Finding the Dual of the Tetrahedral-Octahedral Space Filler

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Abstract

The goal of this paper is to illustrate how octahedra and tetrahedra pack together to fill space, and to identify and visualize the dual to this packing. First, we examine a progression of 2-D and 3-D space-filling packings that relate the tetrahedral-octahedral space-filling packing to the packing of 2-D space by squares. The process will use a combination of stretching, truncation and 2-D to 3-D correspondence. Through slicing, we will also relate certain stages of the process back to simple 2-D packings such as the triangular grid and the 3.6.3.6 Archimedean tiling of the plane. Second, we will illustrate the meaning of duality as it relates to polygons, polyhedra and 2-D and 3-D packings. At a later stage, we will reason out the dual packing of the tetrahedral-octahedral packing. Finally, we will demonstrate that it is indeed a 3-D space filler in its own right by showing different construction methods.

1. Introduction

We are all familiar with simple shapes such as the tetrahedron, cube and octahedron. We are less familiar with their symmetries. Duality can help us understand the symmetries of one shape by considering the symmetries of its dual. For example, the symmetries of an octahedron are the same as those of its dual, the cube. We can observe that both shapes have rotational symmetry of order 3 from the triangular faces of the octahedron, and of order 4 from the square faces of the cube.

Space-filling packings also have duals, which must also be space-filling packings with the same symmetries. Finding the dual of a packing can be useful for finding new packings given known ones, and in helping us visualize and understand a packing. Finding duals is one example of a mathematical connection between shapes and packings. We will also make other connections between packings using processes of circumscription, distortion, and truncation. These connections help illustrate not just the geometric properties of the objects but the transformations and relationships between them.

As we have done in previous papers (see bibliography), we will take this approach and apply it to space-filling packings; we will identify and visualize the dual packing to the tetrahedral-octahedral packing via a progression of related packings. In particular, we will make use of the properties of the cubic packing: It is easy to visualize and the cube is related to the octahedron by duality and the tetrahedron by circumscription.

2. Tilings and packings:
progressing from the square grid, to the cubic packing, to the tetrahedral-octahedral packing

2.1 From the square tiling to the rhombic tiling. Squares tile, row by row, four touching at each vertex. Given such a tiling, it is simple to transform it into a rhombic tiling by stretching it along a diagonal as in figure 1.
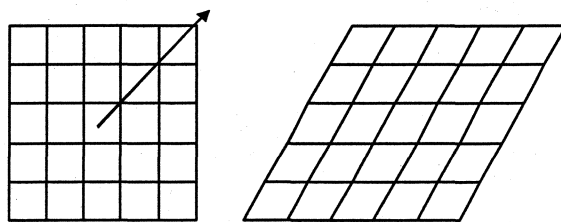

Figure 1: From the square tiling to the rhombic tiling

2.2 From the rhombic tiling to the triangle and hexagon-triangle tilings. When the rhombi have angles of 60 and 120 degrees they can be truncated as shown to yield a grid with equilateral triangles or the 3.6.3.6 Archimedean tiling of triangles and hexagons that pack to fill the plane, as in figure 2.
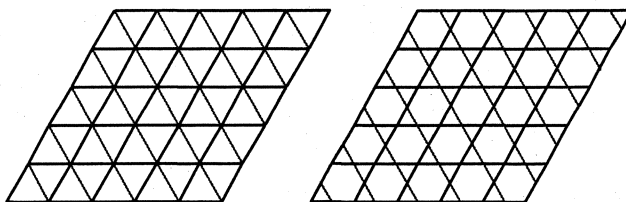

