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Mathematical Connections
in Art, Music, and Science

Play with Infinity

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Abstract

Various approaches to recursive generation of some Arabian-Andalusian Zellij and Muqarnas patterns, inspired by a visit to the Alhambra Palace in Granada.

1. A self-similar tiling suggested by a Zellij panel in the Alhambra Palace.

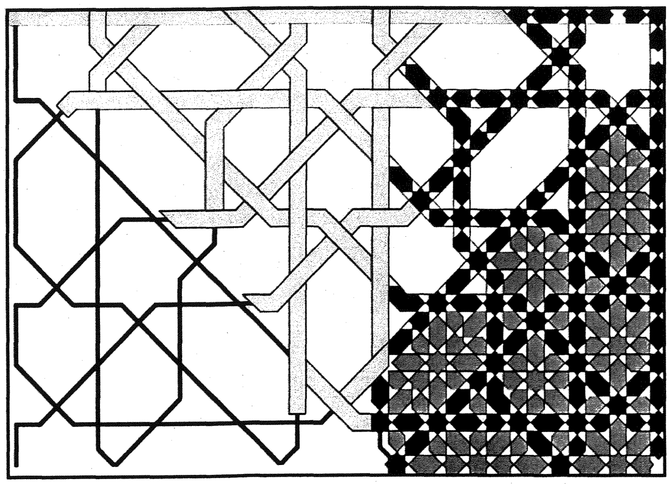
1.1. The Zellij panel. On a wall of the Comares entrance hall there is a very special Zellij panel (A Zellij is a mosaic made up of hand cut pieces of glazed ceramic, which are also called Zellij).
An almost identical panel exists in the Seville Alcazar and in the Cordoba mosque (both in Andalousia), but as far as I know no where else.
The pattern is centered on a 8-pointed star, with interwoven bands. Inspite of its complex look, it's formed from a very simple shape, subjected to recursive procedure that the visitor can follow mentally.
Usually periodics patterns suggest the infinite by the possible repetition beyond the visible window.
In this case, the infinite can be sensed everywere in the framework.

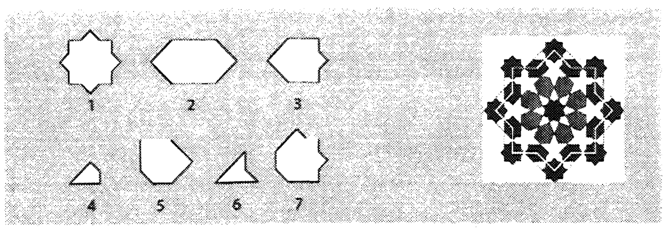
1.2. The method and its extensis. The initial pattern, drawn whith a simple line, is first of all embellished with interlaced bands. These interlaced bands are broken down and divised into smaller Zellijs. The inside surface of each initial tile is subject to the same process [fig. 1].
This leads to a new simple pattern, made up of smallers standard pieces, in the ratio of $(\sqrt{2}-1)/2$.
These lines can be interlaced again generating the pattern of the Zellij panel (in the Cordoba version, since the Alhambra one presents a slight variation [fig. 3]).
Can this process be repeated infinitely ?
We can skip the interlacing step and go directly from a simple design to the next. It will be less adventurous this time, but this is how we are going to define the transformation rules of every piece.

1.3. Transformation rules. [fig. 4] shows the transformation rules for each piece.
The basic pattern included 7 pieces. Piece n° 2 leads to 2 possibles interpretations, symmetrical or not, and bears a new form (n° 8). Pieces 5, 6 and 7 also have 2 interpretations and generate new forms (9, 10 and 11). Minor variations may also occur without structural consequences (see the center of tile n° 7).
In order to continue this process we have to define the transformation rules for the new pieces (8, 9, 10 and 11). The solutions of [fig. 4] does not need any other piece beside the ones already used.

Therefore the process can go on, infinitely...

Is it possible to imagine a similar process for the fascinating Muqarnas domes ?





