The Mathematics of Color-Reversing Decorative Friezes: Façades of Pirgí, Greece

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Abstract

The arts and architecture of the Greek island of Chios in the Eastern Agean reveal that it has long been exposed to a wide range of political and cultural influences. Especially interesting is the village of Pirgí, where the façades of the buildings are ornamented with plasterwork friezes. While analyzing the mathematical structure underlying the frieze designs, we discovered that the frieze artists were intuitively obeying a unique set of color-reversing rules. This new art form is explained and an innovative mathematical structure is introduced as a tool for analysis.

The Project

Pirgí is the most visually striking of the walled medieval villages on the island of Chios, Greece, in the eastern Mediterranean, due to its unique ornamental plasterwork friezes, called *xistá*. The picturesque façades of hundreds of buildings are completely covered with these gray and white decorative friezes. Circles, squares, triangles, and rhomboids are used to create a lively geometry, ranging from the straightforward to the complex, which gives each house its own special face to display to the world.

Dating back to the Genoese occupation of Chios (thirteenth to sixteenth centuries), the technique of *xistá* is remarkably similar to the *sgraffiti* found in Italy from the Renaissance onwards [1]; however, current *xistá* designs have evolved beyond the simple *sgraffiti* form, as a result of the meeting of two cultural traditions. In 1922, at the time of the Greek-Turkish War, when Asia Minor Greeks came to Chios as refugees, they brought with them their rich decorative traditions of elegant rug weaving and colorful embroidery design [2]. This influx had a profound impact on the striking decorative style of *xistá*, which now recalls the custom of hanging decorative woven dowry cloths from windows on feast days.

Our team, consisting of an anthropologist, a mathematician, and an architect, analyzed these splendid friezes. Analysis revealed that the Pirgí artists were intuitively obeying a unique set of color-reversing rules. The goal of our project was to explain this powerful art form, and further, to describe the essential mathematical structure underlying these color-reversing friezes.

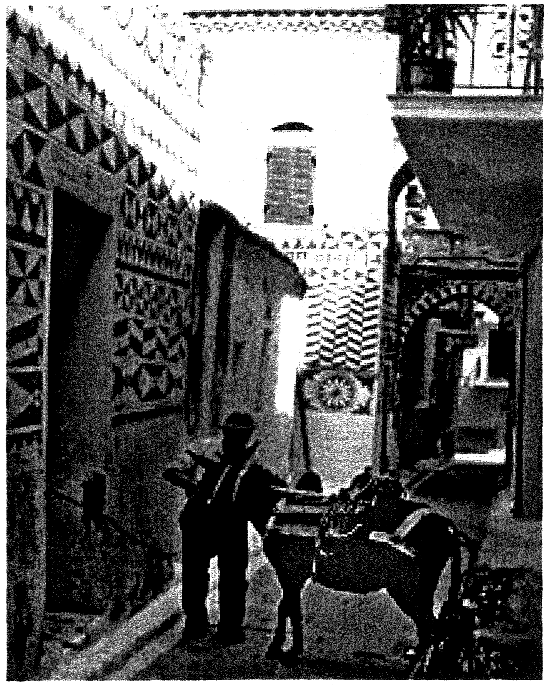
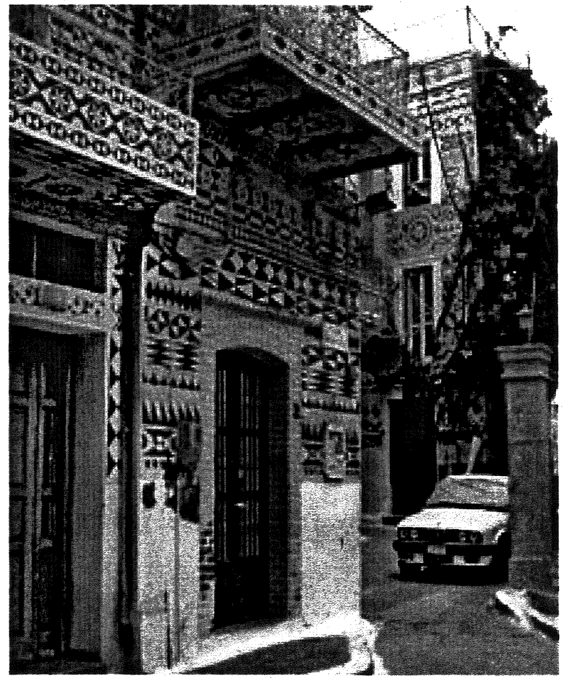
 

