

Quilt Designs Using Non-Edge-to-Edge Tilings by Squares

Gwen L. Fisher
Mathematics Department
California Polytechnic State University
San Luis Obispo, CA 93407, USA
E-mail: glfisher@calpoly.edu

Abstract

Many traditional quilt designs are based on tilings by squares. Most of these traditional patterns use edge-to-edge tilings, meaning that all of the corners and edges of each tile are aligned with all of the vertices and edges of the tiling. Using non-edge-to-edge tilings as the basis of a quilt design allows for many new quilt settings that are not typically found in traditional quilts, while maintaining the convenience and tradition of using square quilt blocks. This paper discusses the mathematics of non-edge-to-edge tilings by squares, the design of such tilings using Microsoft Excel, and the technical details behind physical construction of such quilts. Both infinite tilings of the plane and finite tilings of rectangles are discussed.

1. Introduction

This paper presents a description of the mathematics behind quilt designs that use non-edge-to-edge tilings by squares. This paper also presents some of the technical details involved with the design and construction of these quilts.

There is a section in Grunbaum and Shephard's *Tilings and Patterns* [1] devoted to tilings that are not edge-to-edge. Pictures of such tilings with only square tiles inspired me to create several quilt designs. The concentration on square tiles allows for individual tiles to be independent compositions using traditional quilt designs. Note that while the completed quilt designs presented in this paper incorporate traditional quilt designs, such as Square within a Square, Postage Stamp One Patch, and Amish Bars [2], any square quilt design could be used.

2. Tilings with Squares

A substantial portion of traditional quilts use squares as the basic design element. The reasons for this are more likely technical rather than aesthetic. First, as most beds are rectangular, it makes sense to start with a shape that will tile a rectangle. Second, fabric is traditionally woven using threads that are at right angles. Therefore, to preserve the stability of pieced quilts, fabric is cut along the lines of the weave, commonly referred to as the grain of the fabric. Finally, squares are easy to measure, cut, and sew. Unlike with other shapes, inset seams can be avoided when a piecework pattern of a quilt is limited to squares.

2.1. Definitions. In order to provide a mathematical discussion of non-edge-to-edge tilings by squares, I begin with some definitions.

Definition: A tiling in which all of the corners and edges of each tile are aligned with all of the vertices and edges of the tiling is said to be *edge-to-edge*.

Definition: A *monohedral* tiling is a tiling in which all tiles are congruent. That is, all tiles are the same size and the same shape. Any tiling that is not monohedral is said to be *polyhedral*.

Definition: A tiling is said to be *uniform* if the symmetries of the tiling act transitively on the vertices. In other words, all of the vertices are of the same species.

Definition: A tiling is said to be *equitransitive* if every tile is equivalent under some symmetry to every congruent tile in the tiling. In other words, all tiles of the same size and shape have all of the same neighboring tiles in the same configuration.

2.2. Monohedral Tilings with Squares. The regular tiling of the plane by squares, shown in Figure 1, is monohedral, uniform, and equitransitive. Furthermore, the regular tiling is edge-to-edge.

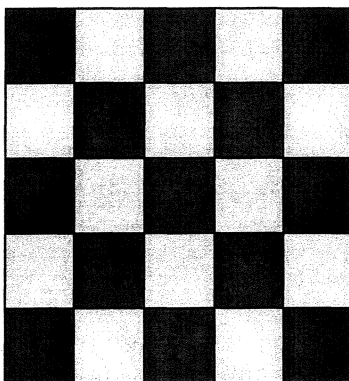


Figure 1: *The Regular Tiling by Squares: $p = 0$*

A monohedral, uniform, equitransitive tiling of the plane that is not edge-to-edge can be constructed as follows. Start with the monohedral tiling with squares, and slide all rows by the same constant distance. Let the ratio of the slide length to the length of the side of the square be the number p . Since p can be any real number, there exist an uncountably infinite number of such tilings. Examples using various values of p are shown in Figures 1 through 3.