Figure 2: The triangle grid and the 3.6.3.6 Archimedean tiling

The four tilings shown in the last two figures have several things in common, the simplest of which is that in each case, all the polygons touch any one of their neighbors at a *single* vertex, or along a *single whole* edge. There is no instance of two tiles touching only along part of an edge, or along more than one edge. This property qualifies the tilings as a regular tessellation[1]. Secondly, an important property that these tilings have in common is that they each have only one type of vertex. That is to say that any vertex from one of the tilings can be replaced, through translation and/or rotation, with any other vertex from the same tiling. The same is true with respect to the tiles. With exception of the last tiling, which contains two different tiles, each tile is congruent to all the others in the same tiling. In the case of the last tiling, which contains two different types of tiles, the hexagons are all congruent and the triangles are all congruent. Finally, an important property to note in this context is that each tile has edges in common with only one type of tile. In the first three tilings this is a trivial property, but in the 3.6.3.6 Archimedean tiling, it is important to note that all the triangles share edges with only hexagons and vice versa.

2.3 From the cubic packing to the parallelepiped packing. Repeating the process shown above, if we started with the cubic packing, we can stretch it, again, to obtain the parallelepiped packing, as in figure 3.
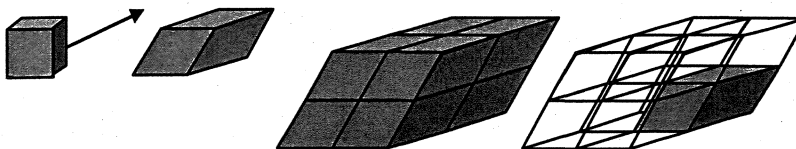

Figure 3: From the cube to the parallelepiped packing

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[1] Following the definition of 'Regular Tessellation' given by authors of high school geometry texts (see Serra 1997).

2.4 Truncating the parallelepiped packing. The truncation stage, in the 3-D version, is a little bit more complex. First, it is useful to see what happens to a single parallelepiped. In figure 4, the parallelepiped is sliced through twice, cutting all the rhombic faces into two triangular faces each. This operation creates three pieces, that is, two tetrahedra and one octahedron.
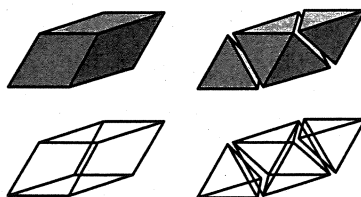

Figure 4: From a single parallelepiped to two tetrahedra and one octahedron

The new solids are packed in the same way that the parallelepipeds were packed, producing the tetrahedral-octahedral packing. A good way of visualizing this result is by slicing at different levels through the packing. In figure 5, the slice was made through the plane of contact between two layers of truncated parallelepipeds, showing a pattern that looks like the regular triangular grid of figure 2 (left). In figure 6, the slicing was made through the middle of a layer, showing the 3.6.3.6 Archimedean pattern of figure 2 (right).
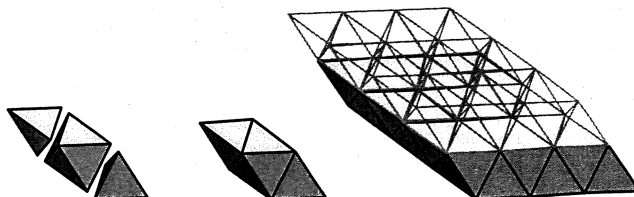

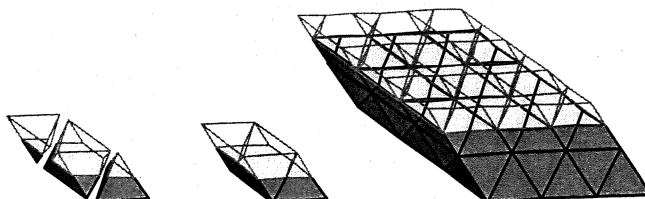**Figure 5: Slicing the packing through the plane of contact**