Figure 1: *Typical scenes of Pirgí.*

The Village Of Pirgí

The site of our study is one of the remaining medieval villages on Chios, Greece. A Genoese defensive tower, now in ruins, stands in the middle of Pirgí. This tower was the highest building in the village and would have been the final refuge of the inhabitants if the village were invaded. Adjacent to the tower is the main square, the *platía*. The few streets linking the *platía* to the gates are twisting and narrow. Even smaller alleyways lead off these streets, often ending in cul-de-sacs. The curving of the streets and alleys, which seldom intersect at right angles, increases the labyrinthine effect of the street plan. The roofs of the houses are all of the same height, vaulted, and connected by bridging arches, thus allowing movement across the village at roof level for escape to the tower.

Xistá

The façades of most of the houses of Pirgí, as well as the church and commercial buildings, are covered by *xistá*. The dark gray and white patterns consist of layers of horizontal strips 15 to 30 cm wide, each with its own design, separated by plain, narrow white strips 3 to 5 cm wide.

The name of this ornamentation means scratching or scraping and refers to the method by which it is produced. Two different coats are applied to create the effect: first, a dark rough coat; then a light smooth coat. The coats are applied in bands across the façade from the top down. Only a small portion is laid on at one time as one man can work only three square meters per day. Sand (originally dark sand from the nearby beach at Emborió), dark cement, lime, and water are mixed

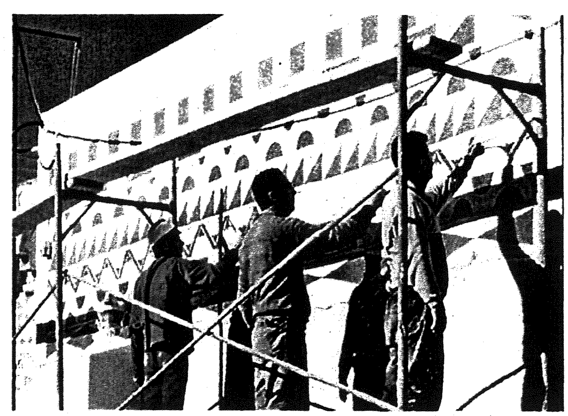


Figure 2: *Xistá craftsmen.*

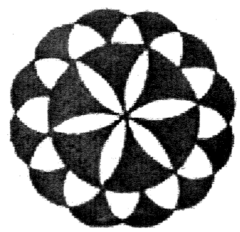
together and applied. The dark mortar is worked with a trowel until the lime rises to the surface, creating a lighter-colored top layer. After drying a few hours, this layer is then covered with a light-colored coat. At this point the local craftsman inscribes the design. After using a straightedge or a stretched thread to mark the horizontal row, geometrical patterns—made by using a compass and straight edge—are conceived and drawn. The craftsman creates the design, improvising, but basing the pattern on past experience. After the leading artist lays down the design, other workmen follow and scratch the white surface with a fork down to the dark gray to reveal the pattern. If the design artist has made the sails of a boat by incising triangles of white, in a later design he may choose to indicate the shadows of the sails by reversing the gray and white. Among the possible designs are mill sails, reels for thread, diamonds or sawteeth, and their shadows.

Symmetry in Decorative Art

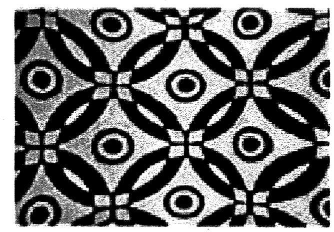
There are three distinct categories of symmetry: point symmetry, frieze symmetry, and space-filling symmetry. The psychological appeal of symmetry arises from the intuitive recognition that there is some underlying set of rules being obeyed.