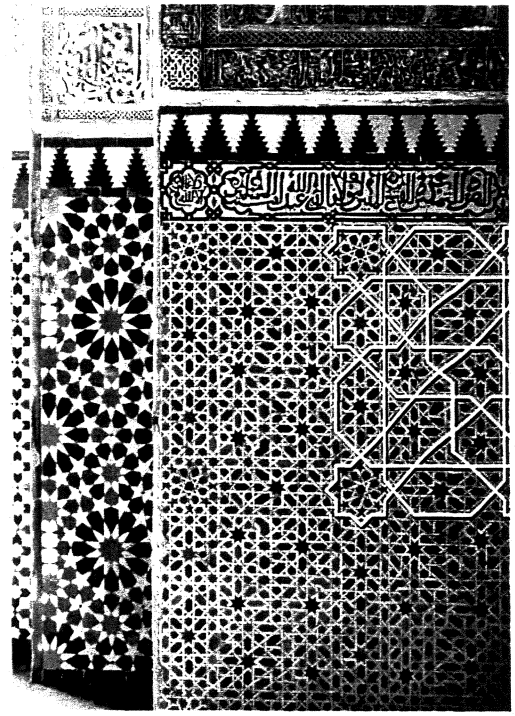


Figure 1: *Simple lines, intrelaced bands, decomposition.* **Figure 2**: *The Alhambra panel.* **Figure 3**: *The 7 tiles of the initial pattern. On the right hand side, the Alhambra variation, not interpreted here.*

