefold axes of a cube (figure 13), through a regular octahedron (figure 16), four plane hexagons (figure 18), a cube (figure 20) and four line-segments again, then the sequence reverses.

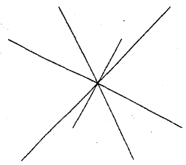


Figure 13

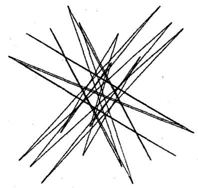
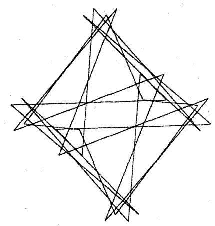


Figure 14



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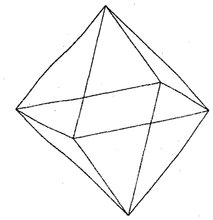

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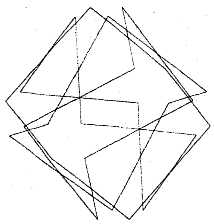

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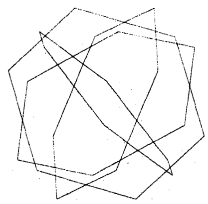

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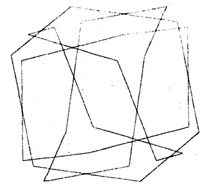

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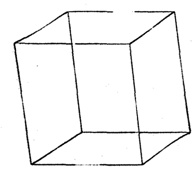


Figure 20

Figures 17 - 20. Stages during the motion of four skew hexagons.

Various polyhedra can be produced by constructing planes defined by the edges and vertices, including a compound of four triangular antiprisms. At one stage they coincide in a single regular octahedron, but they do not coincide when they become regular a second time, producing the regular compound of four octahedra (see e.g. Cromwell[3]).

Throughout their motions these frameworks can be seen as edges rotating in pairs about pivots at their mutual mid-points. Generally no other position for the pivot can be chosen in a consistent way, since if it is nearer to one vertex of the original polyhedron it must be further from another. Of all the Platonic polyhedra only the octahedron has an even number of edges at a vertex, so in this case the pivot can be chosen so it is alternately nearer and further from a vertex, going round the edges in order (figures 21-23). The convex hull of this arrangement has a curiously distorted appearance, but at one stage it consists of a truncated cube (figures 22, 25), which is regular if the pivot is chosen correctly.

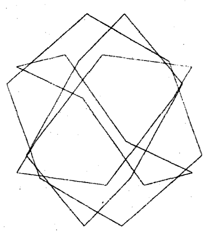
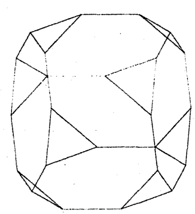


Figure 21



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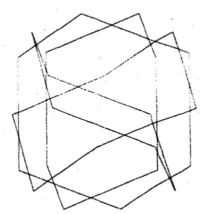
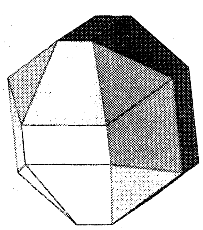


Figure 23

Figures 21 – 23. Skew hexagons turning about points other than the mid-point of the edges.



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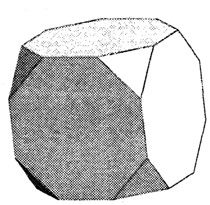

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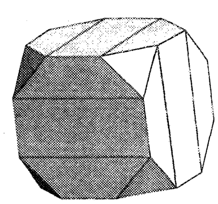


Figure 26

Figures 24 - 26. Polyhedra generated from the frameworks in figures 21–23.

In the limit the pivots can be chosen to coincide with the vertices, producing Buckminster Fuller's jitterbug[4] (figures 27–29), although the complete cycle moves through a stage where the edges are inside the convex hull, and Buckminster Fuller's triangles intersect each other (figure 30). Again various polyhedra can be produced depending on which planes are chosen for the faces (figures 31–34).