The Key to Understanding Symmetry: Genotype Versus Phenotype

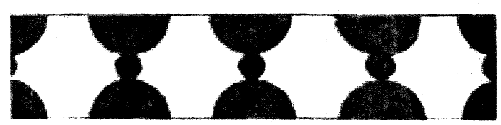
A useful analogy is to consider the visible design of an object to be its phenotype, and the mathematical group structure underlying the design to be its genotype. For example, in Pirgí there are several illustrations of Type 1 friezes, many differing from one another in appearance; that is, they have different phenotypes. But all are in the same family, sharing the same set of actions under which they appear invariant; they have the same genotype, that is, the same group type.



Point Symmetry



Space-Filling Symmetry



Frieze Symmetry

Figure 3: *Examples of categories of symmetry.*

The Group Theory of Friezes

A *frieze* is a pattern that repeats regularly in just one given direction, extending infinitely to the right and left, at least in the imagination. If an entire frieze were moved to the right or left one unit, it would appear exactly as before. In addition, there may be certain ways that a frieze can be rotated or reflected so that the result is indistinguishable from the original. The reason for using mathematical group theory as a tool for understanding friezes is to provide an organizational structure for the collection of all friezes. Each frieze is associated with a group, and the elements of the associated group are the actions that leave the frieze image invariant. The group binary operation between two actions, a and b (designated ab), is the first action, a, followed by the second action, b. Since actions a and b leave the image invariant, so does action ab. There are five basic actions: translation (r), vertical mirror (m_v) horizontal mirror (m_h), half turns (1/2), and glide reflections (g).

In order to be a mathematical group four properties must be satisfied: presence of an identity element, existence of an inverse for each element, associativity, and closure. The theory behind the operations and a formal definition of these four properties can be found in many sources; particularly readable introductions include Farmer [3], Hargittai [4], Washburn and Crowe [5], and Weyl [6]. The first three of the four properties are automatically satisfied for all friezes. The option of doing nothing at all obviously leaves the appearance of the frieze unchanged and is the identity element for the group. Each action has an inverse. In addition, (ab)c is identical to a(bc), so assciativity holds. Thus the only remaining requirement for a collection of elements to be a group is closure, that is, every binary operation of two elements in the collection must give a result that is also in the collection. Certain collections of the five basic actions have this property of closure, while others do not. Collections that do have this property result in friezes and they also form mathematical groups. As such, there exists an important connection between mathematics and friezes.

The Seven Standard Frieze Groups

There are seven different frieze groups, classified according to how much symmetry the particular frieze possesses. Each frieze is invariant under a translation (r)–that is what makes it a frieze.

Mirrors

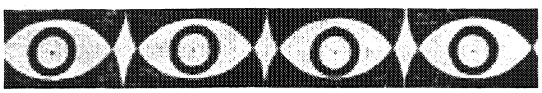
Type 1:	XXXXX	***Horizontal & vertical mirrors & half turns*** $\{m_h, m_v, 1/2\}$
Type 2:	/\/\/\/\/	***Vertical mirrors & half turns & glide reflections*** $\{m_v, 1/2, g\}$
Type 3:	AAAAA	***Vertical mirrors & no half turns*** $\{m_v\}$
Type 4:	EEEEE	***Horizontal mirrors & no half turns*** $\{m_h\}$

No Mirrors

Type 5:	SSSSS	***Half turns*** $\{1/2\}$
Type 6:	qdqdqd	***Glide reflections & no half turns*** $\{g\}$
Type 7:	RRRRR	***Translations only***

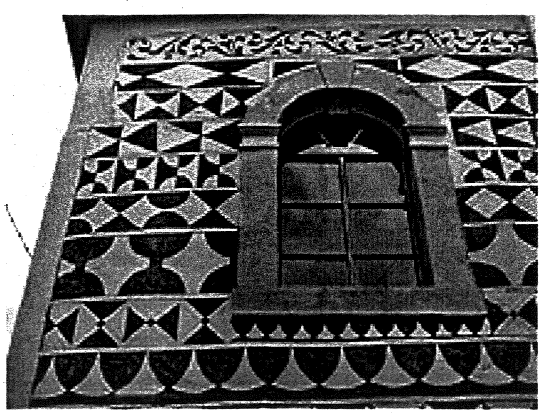
The Friezes of Pirgí

It is interesting that all seven of the standard types of friezes appear in Pirgí.