Figure 6: Slicing the packing through the middle of the layer

Another way to visualize the tetrahedral-octahedral packing is to begin with a tetrahedron inscribed in a cube. The tetrahedron can have one of two orientations within its circumscribing cube, and if one were to assemble the tetrahedra by alternating their orientation, the resulting gaps take the form of octahedra. Figure 7 shows the beginning of such an assembly.
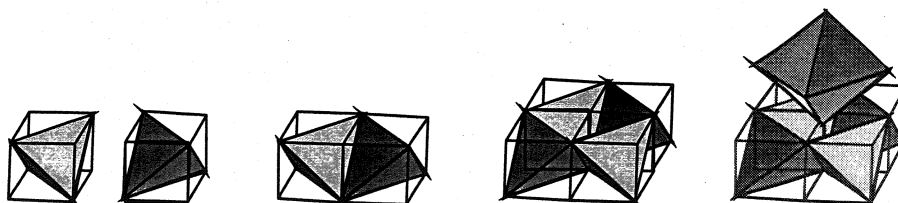

Figure 7: Assembling the tetrahedral-octahedral packing

It is interesting to note that this packing by tetrahedra and octahedra satisfies similar regularity properties to the tilings in figures 1 and 2. Two polyhedra in the packing either touch in one entire common face, one entire common edge or a vertex. Second, all vertices are congruent in that the same number of tetrahedra and octahedra touch it in the same way. As in the case of the tilings above, all the symmetries,

rotational and reflectional, of the tetrahedra and octahedra extend to the whole space filling. Finally, as in the 3.6.3.6 Archimedean tiling, tetrahedra only share faces with octahedra and vice versa[2].

3. Duality

Duality is a property of pairs of geometric objects. It is reciprocal and generally relates two objects that are of the same category: They have the same number of dimensions and the same types of components. For example polygons are duals of polygons, and polyhedra of polyhedra.

3.1 Duals of polygons. The dual of a polygon is defined as the polygon created by placing a vertex at the midpoint of each edge and joining adjacent midpoints by edges. Figure 8 shows the duals of the 5 simplest regular polygons. Notice that there is a one-to-one correspondence between the *edges* of the polygon and the *vertices* of the dual, and vice versa:

Number of edges of polygon = Number of vertices of dual
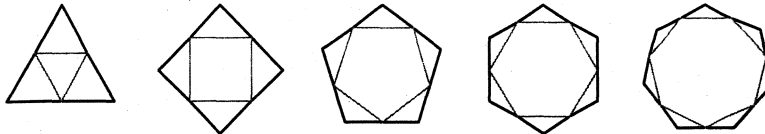Number of vertices of polygon = Number of edges of dual

Figure 8: Polygons and their duals

3.2 Duals of polyhedra. For most simple polyhedra, the dual can be constructed in a very straight forward manner: Place a new vertex in the midpoint of each face and then join the new vertices from adjacent faces with new edges. Finally fill in faces between coplanar edges where these surround a vertex of the initial polyhedron. Figure 9 shows the duals of two well known polyhedra.

This sets up a correspondence between the *edges* of the polyhedron and the *edges* of the dual, *faces* of the polyhedron and *vertices* of the dual, and vice versa. Now we can write:

Number of faces of polyhedron = Number of vertices of dual
Number of edges of polyhedron = Number of edges of dual
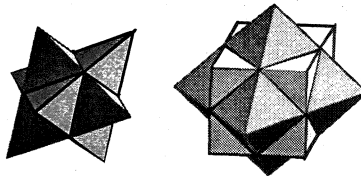Number of vertices of polyhedron = Number of faces of dual

Figure 9: The tetrahedron and the cube, with their duals

3.3 Duals of tilings and space-filling packings. The dual of a tiling of the plane is made by placing a new vertex in the center of each polygon, and joining the vertices by an edge whenever those vertices are at the center of two polygons that share an edge.

Vertices of tiling correspond to *faces* of dual
Edges of tiling correspond to *edges* of dual
Faces of tiling correspond to *vertices* of dual

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[2] This allows us to derive the property that there are twice as many tetrahedra as octahedra since tetrahedra have half as many faces as octahedra.