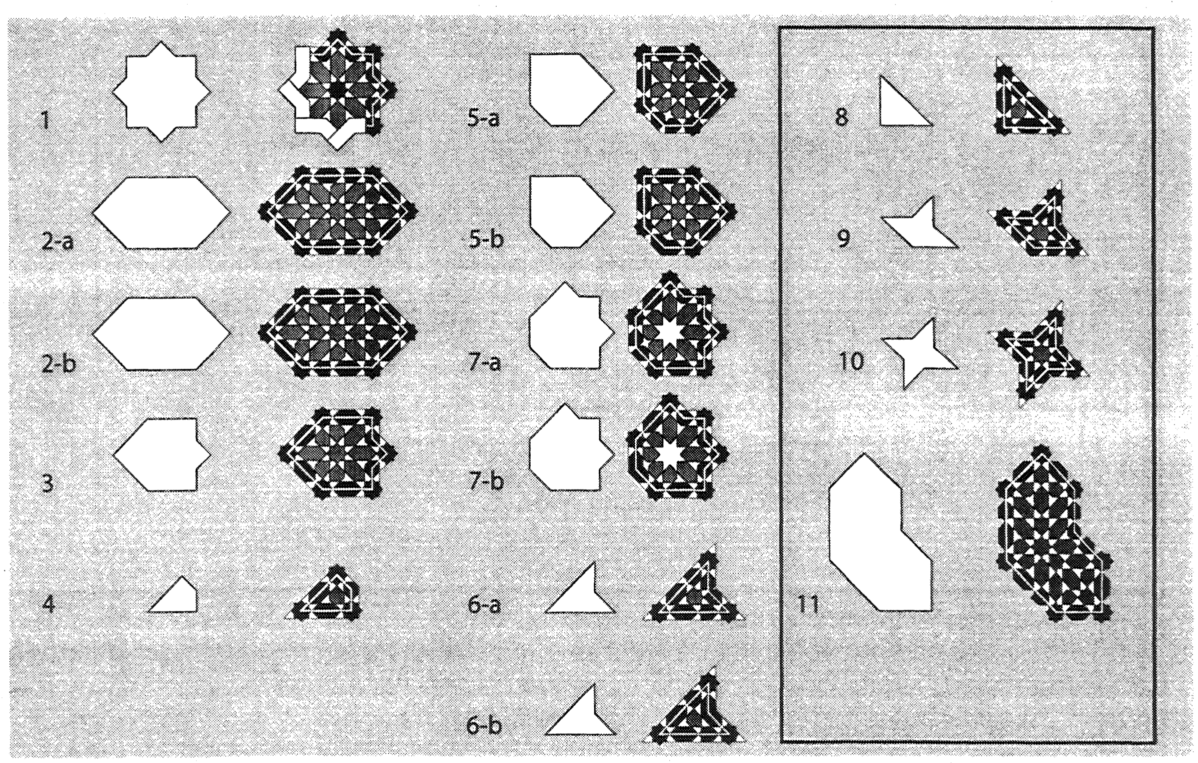


Figure 4: *Transformation rules for the 7 initial tiles and for the 4 new ones.*

2. A recursive generation of muqarnas domes, in connection with zellij patterns

2.1. Principles and definitions. Muqarnas, one of the major elements of traditional architecture decoration, sometimes called "stalactites", are used for friezes, corbels, arches or domes. Here, we consider the modular technique used in Morocco and in Andalousia, consisting in the assembly of small wooden units. Muqarnas are also observed in works of molded and engraved plaster.
A complete description of the geometry of such an assembly is given by a flat projection using a coding for the different units [fig. 14].
These drawings are based on an isometric network similar to the "octogonal tiling" ([3], [4] and [fig. 8]), witch is a non periodic tiling using squares and "diamonds" (45-degrees rhumboi). Not every Muqarnas pattern can be strictly reduced to this tiling, there are a lot of variations. However, each part of an octogonal tiling can lead to an arrangement of Muqarnas. That is the principle we shall apply here, in order to get a recursive process for generating domes.
Because the octogonal tiling can also define the basis for Zellij patterns, we obtain in the same way a correlation between the 3D-decoration of Muqarnas and the 2-D decoration of Zellij.
We shall limit this study to the simplest non-trivial Muqarnas structures, which employ only the 4 main shapes of Muqarnas units [fig. 14]. The generated structures can lead to variations by substitution, with the use of more differents shapes [fig. 14].

2.3. The recursive process. Let us use the inflated construction of the octogonal tiling. Each step of this construction, begining here with 8 diamonds (step 1, 2, 3... see [fig. 8]) can be interpreted as a projection of a Muqarnas dome. The number of possible interpretations increases from one step to the next.
These interpretations involve orienting the vertices of the network according to the orientation of the Muqarnas edges [fig. 14 and 15]. We, then, could use a conventional notation for drawing the map of the dome.
On the other hand, each step of the process leads to a Zellij pattern by using a simple decoration of the squares and diamonds [fig. 7, 9, 10]. These patterns easely produce variations by locally substituting some motifs from the traditional repertory of Zellij [fig. 13]. Nevertheless, these recursive generation of Zellij have nothing in commun with the one of the first part of this paper.

2.4. Number of stories in such a dome. From one such given octogonal tiling we can define domes with different numbers of stories. But this number is necessarily even if we only use the 4 main pieces of muqarnas. See examples for step 3 in [fig. 15].

2.5. Another method. To generate models of periodic or non periodic tilings and associated Muqarnas structures, we can also use directly squares and diamonds. Here is my set of 4 double side pieces. On one side is the Zellij-style decoration, on the other side is the coding of the 4 muqarnas pieces. The linking is constrained to force a coherent fitting of the edges, automatically leading to a map of a muqarnas structure.

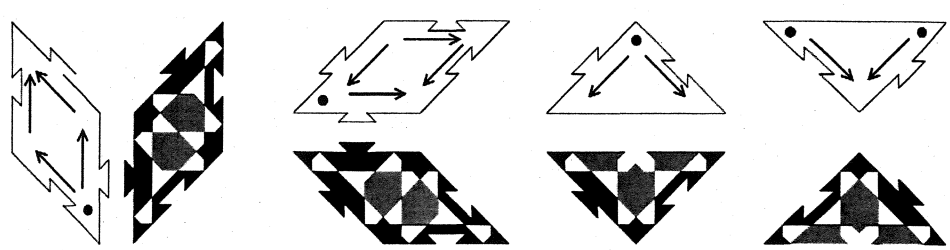


Figure 5: *My set of 4 double-side pieces. One side for Zellij, the other side for Muqarnas (The arrows gives the edge's orientation of the associated Muqarnas units)*

