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Two Perspectives on Inversion

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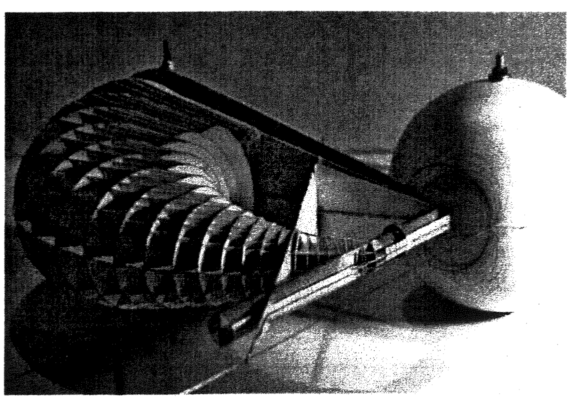
Abstract

When working with mathematical or algorithmic art, it is inevitable that someone else is using similar techniques. However, this does not mean that the results are the same. By chance I went to an exhibition of the sculptural work of the artist John Pickering, whose work revolves around a mathematical transformation called inversion. I have used inversion, both in the plane and space. This paper compares results and techniques.

Introduction

Artists have many tools to transform their ideas into something concrete, be it a sculpture, picture or building. This is also true of mathematical tools for transformation and is hardly surprising that many people seize on the same tools although the results may be different. I have been sculpting three dimensional surfaces using a technique which slices them. My Sliceforms were presented at the first Bridges Conference [1] and are a set of planes which slot into one another so that not only is a surface modelled, but it is one that is continually deformable. I have used many techniques including the mathematical transformation called inversion which is described below. I have also presented a paper which involved inversion at the 2002 Bridges Conference [2]. Robert Dixon has some artistic work using inversion [3] which inspired Robert Krawczyk in producing curving Spirolaterals [4]. Shortly before Xmas, I was looking for an exhibition to take a visitor to London and by chance I decided upon the last day of the Architectural Association's show of John Pickering's "The Inversion Principle". Although I know they have high quality exhibitions, I had no idea that the subject would be the same inversion that I had been using. I was even more astounded when I saw that he had been modelling with slices, but not quite in the way that I have been but even with the same objects (see figure 1). As a result we made contact and have compared ideas and techniques, which this paper summarises.



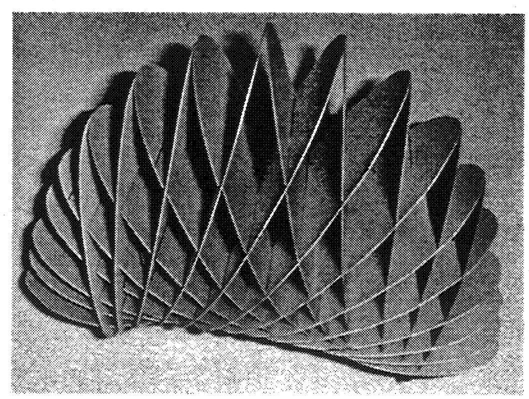


Figure 1, Inversion of a cylinder (left John Pickering, right John Sharp)

Inversion

Inversion using a circle is a well known transformation of points to points in the plane. It is normally defined (see figure 2) by saying that a point P is inverted with respect to a circle, radius r and centre O, to give a point P_1, so that O,P and P_1 are in the same line and $OP.OP_1 = r^2$. Another definition is that P_1 is obtained by taking the polar of P with respect to the circle and then intersecting the line OP with the polar line. The transformation can also take place in space using a sphere, so that whereas in the plane all points outside the circle are mapped inside (and vice versa), in space this is true for the sphere and figure 3 then becomes a representation of a cross section of a sphere.

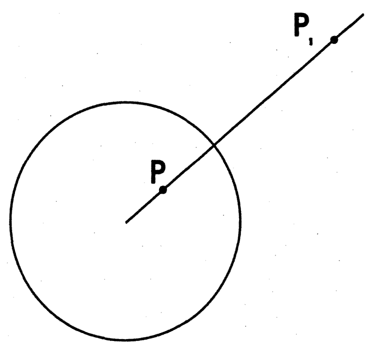


Figure 2, inversion of a point

The geometry of inversion in the plane is such that straight lines (except those through the centre of the circle which remain lines) are transformed into circles through the centre; circles become circles and other curves different curves [5]. In space planes become spheres (except those through the centre of the sphere which remain planes) and spheres become spheres; straight lines become circles, again except in the case where they go through the centre of the sphere. Knowing these properties helps in determining and calculating a particular inversion. The literature on plane inversion is huge, since it offers techniques for the easy solution of many problems involving circles; a good sample of such cases in given in [6]. However, inversion in space is not as common; even the book which has inspired both John Pickering and myself [7] only mentions inversion of the torus to give Dupin cyclides.