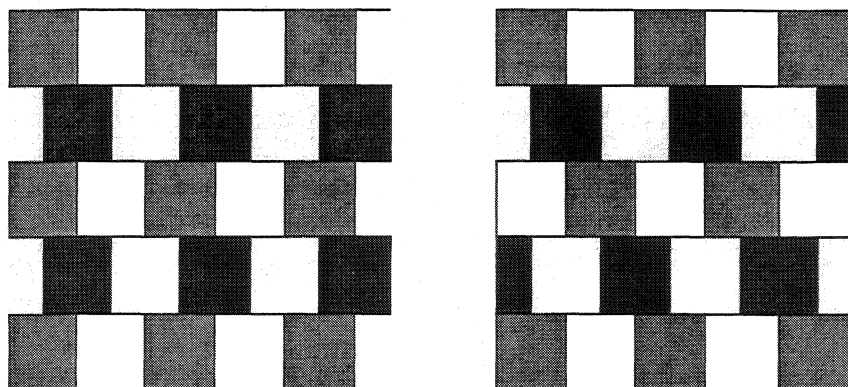


Figure 2: *The Monohedral, Uniform, Equitransitive Tilings by Squares: $p = 1/2$*

The tiling in the left of Figure 2 appears to have vertical stripes, whereas the tiling on the right appears to have diagonal stripes. However, both tilings in Figure 2 are actually the same tiling. They look different because they differ by coloring.

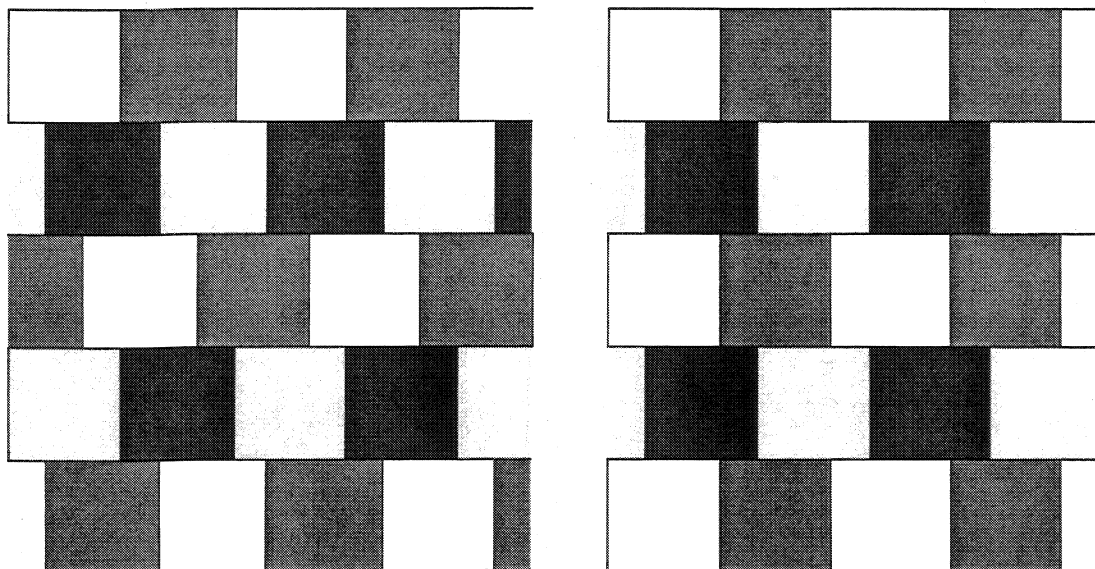


Figure 3: *Two Monohedral, Uniform, Equitransitive Tilings by Squares: $0 < p \leq 1/2$*

The tiling in the left of Figure 3 appears to have diagonal stripes, whereas the tiling on the right appears to have vertical stripes. However, in contrast to the examples in Figure 2, the two tilings in Figure 3 really are different. The left tiling can be generated from the regular tiling of squares by sliding each row to the right by ratio p . In contrast, to generate the right tiling, the rows alternate the direction of the slide: first right, then left, then right, and so on.

Remarkably, not all uniform, equitransitive tilings of the plane by squares are monohedral. Figure 4 shows such a tiling with squares of two different sizes. In Figure 4, the ratio p is the ratio of the side length of the smaller square to the side length of the larger square. Again, since p can be any real number, there exists an uncountably infinite number of such tilings.