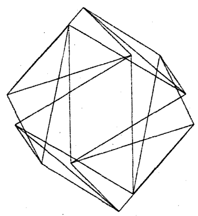
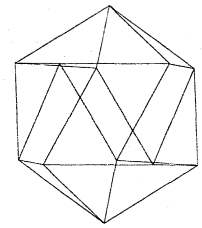
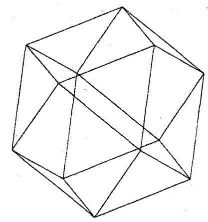
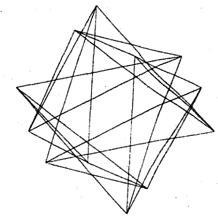





Figure 27 **Figure 28** **Figure 29** **Figure 30**

Figures 27 – 30. Skew hexagons turning about vertices.

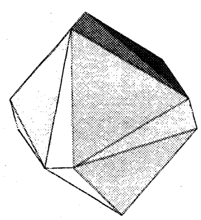
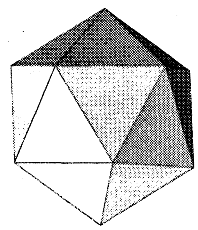
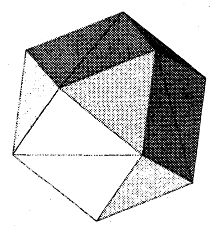
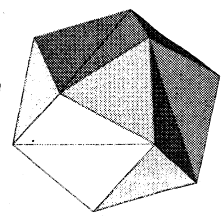





Figure 31 **Figure 32** **Figure 33** **Figure 34**

Figures 31 – 34. A series of jitterbug polyhedra with non-intersecting faces.

The Dodecahedron/Icosahedron

The Petrie polygons of a dodecahedron consist of six skew decagons, which move through line-segments oriented by joining the opposite vertices of the icosahedron (figure 35), the dodecahedron (figure 37), six decagonl planes (figure 39), the icosahedron (figure 41) then six line-segments, and the return journey. A great variety of polyhedra can be produced.

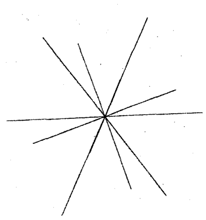
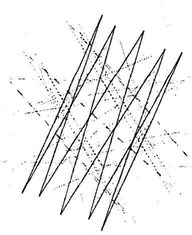
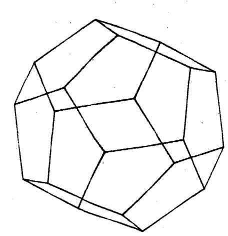




Figure 35 **Figure 36** **Figure 37**

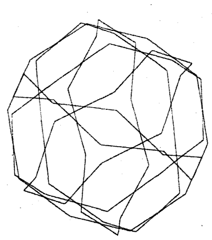
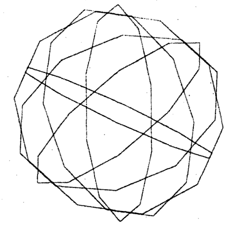
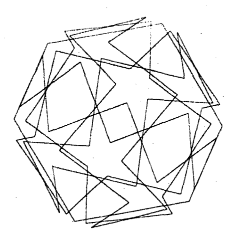
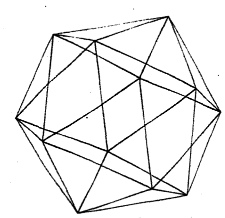





Figure 38 **Figure 39** **Figure 40** **Figure 41**

Figures 35 – 41. Stages during the motion of six skew decagons.

An alternative set of Petrie polygons consists of ten skew hexagons (oriented parallel to the faces of an icosahedron), which move through a similar sequence of figures (figures 42–48), although with a small stellated dodecahedron instead of a dodecahedron. One choice of faces produces a compound of ten triangular antiprisms, forming two different regular compounds of ten octahedra at different stages . A rather unexpected polyhedron occurs when their faces coincide in pairs in the great dodecahedron (unfortunately it has proved impossible to produce a convincing diagram to illustrate this).