John Pickering and his technique

John Pickering was trained as an artist and worked as a sculptor carving in stone, for example at Birmingham Cathedral. However, he began to suffer from rheumatoid arthritis and periods of illness freed him up to work on his own and larger work became difficult. One of his brothers is an architect and another is a teacher of science and mathematics, so that it is not surprising that he was influenced by the ideas of geometry, topology and the work of people like Buckminster Fuller and Le Corbusier. He started out making polyhedral and topological models, but found the latter a dead end artistically. He came to inversion through looking at geometry books, particularly Lockwood [5] and Hilbert and Cohn Vossen [7]. He found inversion much better for working in space partly because of the way it produces sensuous curves, even from lines. Its purity of form produced a balance between his aesthetic and intellectual needs.

Thus he has no mathematical background, and would not consider himself a mathematician. However, having talked to him, I believe he is a natural geometer. I have found the same intuition and visual understanding as in the talks I have had with Brent Collins at previous Bridges Conferences in Kansas [9]. Whereas Brent's work is art which can be interpreted mathematically, John Pickering's work is art which has an architectural as well as mathematical dimension. Unlike Brent, because he is performing a precise mathematical transformation, he needs to perform calculations. He does not use a computer. All his calculations are with a pocket calculator. That is not to say that he does not understand the geometry and uses it to simplify what and where he needs to calculate, and in fact all he needs are the inversion formula and Pythagoras' theorem. As I have also used in my work, he knows that circles and lines are inverted into circles and once he has found the necessary circles he uses them and their intersections. In his sculptures he also make slight aesthetic "adjustments", so that if you are looking at them purely from a mathematical point of view, they do not make the sense you feel they ought to. His artistic intuition does not make you query the result, only when you probe mathematically does the pure logic break down.

Most of his work involves the inversion of a whole or part of cylinder or a cone. This results in a Dupin cyclide which David Wells [6] defines thus:

> All the spheres that touch three fixed spheres (each in an assigned manner, either externally or internally) form a continuous chain whose envelope is a Dupin cyclide. The centres of all the tangent spheres lie on a conic, so an alternative definition of a Dupin cyclide is the envelope of all spheres having their centres on a given conic and touching a given sphere. A third definition is as the envelope of spheres with their centres on a given sphere and cutting a given sphere orthogonally. A torus is a special case of a Dupin cyclide, and also, surprisingly, every Dupin Cyclide is the inverse of a torus.

John Pickering does not need to know this, he simply inverts the cylinder or cone and the centre of inversion determines the shape of the cyclide. He works with planes and had not even seen the connection between the sets of spheres and either the cylinder or cyclide until I pointed it out. As with my Sliceform work described below, he has inverted a circle or line and then treats the result as the border of a plane.

The following description of a current work has been influenced by my pointing out that inverting a disk results in the cap of a sphere and not a plane. He has thus added some sphere caps, which he makes from plaster. His sculptural experience with carving and working with large plaster forms means that these are perfectly smooth spherical surfaces, although quite small. The cyclides, by contrast, are open and his architectural bias results in a three dimensional grid, which is rigid. When talking about this grid, we have called the two types of planes walls and floors. The new work (incomplete) is shown in two views in figure 3.