3.4 Self-duality of the square tiling. Figure 10 shows that the dual of a square tiling is another square tiling. The two tilings differ by a translation in the direction of the diagonal, of length one half of a diagonal.
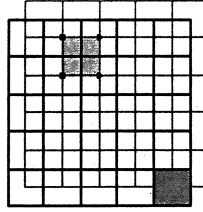

Figure 10: The square tiling is a self-dual

3.5 Duality of the triangular and hexagonal tilings. It is not always true that dual tilings differ only by translation. The regular tilings of the plane by triangles and by hexagons are dual to each other, as illustrated in figure 11.
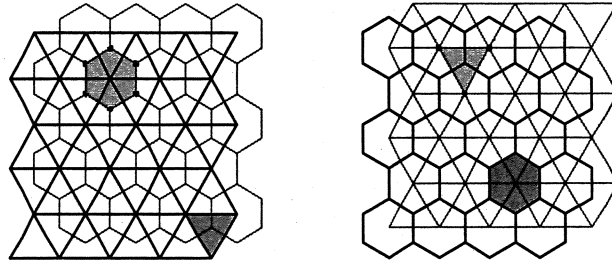

Figure 11: Triangular and hexagonal tilings are dual

3.6 Dual of the 3.6.3.6 Archimedean tiling: a 60-120 rhombic tiling. Figure 12 shows this Archimedean tiling, containing two different types of tiles, but with a dual tiling containing only one type of tile, the rhombus, in three different orientations[3].
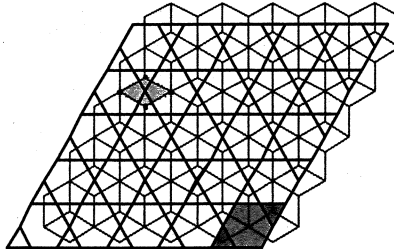

Figure 12: Dual of the 3.6.3.6 Archimedean tiling.

3.7 Duals of 3-D space-filling packings. The dual of a 3-D packing is made by placing a new vertex in the center of each polyhedron, and joining these vertices by an edge whenever they are at the center of two polyhedra that share a face. Then the faces of the dual are filled in. This forms polyhedra whose centers are at the vertices of the original packing. This sets up the following four correspondences:

Vertices of packing correspond to *centers of polyhedra* of dual
Edges of packing correspond to *faces* of dual
Faces of packing correspond to *edges* of dual
Centers of polyhedra of packing correspond to *vertices* of dual

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[3] The dual has only one type of tile because the 3.6.3.6 tiling has only one type of vertex.

3.8 Self-duality of the cubic space filler. Figure 13 shows a cubic packing and its dual, another cubic packing. The method we will use to find dual packings will use the one-to-one correspondences between centers of polyhedra in the packing and vertices of the dual. Conversely, a polyhedron in the dual will have its center at a vertex of the packing and its vertices will be at the centers of the polyhedra meeting at that vertex. As eight cubes meet at a vertex in the packing, we know that their eight centers form vertices of a polyhedron of the dual. The cube has eight vertices so the dual of a cubic packing is another cubic packing.
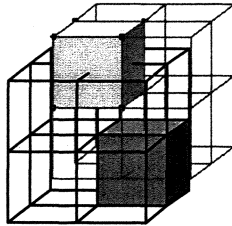

Figure 13: The cubic packing is self-dual

4. The dual of the tetrahedral-octahedral space filler

To find the dual of the tetrahedral-octahedral space filler using the same method as above, we first need to know how many tetrahedra and octahedra meet at each vertex, and in what configuration[4]. We already know that tetrahedra only share faces with octahedra and vice versa, but we need to know what happens at a vertex. To do this, we will assemble the tetrahedra and octahedra that meet at a vertex. Since the parallelepiped packing is derived from the cubic packing, let us begin with that. In the case of the cube packing, eight cubes meet at a vertex, and there are eight vertices on a cube. As cubes can be packed using translation only, we can say that each vertex of the packing contains all the vertices of a cube. This does not change after deformation to the parallelepiped packing since the polyhedra are not restacked. We now know that each vertex of the packing contains all the different vertices of a parallelepiped. It is therefore enough to count the eight vertices of one parallelepiped to know how many meet at a vertex.
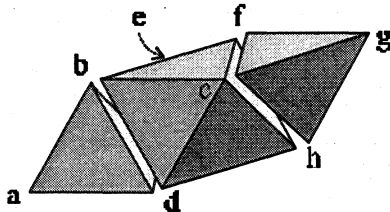