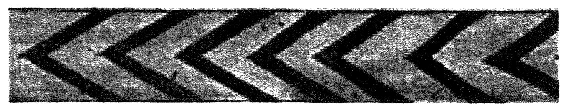


Type 1 *Horizontal and vertical mirrors, half turns* $\{m_h, m_v, 1/2\}$

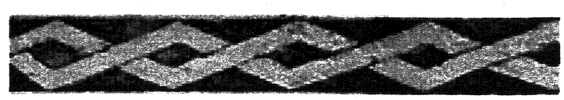
Type 2 *Vertical mirrors, half turns, glide reflections* $\{m_v, 1/2, g\}$



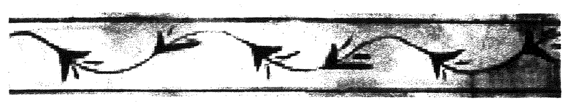
Type 3 *Only vertical mirrors (bottom frieze)* $\{m_v\}$



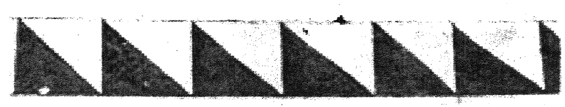
Type 4 *Only horizontal mirrors (rare)* $\{m_h\}$



Type 5 *Half turns, no mirrors* $\{1/2\}$
(Note: under/over cross)



Type 6 *Glide reflections* $\{g\}$ *and no half turns*



Type 7 *Translation only, marching right triangles*

Figure 4: *Types of frieze symmetry.*

These seven frieze types occur in varying degrees of frequency in Pirgí. We separated the friezes into two categories: friezes on the five remaining houses that had retained their older frieze patterns (Old Friezes), and friezes on a selection of thirteen of the approximately six hundred other houses (New Friezes). A total of 417 friezes were analyzed.

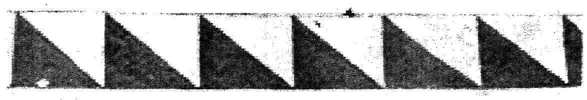
Figure 5: *Relative occurences of the seven standard frieze types.*

	Type 1	Type 2	Type 3	Type 4	Type 5	Type 6	Type 7
Old	41	9	23	8	14	7	6
friezes	38.0%	8.3%	21.3%	7.4%	13.0%	6.5%	5.6%
New	129	24	57	28	30	30	11
friezes	41.7%	7.8%	18.4%	9.1%	9.7%	9.7%	3.6%
Total	170	33	80	36	44	37	17
	40.8%	7.9%	19.2%	8.6%	10.6%	8.9%	4.1%

New Approach: Color-reversing Friezes

The seven-group analysis presented above provides a good start to understanding the friezes of Pirgí; however, during our analysis, a strong sense emerged that the Pirgí friezes have more subtlety than can be described using only the seven-group analysis. A good example is the frieze below.

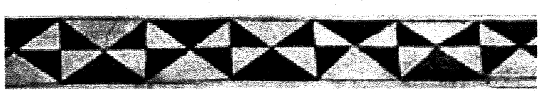
Figure 6: *The marching-right-triangle frieze.*



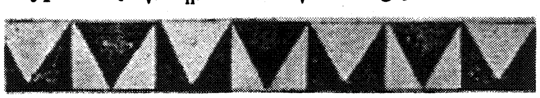
Since this frieze possesses no mirror images, no half-turns, and no glide reflections, it is Type 7 (no symmetry at all). Still one feels that there is some visual appeal that has not yet been taken into account. We are not the first to feel that some situations call for more than the standard seven group analysis can provide. Knight [7] has studied some interesting ambiguities and curiosities of frieze typology. As a result of this and other similar friezes, we knew there was more of interest to be found. We undertook to discover the particular rules of this new art form that the artists of Pirgí were intuitively obeying. We therefore introduced a type of group analysis to incorporate the possibility of color-reversal. When color-reversal is included as a distinguishing component, then there are seventeen new group types to add to the original seven group types. These patterns with two-color (black and white) symmetry were first classified by Woods [8]. This classification is shown in Grünbaum and Shephard [9].