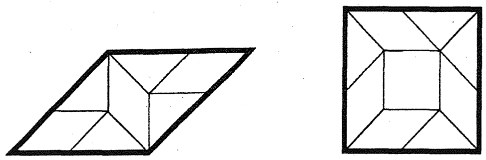


Figure 6: *Rules of inflation for square sand diamond.*

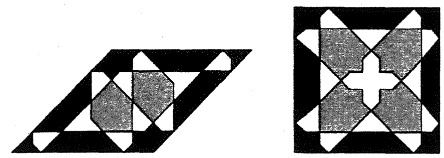


Figure 7: *A zellij-style decoration.*

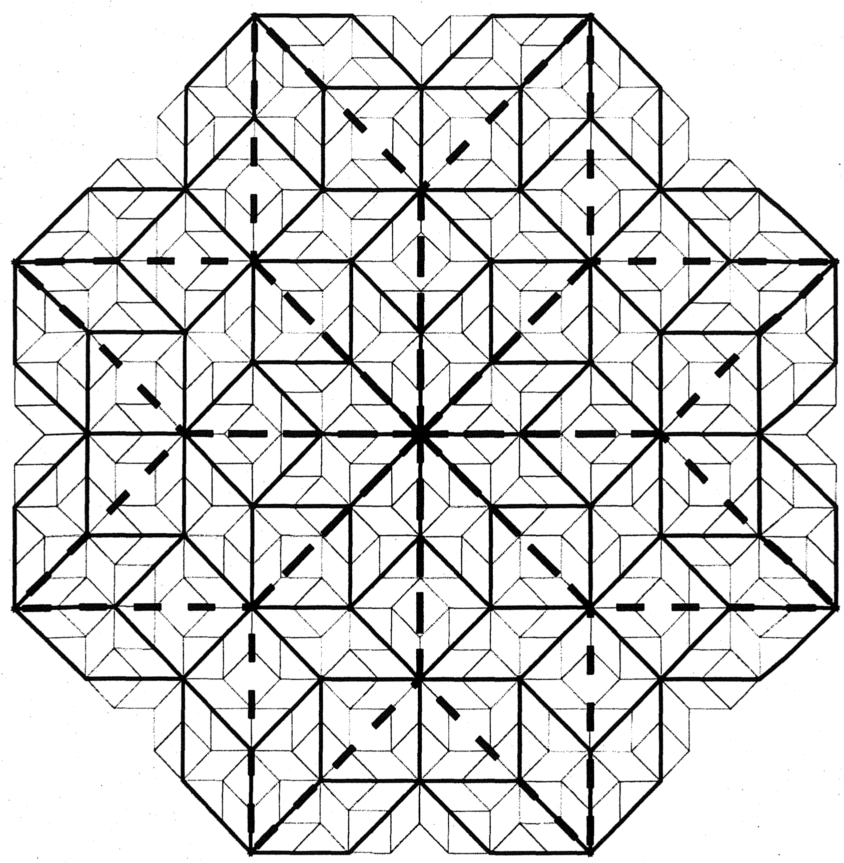


Figure 8: *Inflated construction of the octogonal tiling : step 1 (in dot lines), 2 (bold lines) and 3.*