Figure 14: A subdivided parallelepiped

We are ready now to replace the parallelepipeds with groups of tetrahedra and octahedra. As each vertex of the parallelepiped will meet each vertex of the space filler, we can now count how many tetrahedral and octahedral vertices will meet at each vertex of the space filler. Using the subdivided parallelepiped of figure 14, we can count the vertices by going around the original parallelepiped, vertices a through g: (a) 1 tetrahedral vertex, (b) 1 tetrahedral and 1 octahedral, (c) 1 tetrahedral and 1 octahedral, (d) 1 tetrahedral and 1 octahedral, (e) 1 tetrahedral and 1 octahedral, (f) 1 tetrahedral and 1 octahedral, (g) 1 tetrahedral, and (h) 1 tetrahedral and 1 octahedral. As each vertex comes from a separate polyhedron in the space filler, this tells us that a total of 8 tetrahedra and 6 octahedra meet at each vertex of the space filler.

The resulting fourteen polyhedra, being packed one by one around a vertex, are shown in figure 15. The position of the centers of each tetrahedron and octahedron are shown as triangles and rhombi respectively. Beginning with one octahedron, we have assembled what amounts to a double edge length octahedron by

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[4] As in the 3.6.3.6 tiling, there will only be one type of polyhedron in the dual packing because there is only one type of vertex in the tetra- and octahedral space-filler.

adding first the four tetrahedra that touch the original octahedron, then four octahedra touching those, then another set of four tetrahedra, and finally, an octahedron. Shown in the middle of the figure is the polyhedron created by using the centers of the fourteen polyhedra as vertices: the dual polyhedron. Known as the rhombic dodecahedron, it is made up of twelve rhombi meeting at fourteen 3- and 4-vertices corresponding to the centers of the tetrahedra and octahedra, as shown in the diagram.
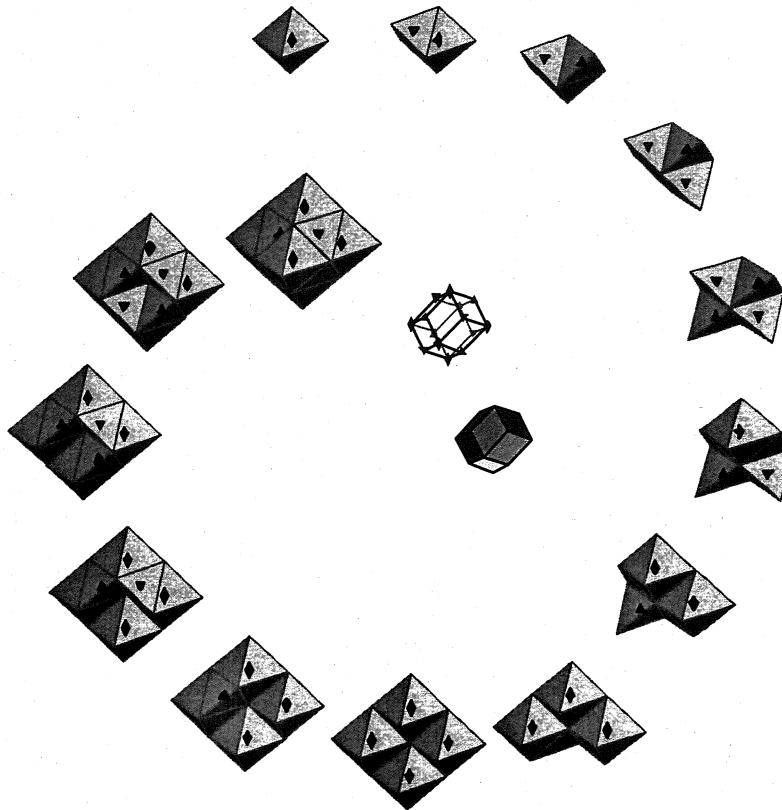