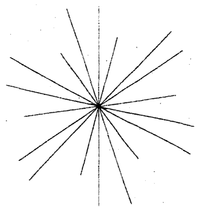
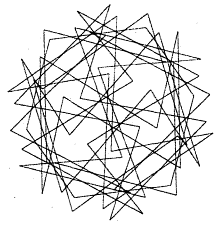


Figure 42



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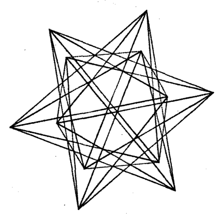

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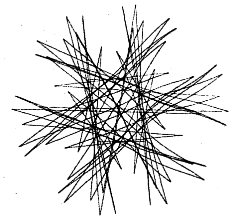

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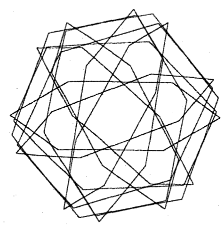

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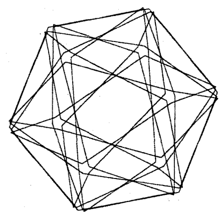

Figure 47

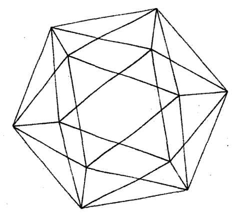

Figure 48

Figures 42 – 48. Stages during the motion of ten skew hexagons.

Conjugate Frameworks

Obviously when the framework collapses into line-segments it is possible to join the ends to form the edges of a Platonic polyhedron, the four line-segments of the octahedron/cube defining the vertices of a cube, the six collapsed decagons the vertices of an icosahedron, and the ten collapsed hexagons the vertices of a dodecahedron. What is not so obvious is that it is possible to choose among the coincident vertices on a combinatorial basis, so that a new framework (the *conjugate*) is produced that goes through exactly the same sequence as the original one in completing a full cycle. Unfortunately the figures are quite complicated and probably need 3-D computer imaging to be readily understood.

The tetrahedron is a special case, since when it collapses into three line-segments the vertices define an octahedron, which does not occur in the tetrahedral framework cycle.

If a framework and its conjugate move through a complete cycle there are certain stages where the two frameworks coincide. When this happens with the octahedron/cube the frameworks lie along the edges of four star-hexagons, {6/2}, lying in diametral planes oriented tetrahedrally (i.e. parallel to the faces of a tetrahedron) (figure 49). This type of configuration was called a nolid by Holden[5]. In the case of the jitterbug, it coincides with its conjugate when the frameworks lie along the edges of a regular octahedron.

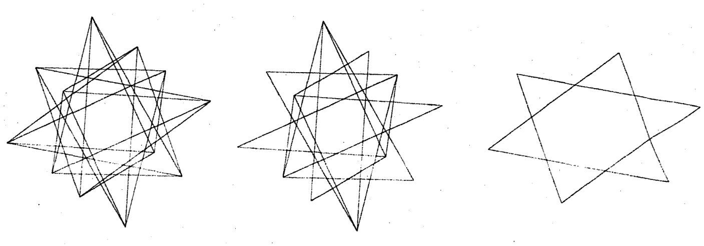

Figure 49. The configuration when the octahedral/cubic framework coincides with its conjugate, and its appearance when one of the star-hexagons is removed.

The dodecahedral/icosahedral frameworks are even more complicated and it can be quite difficult to understand what is happening. The two sets of six decagons coincide along the edges of ten star-hexagons, {6/2}, lying in diametral planes oriented icosahedrally (figure 50), and again along the edges of six sets of intersecting pentagons, {10/2}, oriented dodecahedrally (figure 51). The two sets of ten hexagons also coincide along the edges of six sets of intersecting pentagons, {10/2}, oriented dodecahedrally, and again along the edges of six sets of intersecting pentagrams, {10/4}, oriented dodecahedrally (figure 52).

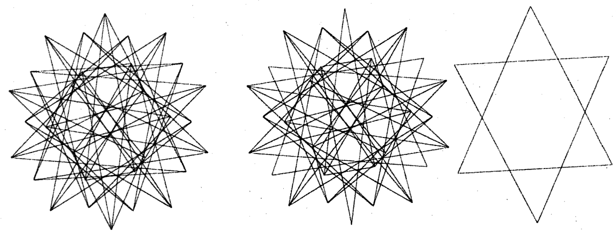

Figure 50. The configuration when the dodecahedral/icosahedral framework coincides with its conjugate, and its appearance when one of the star-hexagons is removed.