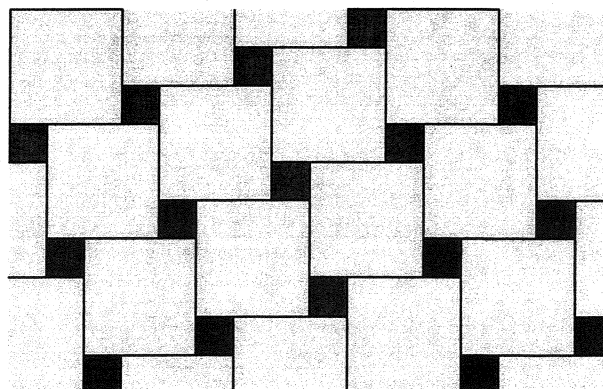


Figure 4: *A Polyhedral, Uniform, Equitransitive Tilings with Two Squares: $0 < p < 1$*

2.3. Polyhedral Tilings with Squares. A polyhedral tiling is one in which not all of the tiles are congruent. Since all squares are similar, a polyhedral tiling with squares contains squares of various sizes. There are many distinct types of tilings of the plane by squares that are equitransitive but not uniform. So, in order to limit this discussion, the condition of uniformity, which is about vertices, is exchanged for a different condition on the edges.

Definition: A tiling is *unilateral* if each edge of the tiling is a side of at most one polygon.

There are five types of unilateral, equitransitive tilings that use exactly three squares. All five types of these tilings have the property that the sum of the side lengths of the two smallest squares equals the side length of the largest square. That is, the side lengths of the three squares are x , y , and $x + y$. In particular, all five types of these tilings can be represented with squares with side lengths in the proportion 1:2:3. Examples of two types of these tilings are shown in Figure 5.