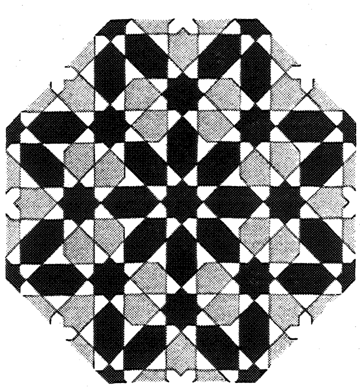


Figure 9: *step 1, zellij decoration.*

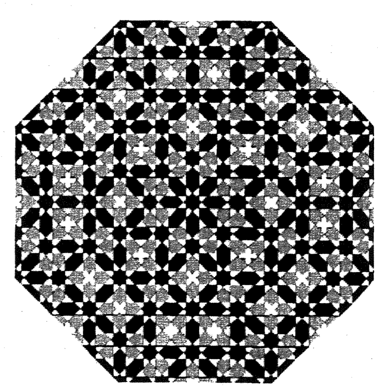


Figure 10: *Step 2, zellij decoration .*

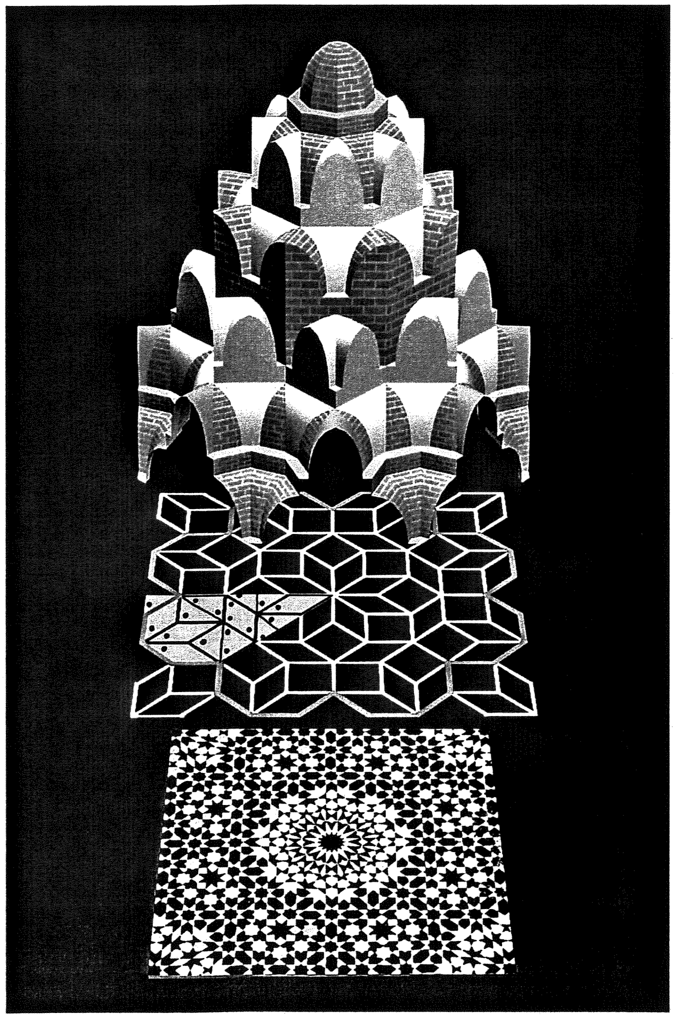
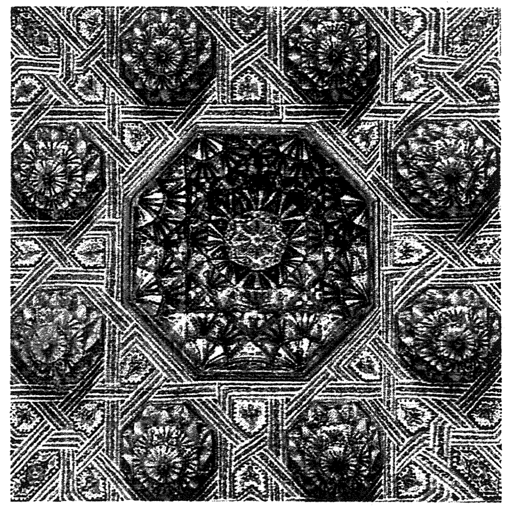
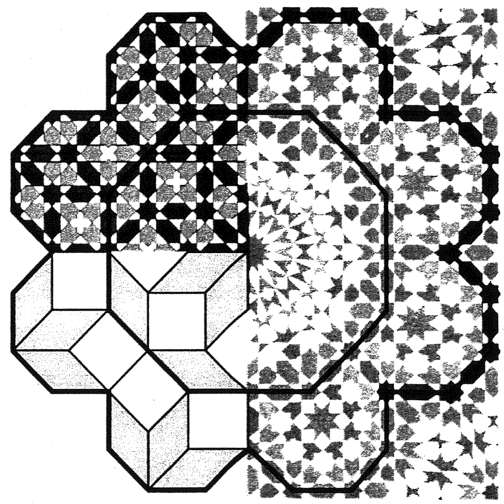




Figure 11: *The step 2-octogonal tiling (in the middle), projected onto a dome and used as the basis for a zellij.* **Figure 12:** *A small dome in Telouet (Morroco), built according to this plan.* **Figure 13:** *The 16-pointed star implantation.*