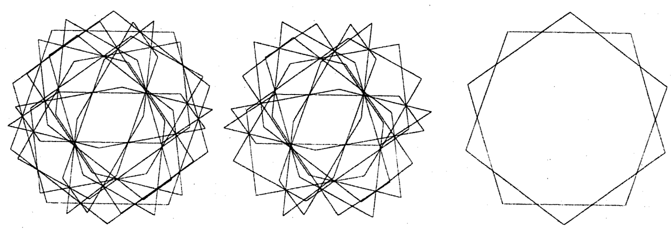

Figure 51. The other configuration when the dodecahedral/icosahedral framework coincides with its conjugate, and its appearance when one of the star-decagons is removed.

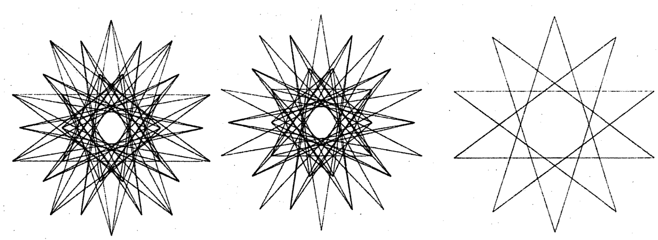

Figure 52. The configuration when conjugate frameworks of ten skew hexagons coincide, and its appearance when one of the star-decagons is removed.

Secondary Frameworks

There is another way to connect the vertices of a framework, this time to produce the dually generated framework, i.e. the vertices of the six decagons will generate the ten hexagons, in two ways, since it will also generate the conjugate framework. The framework of four hexagons is slightly different since when the hexagons collapse into four line-segments, defining the vertices of a cube, this defines a compound of two tetrahedra (a stella octangula). In this case the frameworks never coincide.

The dodecahedral/icosahedral frameworks coincide in both possible combinations, in one case along the edges of an icosahedron, in the other along the edges of a compound of ten tetrahedra (figure 53).

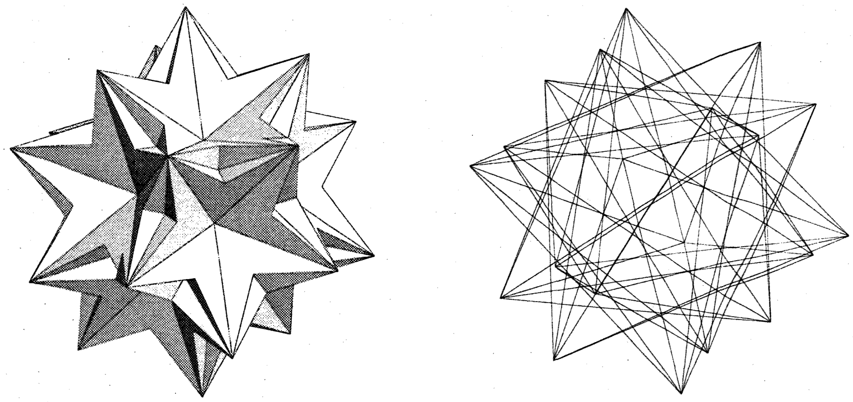

Figure 53. The regular compound of ten tetrahedra.