Figure 15: A vertex of the tetrahedral-octahedral space filler

In figure 16, the rhombic dodecahedron is shown, in wireframe and solid form, with its packing.
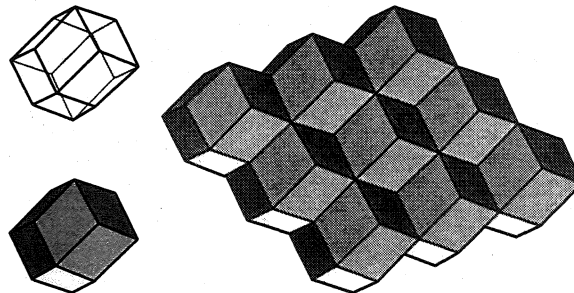

Figure 16: The rhombic dodecahedron and its packing

4.1 Visualizing the space-filling packing of rhombic dodecahedra. There is an alternate way of visualizing the packing of the rhombic dodecahedron. Picturing a 3-D checkerboard we replace every 'black' cube with six pyramids as shown in figure 17. These pyramids are then attached, by their base, to the neighboring 'white' cube. If this is done throughout the packing, the result is the rhombic dodecahedral packing.
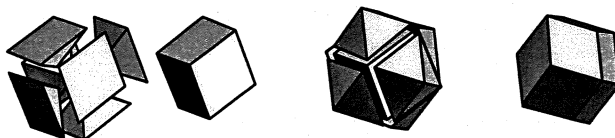
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Figure 17: Constructing the rhombic dodecahedron

The rhombic dodecahedron occurs in nature as a Garnet crystal. To see a rhombic dodecahedron online, visit http://mathworld.wolfram.com/RhombicDodecahedron.html.

5. Conclusion

In addition to identifying the rhombic dodecahedron as the shape in the dual to the tetrahedral-octahedral space filler, we also made other connections between packings. We have shown three ways to connect the cubic packing to the tetrahedral-octahedral packing.

The first connection was made (figure 3) by deforming the cube into a parallelepiped via a stretching process that extends to deform the cubic packing into a packing by parallelepipeds. Then by truncation, this becomes the tetrahedral-octahedral space filler (figure 4). The second connection was made in figure 7 where we used a packing of cubes each containing a tetrahedron to position the tetrahedra within the tetrahedral-octahedral packing. The third connection was made to visualize the dual, the rhombic dodecahedron using the 3-D checkerboard construction.

Just as duality can help us understand the symmetries of one shape by considering the symmetries of its dual, here, duality has enabled us to identify an unexpected packing from a given one. Not only did we identify this dual packing, but we found it contained only one type of polyhedron, the rhombic dodecahedron.

Bibliography

Grünbaum, B., Shephard, G. C., *Tilings and Patterns*, W. H. Freeman and Company, New York, 1987.

Knoll, E., From a Subdivided Tetrahedron to the Dodecahedron: Exploring Regular Colorings, *Bridges: Mathematical Connections between Art Music and Science*, Baltimore, Maryland, 2002.

Knoll, E., From the Circle to the Icosahedron, *Bridges: Mathematical Connections between Art Music and Science*, Winfield, Kansas, 2000.

Knoll, E., Decomposing Deltahedra, *International Society of the Arts, Mathematics and Architecture (ISAMA) conference*, Albany, New York, 2000.

Knoll, E., Developing a Procedure to Transfer Geometrical Constraints from the Plane into Space, *Journal for Geometry and Graphics*, Vol.2, no.2, 1998.

Serra, Michael, *Discovering Geometry: An Inductive Approach*, Key Curriculum Press, Emeryville, California, USA, 1997.