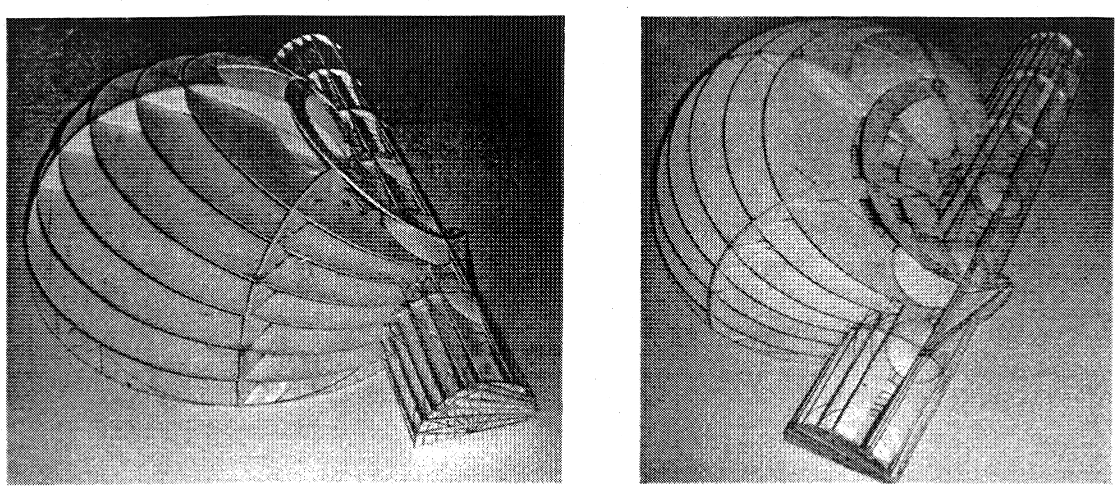


Figure 3, work in progress

Most Dupin cyclides you see in books like [6], [7] and [10] are like the other examples of John Pickering's work, that is like an asymmetrical torus, or come to a point like the right hand grid of my Sliceform in figure XX. Because the cylinder being inverted intersects the inverting sphere (not in the sculpture, but having a centre not far outside the cylinder) the cyclide looks almost complete. In the right hand view you can also see caps of spheres which appear like disks in the photograph.

Figure 6 shows the end of the cylinder and how intersecting planes have been chosen for inversion.

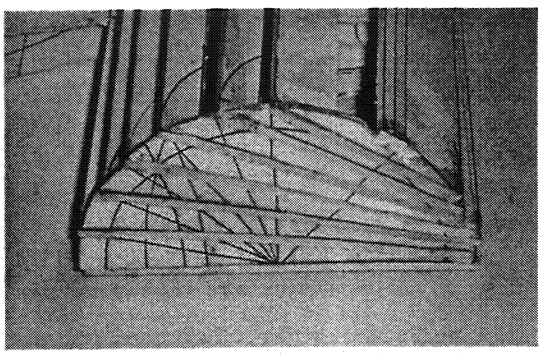


Figure 4 planes of the cylinder to be inverted

Further examples of John Pickering's work

The following examples of John Pickering's work show how he has given life to a variety of inverted forms. They need to be seen as sculptures in space to appreciate them fully.

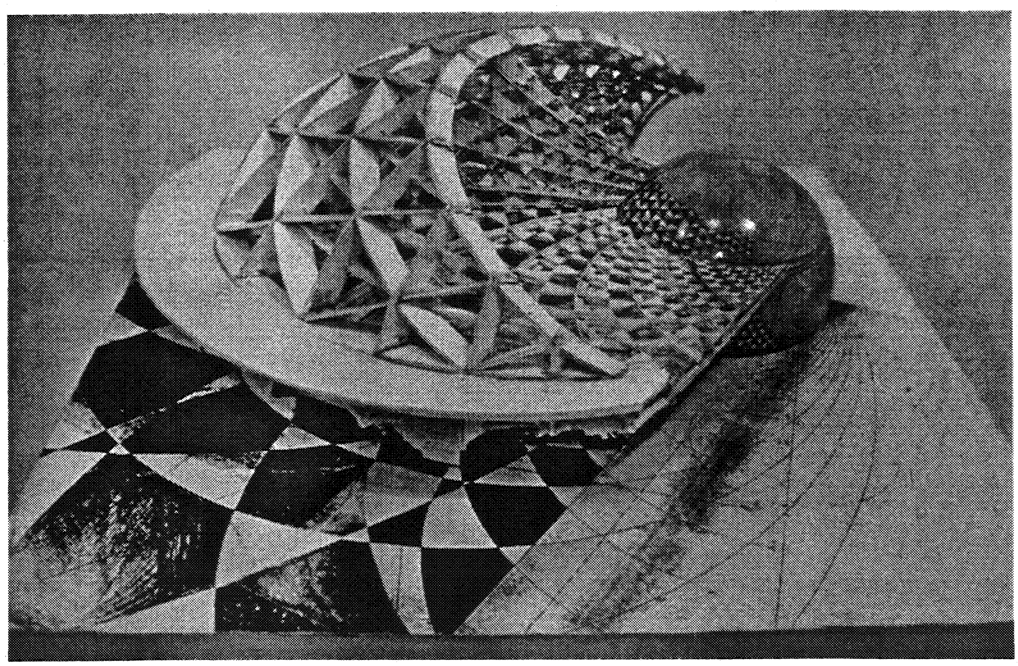


Figure 5, inversion of space filling polyhedra.