Alternative Stereo Pair Figures

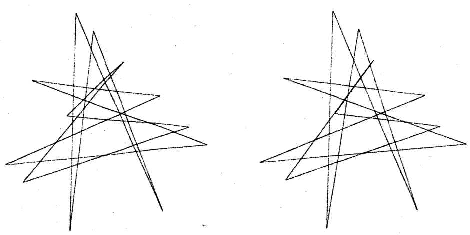

Figure 3a

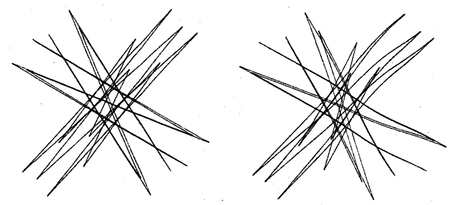

Figure 14a

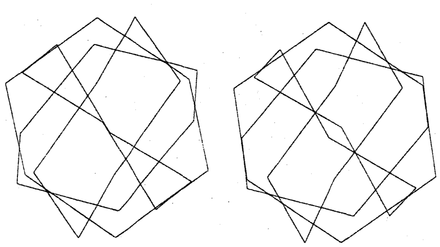

Figure 21a

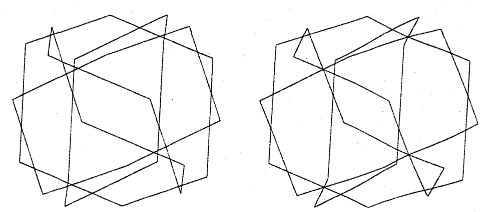

Figure 23a

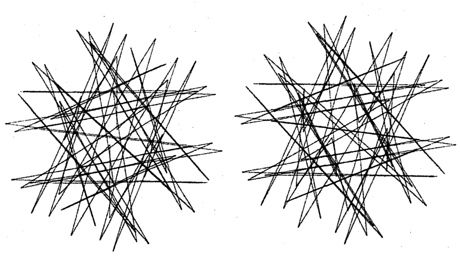

Figure 36a

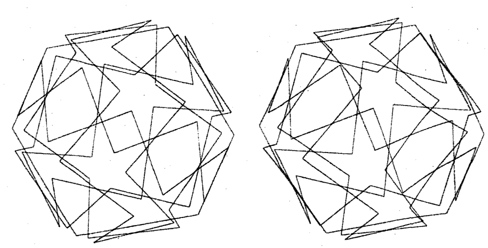

Figure 40a

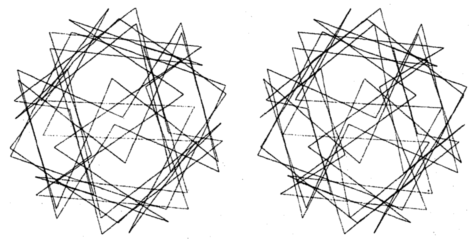

Figure 43a

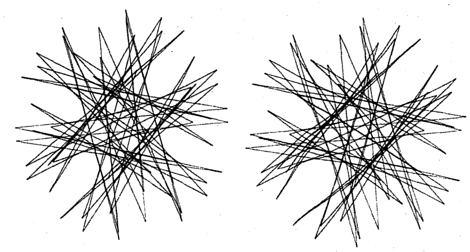

Figure 45a

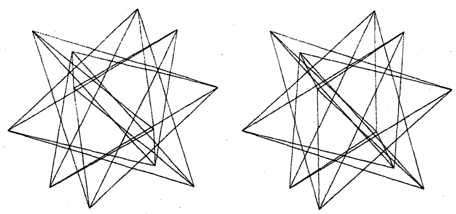

Figure 49a

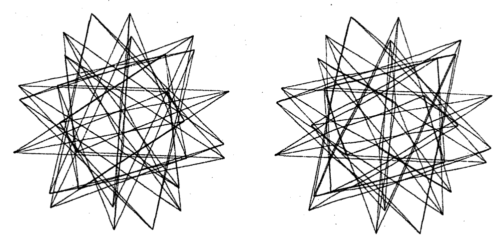

Figure 50a

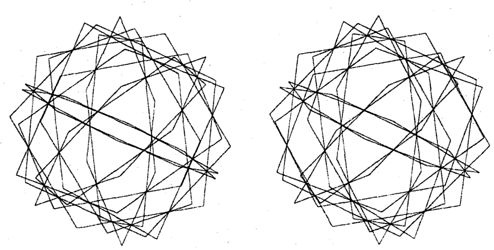

Figure 51a

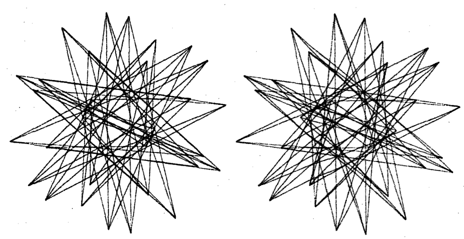

Figure 52a

Computer Realisation

Animations of the frameworks, framework combinations and some of the related polyhedra transformations have been produced using VRML. This paper has been written using TechWriter (for RISCOS machines), and the diagrams have been produced using !PolyDraw (also for RISCOS machines).

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