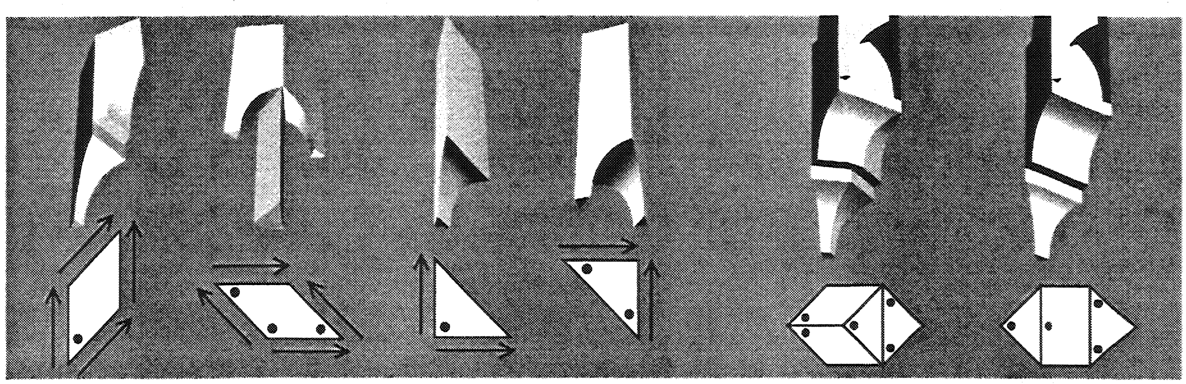


Figure 14: *The 4 main units of muqarnas and their standard projective representation. The dots indicates the "feet" of the units (the arrows indicate the edge's orientation). On the right hand side, an arrangement and its variation with a new unit, projected onto a rectangle.*

3. Application

3.1. The projet. Within the framework of the project of an exibition scheduled for 2004 in Paris, we have design an octogonal stand which will be used to shelter a display of Arabian-Andalusian traditional architecture.
We want all the volume proportions, and the decorative elements to be logically interconnected. This is possible by the use of diamond-square octagonal tiling.

3.2. Description. We started from the step 3 of the octogonal tiling inflated construction [fig. 8]. Because of some space constraints we have chosen a 6-storied dome [fig. 15, 16, 17]. On the floor, an original pattern emphasize the vertices of the octogonal tiling an suggest a Zellij extension. Walls are embellished with the famous traditional simple motif well known as "shoulder and step", whose lines can fit together with the Muqarnas profile in order to produce a continuous connection with the dome.

References

[1] J. M. Castera, *Arabesques,* ACR, Paris, 1996 (fr) and 1999 (en).

[2] J. M. Castera, H. Jolis, *Géométrie douce,* Atelier $6^{1/2}$, Paris 1992.

[3] D. Gratias, A. Katz, *Tiling and Quasicrystals, in Lectures on Quasicrystals,* Les éditions de physique, Les Ulis, 1994.

[4] B. Grünbaum, G. C. Shephard, *Tiling and Patterns,* W.H. Freeman, San Francisco, 1987.

[5] A. Paccard, *Le Maroc et l'artisanat traditionnel islamique dans l'architecture,* éditions Atelier 74, Saint-Jorioz, France, 1980.

[6] M. Sakkal, *An introduction to Muqarnas domes Geometry, in Stuctural Topology #14,* Université du Quebec à Montréal, 1988.