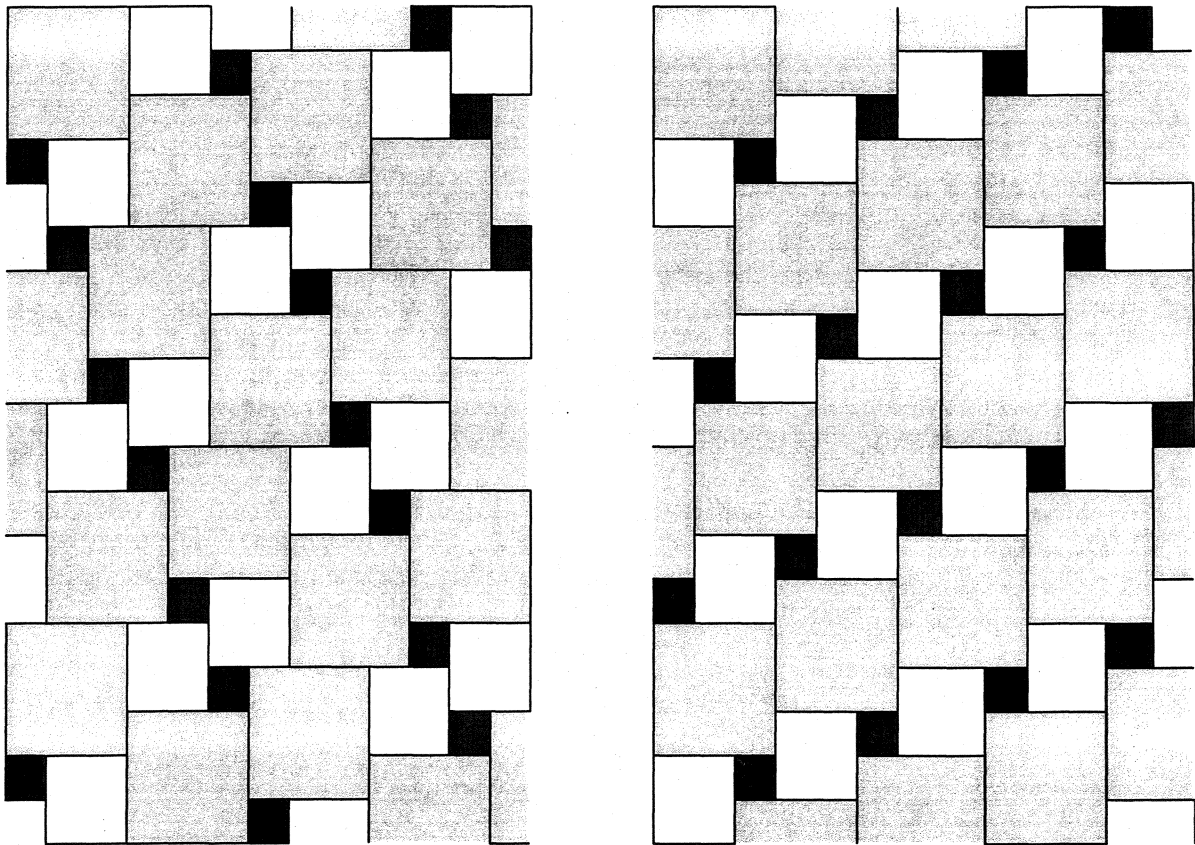


Figure 5: *Unilateral, Equitransitive Tilings that Use Exactly Three Squares*

There are also many examples of unilateral, equitransitive tilings by more than three squares. Quilt designs incorporating such examples are shown in Figures 6 and 7. In both of these figures, the square tiles are individual compositions, which are composed of traditional quilt designs. Thus, the overall intricacy of the quilt design is more complex than is the unilateral, equitransitive tiling on which the design is based. The design in Figure 6 is based on a tiling using four different squares with side lengths 1, 2, 3, and 6. Similarly, the tiling in Figure 7 incorporates six different squares with side lengths 1, 4, 7, 8, 9, and 10.

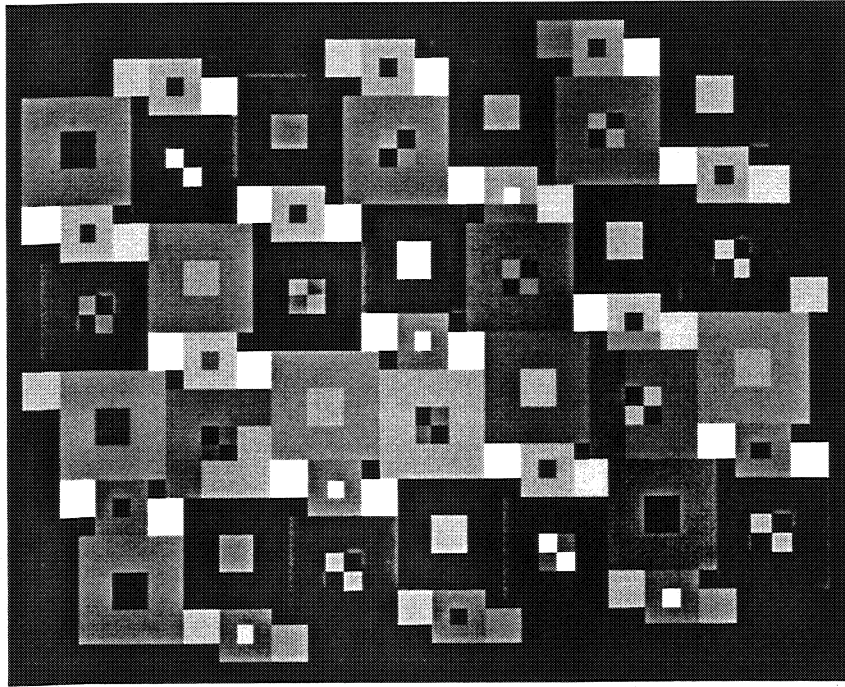


Figure 6: *Unilateral, Equitransitive Quilt Design that Uses Exactly Four Squares*