Friezes with the Color-reversing Action r′

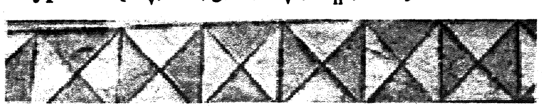
The simplest color-reversing action is to repeat the pattern one unit to the right, but with the colors reversed. Five examples of such friezes are illustrated below:



Type 1′ $\{m_v, m_h, 1/2, r', m_v', 1/2', g'\}$

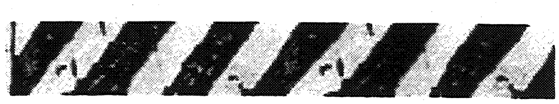


Type 2′ $\{m_v, 1/2, g, r', m_v', m_h', 1/2'\}$



Type 3′ $\{m_v, r', m_v'\}$

Type 4′ $\{m_h, r', g'\}$



Type 5′ $\{1/2, r', 1/2'\}$

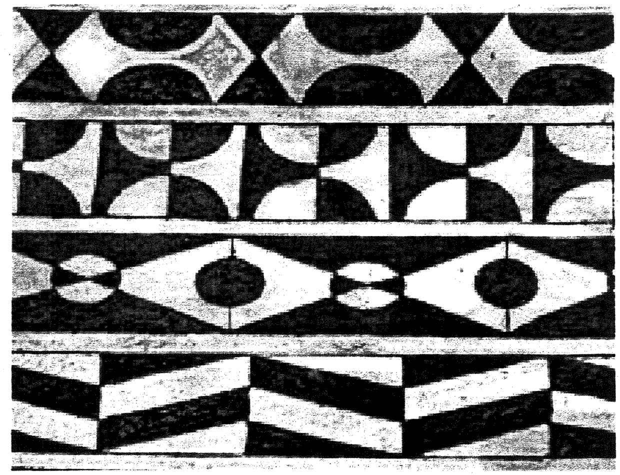
Figure 7: *Color-reversing friezes.*

The primes on any of the actions indicate color-reversal. For example, m_v' is a vertical mirror image where the design on one side of the mirror is reflected **and** color-reversed on the other side. Type 6′ $\{g, m_v', 1/2'\}$ does not appear in Pirgi′.

In addition to the friezes listed above with the r′ action (translation with color-reversing) there exist exactly six other color-reversing friezes having more than one action. These are illustrated below. For example, in Type 7′, while there are no places with normal (non-coloring-reversing) vertical mirrors, there

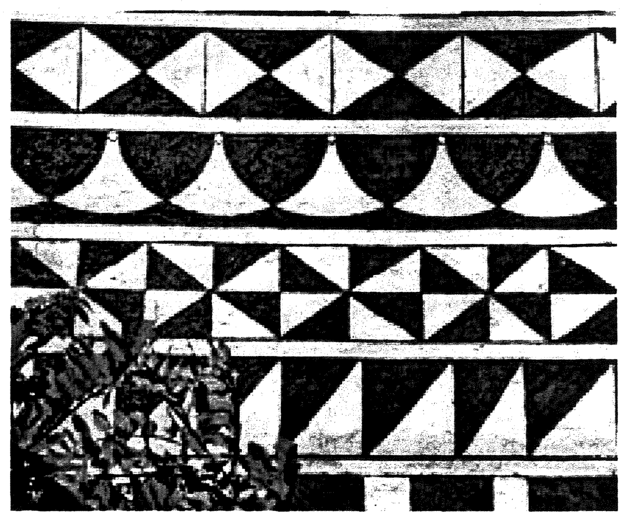
Friezes (from top to bottom)

Type 1 Non color reversing
Type 7′ $\{m_h, m_v', 1/2'\}$
Type 1 Non color reversing
Type 10′ $\{g, m_v', 1/2'\}$



Friezes (from top to bottom)

Type 1 Non color reversing
Type 3 Non color reversing
Type 8′ $\{1/2', m_v', m_h'\}$
Type 4″ $\{1/2'\}$ (see next section)



Friezes (from top to bottom)

Type 9′ $\{1/2, m_v', g'\}$
Type 2 Non color reversing
Type 4′ $\{m_h', r', g'\}$
Type 11′ $\{m_v, m_h', 1/2'\}$
Type 12′ thin curved frieze $\{m_v, 1/2', g'\}$