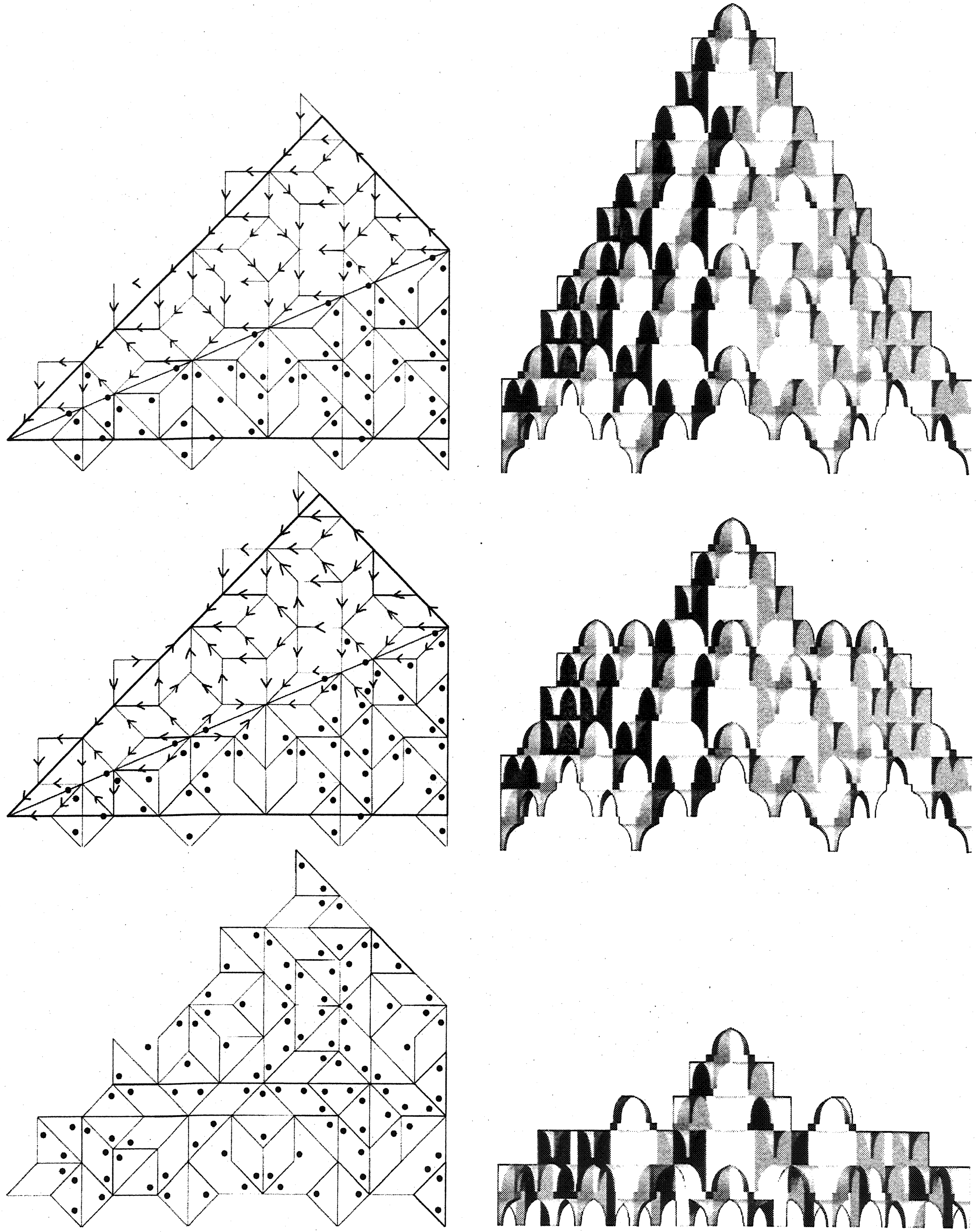


Figure 15: *Differents interpretations of the same octogonal tiling (step3 in [fig. 8]), as a map of Muqarnas dome. Top : with the maximum number of stories (14). Middle : a 10-storied dome. Bottom : a 6-storied dome.*