Figure 5 is an inversion of a set of tetrahedra and octahedra which together fill space. The planes of this space packing give rise to a triangular tessellation and the lines of these tessellations invert to circles. The inverting sphere is shown with selected lines of the inversion.

As well as inverting cylinders, he has also worked with circular cones, since these also can be seen as a set of circles centred along an axis. You need to walk around the sculpture rather than just look at the photograph in figure 6. Even so it may not make complete sense, since he has made modifications for artistic effect. He has two cones of which the smaller is not obvious. These cones are made as follows: consider a pair of lines and the two bisectors of the angles between the lines; if these bisectors are axes of rotation, then the pairs of lines rotated about each of these axes in turn gives rise to a cone with their vertex at the intersection of the lines. The large cone is obvious, but there is a smaller one in the centre. The inversion of these cones goes over infinity because the centre of inversion lies on the cones, so that the cyclide produced can only be partly made, as the plaster base with geodesic lines on it at the base. There is cylinder which has been inverted to a cyclide which is nearly a torus and he has also moved part of the inversion of the cones to the kidney shaped object in the centre so that it does not interfere with the other parts. This sculpture is an indication of how he creates objects and places them together in a harmonious composition.

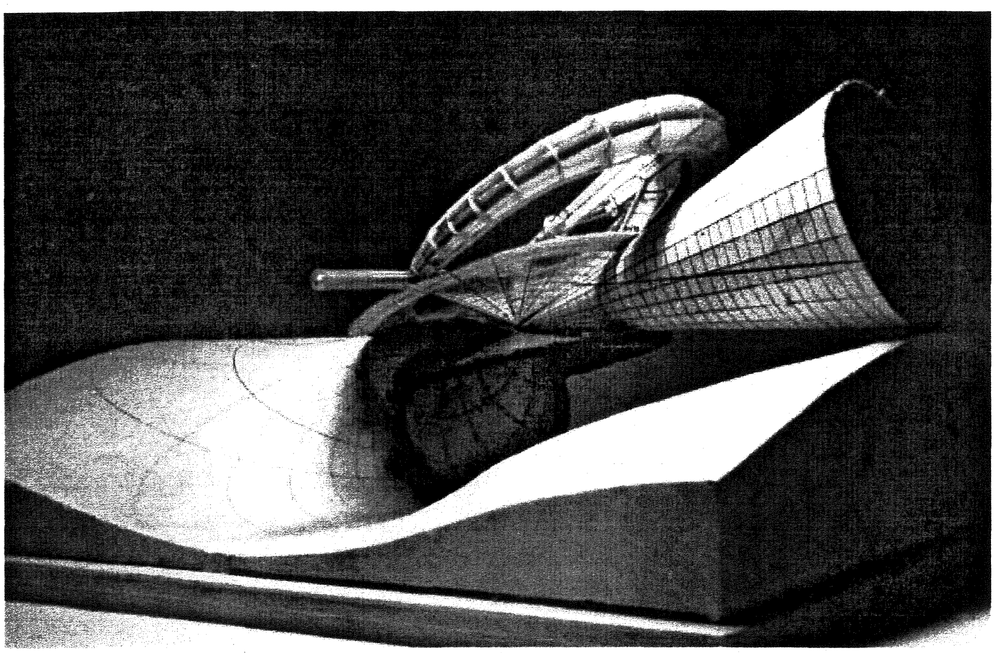


Figure 6 inversion with cones

This technique of enhancing the composition is also evident in figure 1, where you can just see some additional ellipses nearly intersecting the cyclide. Consider the cyclide as a building, made up of two sets of planes: "walls" and "floors". The "walls" are circles formed by inverting circles perpendicular to the axes of the original cylinder. The "floors" are planes formed by inverting plane sections of the cylinder, very much line the ones of the work in progress model shown in figure 4. Then if these floor planes intersect the cones which are defined by the centre of inversion and the wall circles, an ellipse results. Because the construction of the sculpture is only a part of the complete geometrical figure, which he sees in his head, the parts he chooses to create with his hands do not always look connected, but he manages to get the aesthetics just right. To me, this makes the sculptures grow on you as you look at them and understand them more.