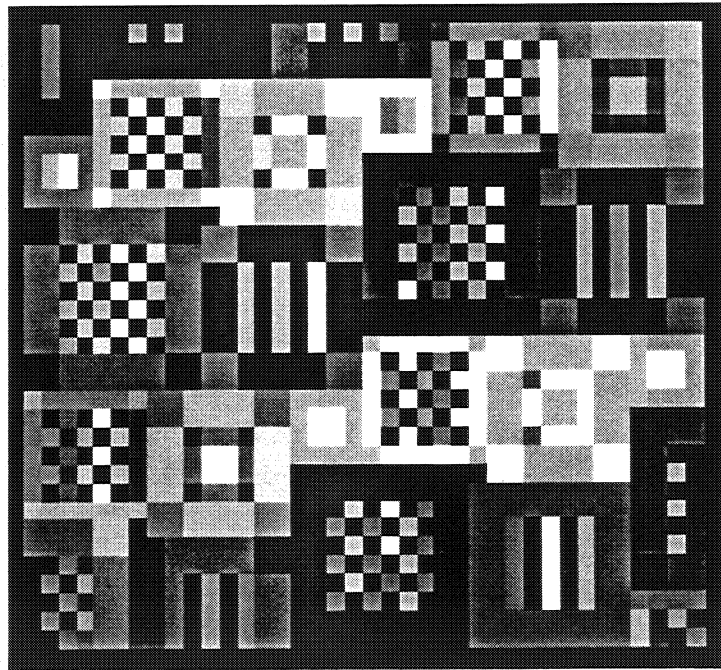


Figure 7: *Unilateral, Equitransitive Quilt Design that Uses Exactly Six Squares*

2.4. Squared Rectangles. A squared rectangle is a rectangle that is tiled by more than one square, where all of the square tiles are different sizes. Consequently, all squared rectangles are not edge-to-edge. The smallest squared square was discovered by A. J. W. Duijvestijn and requires 21 different squares. The smallest squared rectangle was discovered by Z Moron in 1925 and has only 9 squares [3]. This squared rectangle measures 32 by 33 units and has squares with side lengths 1, 4, 7, 8, 9, 10, 14, 15, and 18. Figure 8 shows a quilt design that uses this smallest squared rectangle as the basis for its layout. Border designs were added to help balance the composition.