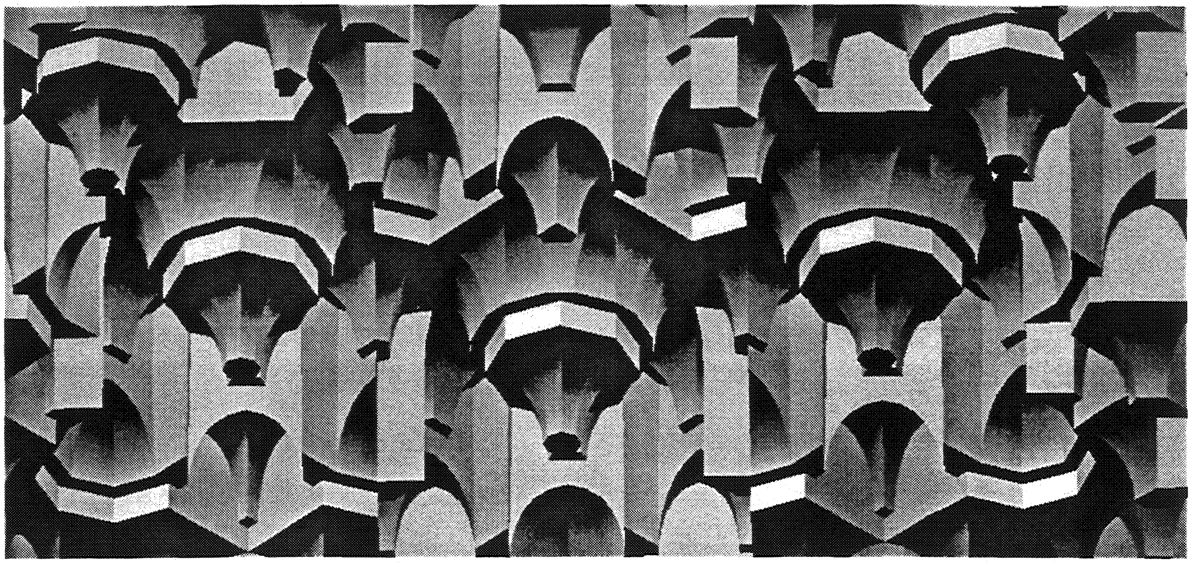


Figure 16: *An inside view.*

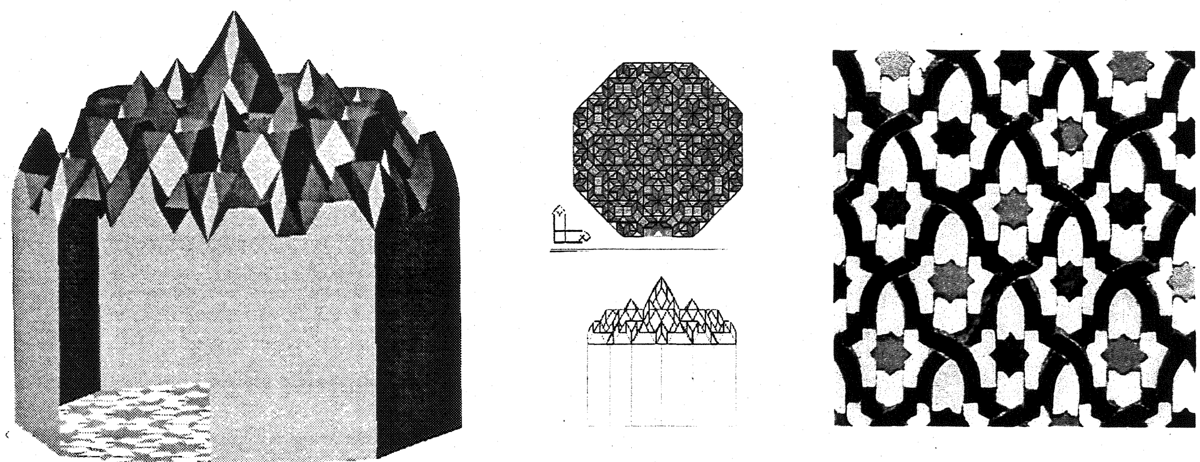


Figure 17: *A simple representation, and the motif for the walls*

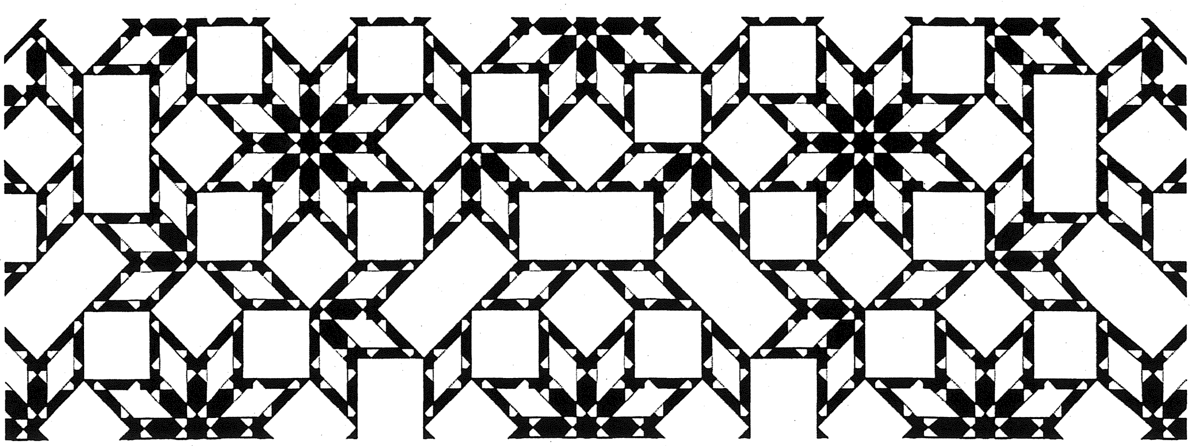


Figure 18: *An original pattern for the floor.*