In figure 7, he has thought more about the construction more architecturally. So instead of the planes of the "floors" being slanted as in figures 1 and 3, they have been constructed at chosen positions parallel to the base of the sculpture.

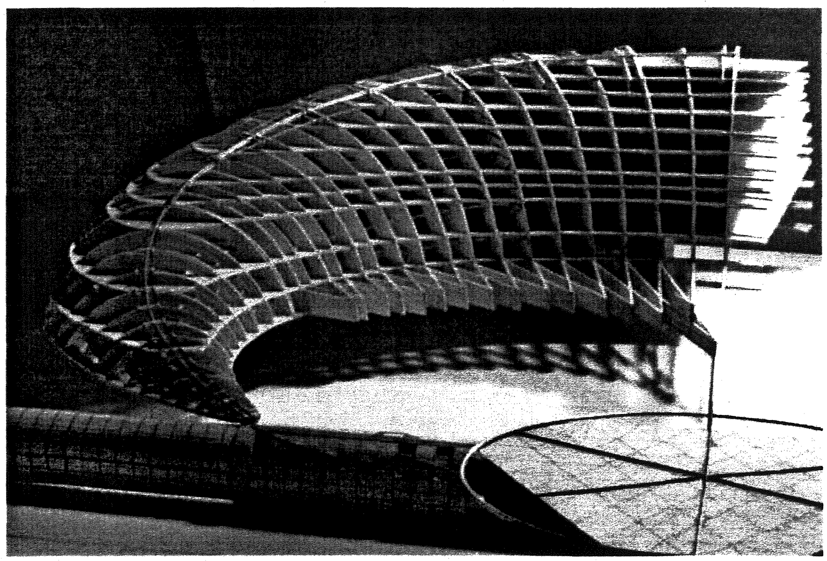


Figure 7, cyclide looked at more architecturally

The cylinder which is being inverted, at the front of figure 7, is shown in more detail in figure 8.

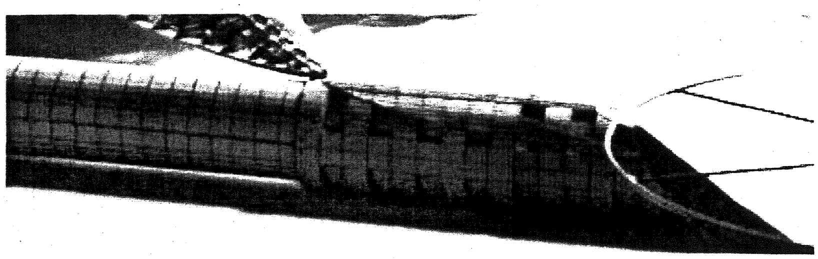


Figure 8, close up of cylinder

This has a set of "windows" designed into it. They position is decided by inverting the edges of the floors in the main structure back onto the cylinder. The floors were chosen on the main cyclide rather than constructed by inversion from the cylinder. He has explored the geometry of the shape of the curves and thinks they may be such that when developed (that is if the cylinder is cut open and folded flat) form an ellipse. Even if this is not the case, they are very close to being ellipses and so he constructed them by making ellipse templates.

My technique

My use of inversion is mainly in the plane and I use a computer to plot points or curves. I have only made three spatial models, since I just explored the technique as one of many. I used the property of circles always going to circles. I took three cases of surfaces being defined as circles (an elliptic cylinder and two inversions of an ellipsoid).

To get the complete surface of inversion of the elliptical cylinder, it is necessary to have an infinite number of circles, but even a few show the surface and the way it curves round in a ring. This surface is similar to a Dupin cyclide which is formed by inverting a torus. The left diagram of figure 9 shows a cross section of a cylinder where each line represents a circle. The right diagram shows how the cylinder represented by the two lines which is tangent to the circle is inverted in the circle on the left to give two circles which are tangent at the centre of the inverting circle. These two circles are the limiting circles of the section of the inverted solid which is show as a grid diagram at the right. Each line of the grid represents the inversion of a circle of the cylinder. The Sliceform model is shown in figure 1 (right). Further information is given in my forthcoming book on creating Sliceform models [8].