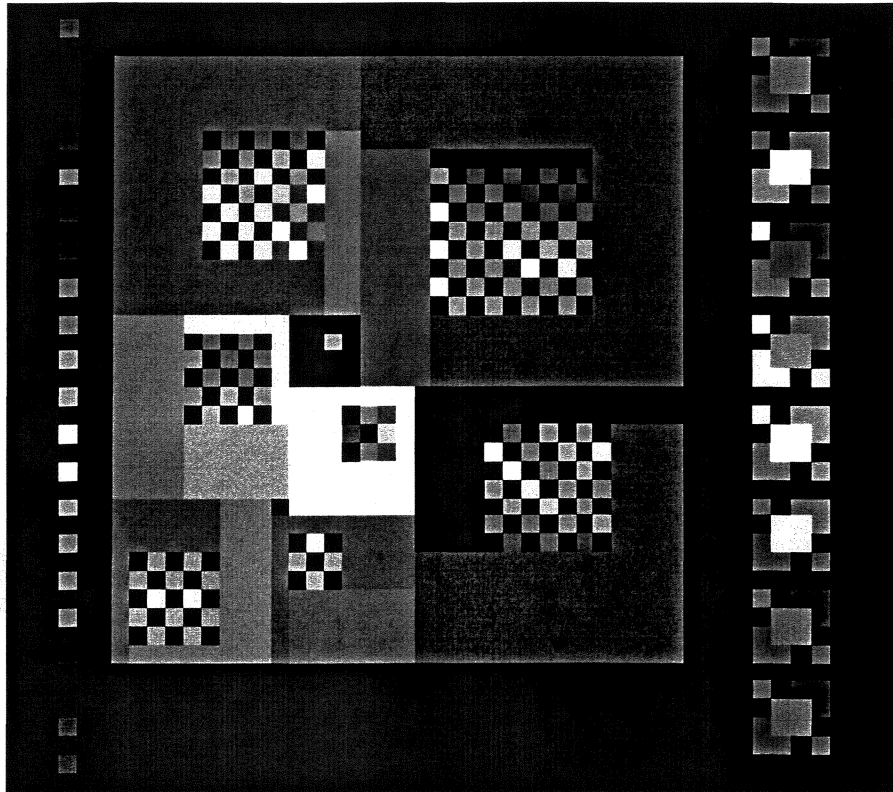


Figure 8: *Quilt Design Using a 32 by 33 Squared Rectangle*

3. Designing the Quilts

3.1 Coloring the Squares. For the purposes of quilt design, the pattern of colors matters. As demonstrated in Figure 2, the same tiling with different colorings may have different visual effects. For my quilt designs, I originally required each square of the same size to have the same color, while squares of different sizes had different colors. As the design of the quilt evolved, “color” later became design. That is, squares of the same size have the same interior design. The reason for this modification is to create a more complex visual design that can incorporate a much wider range of colors. The designs used in the interiors of the squares in Figures 6, 7, and 8 are based on traditional Amish quilt designs such as Square within a Square, Postage Stamp One Patch, and Amish Bars [2]. Additionally, each block incorporates simple borders.

3.2. Using MS Excel to Design Patterns with Squares. The use of Microsoft Excel to design patterns with squares is remarkably easy. First, adjust the column width to be equal to the row height to make all

cells into squares of the same size. Then, use the “fill color” palette to color in individual squares. Make larger squares as composites of smaller squares. Patterns can be easily repeated using the cut and paste commands.

4. Sewing Fabric Designs that Are Not Edge-to-Edge

The method of construction for quilts incorporating non-edge-to-edge tilings by squares requires various sewing sequences, depending upon the complexity of the design. In particular, tilings that are nowhere edge-to-edge require more complicated sewing sequences than do simpler designs. In Figures 2 and 3, the vertical edges of the tiling are the same line segments that make the vertical sides of the tiles. In contrast, no horizontal edge of a tile shares a complete edge with another tile. One may consider such a tiling to be only partially edge-to-edge. In this case, sewing together patches of fabric is done in the same manner as sewing the regular tiling shown in Figure 1; first, sew all squares into horizontal rows, and then join the rows.

In contrast, the designs in Figures 4 through 8 cannot simply be sewn in simple rows because all of these designs are nowhere edge-to-edge. Generally, the sewing of fabric designs that are nowhere edge-to-edge requires some planning before sewing. This is due to the fact that no edge of a tile is the edge of another tile. The essential problem for sewing is to determine how to sew together a configuration as shown in Figure 9.

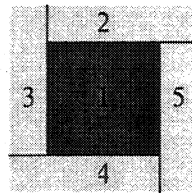


Figure 9: *The Problem in Sewing Non-Edge-to-Edge Tilings*

While the block in Figure 9 has only four edges, this block actually requires five lines of stitches because one of the edges needs to be sewn in two stages. More explicitly, Patch 1 is first partially sewn to Patch 2, stopping about an inch before the vertex at Patch 5 is reached. See the bold line in Figure 10 for the partial seam line.

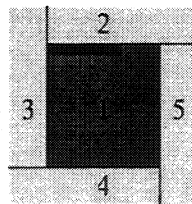


Figure 10: *Sewing the First Seam*

Then Patch 3 is sewn to Patch 1-2 as shown in Figure 11.

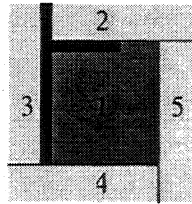


Figure 11: *Sewing the Second Seam*

Continuing in the same manner, Patch 4 is sewn to Patch 1-2-3, and Patch 5 is sewn to Patch 1-2-3-4. Finally, complete the first seam of Patch 2 that was started in Figure 10. This completes the sewing sequence.

5. Conclusion

Mathematicians have discovered numerous non-edge-to-edge tilings by squares. Some of these tilings use squares that are all the same size, whereas other tilings use squares of many different sizes. Non-edge-to-edge tilings offer quilt designers an expanded set of quilt layouts that are not afforded by traditional edge-to-edge layouts. At the same time, patches within the quilts can still be limited to squares, thereby maintaining most of the convenience and all of the tradition of using square blocks.

References

- [1] Branko Grunbaum & G.C. Shephard, *Tilings and Patterns*, W. H. Freeman and Company: New York, 1987.
- [2] Dennis Duke & Deborah Harding, *America's Glorious Quilts*. Hugh Lauter Levin Associates: Hong Kong, 1987.
- [3] David Wells, *Penguin Dictionary of Curious and Interesting Geometry*. Penguin: U.S.A., 1992.