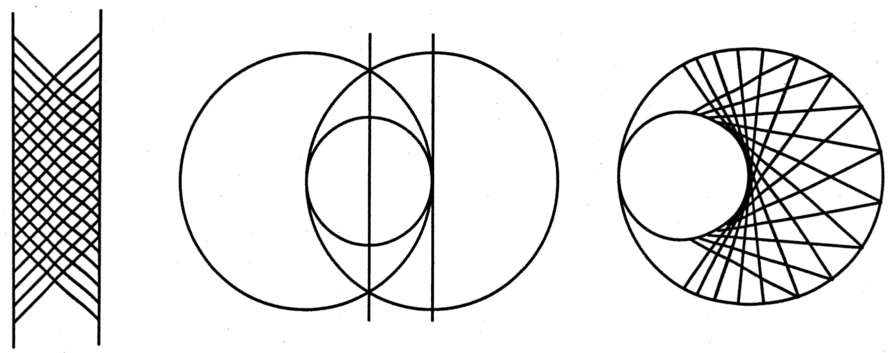


figure 9, grids for Sliceform inversion of cylinder

For the ellipsoid, the resultant surface depends on the position of the inverting circle. So in figure 10, the ellipsoid is seen in cross-section with each line of the grid representing a circular slice. Inverting with a circle along the major axis gives an egg shape; a magnified version of the inversion, together with a typical slice is shown at the right of the figure. Inverting with the centre of the circle along the minor axis, shows a totally different shape.

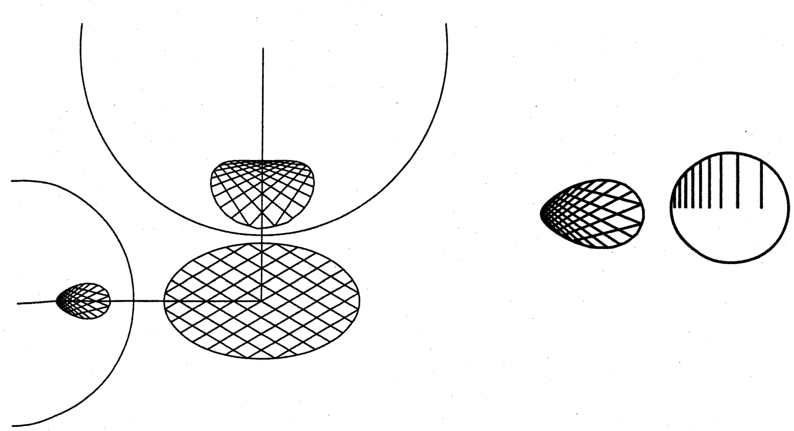
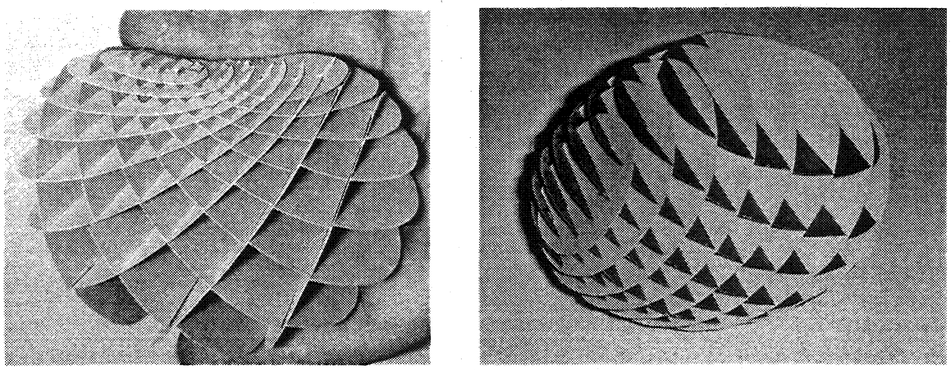


figure 10, grids for Sliceform inversion of ellipsoid

The two Sliceform models are shown in figure 11.



figures 11a and 11b Sliceform inversion of ellipsoids
figure 11a shows inversion with the centre of the circle along the minor axis
figure 11b shows egg shape when centre is along major axis

Conclusions

The two methods show here that inversion in space has lots of potential for sculpture and architecture. These forms do not seem to have been used before to our knowledge, despite many artists being influenced by geometrical models made at the end of the nineteenth century [10].

References

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11. More information about John Pickering can be found at `www.johnpickering.net`