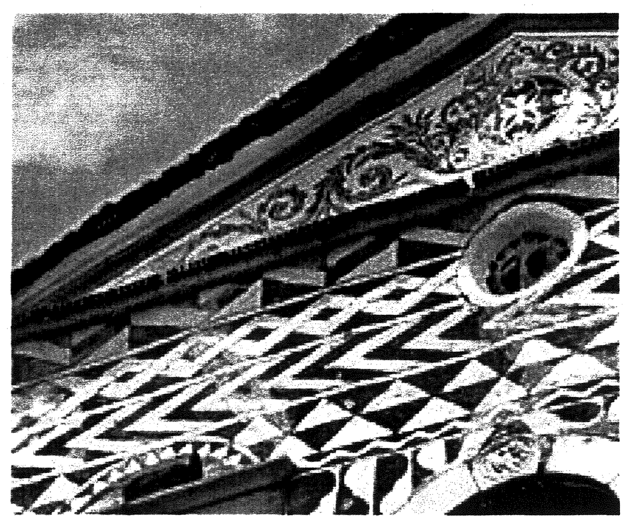


Figure 8: *Examples of friezes from Pírgi.*

are places where if a vertical reflection **and** color reversal (m_V') are performed, the frieze image appears identical. There are also places where if a half-turn **and** color-reversal (1/2′) are performed, the appearance of the frieze is invariant. Notice that some of the friezes in Figure 8 are of the usual non-color-reversal type discussed earlier and so are not relevant to this section.

Three Categories of Friezes

When color-reversing actions are permitted, there can be exactly twenty-four frieze groups. We can collect all of these twenty-four frieze types into three categories. There are the seven standard non-color-reversing frieze groups, "the Magnificent Seven." In addition there are the twelve frieze groups in which color reversal forms the basis of a new art form, the "Grecian Groups." Finally there are the five so-called "Boring Groups" that have only one generator (along with r), and so are unattractive to the Greek eye.

	Magnificent Seven							The Grecian Groups												Boring Groups				
	1	2	3	4	5	6	7	1'	2'	3'	4'	5'	6'	7'	8'	9'	10'	11'	12'	1"	2"	3"	4"	5"
m_v	x	x	x					x	x	x			x	x										
m_v'								x	x		x	x	x		x	x				x				
1/2	x	x			x			x	x		x				x		x							
1/2'								x	x	x		x		x		x	x				x			
m_h	x			x				x				x						x						
m_h'									x	x	x								x			x		
g		x				x			x							x			x					
g'								x						x	x			x					x	
r'								x	x				x				x	x	x					x
actions	3	3	1	1	1	1	0	7	7	3	3	3	3	3	3	3	3	3	3	1	1	1	1	1

Figure 9: *Categories of frieze types.*

These various frieze types appear with different degrees of frequency at Pirgí, as seen in the chart below.

	Magnificent Seven							The Grecian Groups												Boring Groups				
	1	2	3	4	5	6	7	1'	2'	3'	4'	5'	6'	7'	8'	9'	10'	11'	12'	1"	2"	3"	4"	5"
Old	34	4	5	1	2	2	0	7	5	5	4	5	5	8	5	5	3	2	0	0	6	0	0	0
Friezes	31.5%	3.7	4.6	0.9	1.9	1.9	0	6.5	4.6	4.6	3.7	4.6	4.6	7.4	4.6	4.6	2.8	1.9	0	0	5.6	0	0	0
New	94	2	18	1	2	9	0	35	22	5	19	27	13	21	3	21	6	0	0	0	11	0	0	0
Friezes	30.4%	0.6	5.8	0.3	0.6	2.9	0	11.3	7.1	1.6	6.1	8.7	4.2	6.8	1.0	6.8	1.9	0	0	0	3.6	0	0	0
Total	128	6	23	2	4	11	0	42	27	10	23	32	18	29	8	26	9	2	0	0	17	0	0	0
%	30.7%	1.4	5.5	0.5	1.0	2.6	0	10.1	6.5	2.4	5.5	7.7	4.3	7.0	1.9	6.2	2.2	0.5	0	0	4.1	0	0	0

Figure 10: *Frequency chart for all twenty-four frieze types.*

Remarks

The rules of the decorative art form invented by the artists of Pirgí are well defined and fairly rigid. It is fascinating in such situations to observe whether contemporary artists are trying to stretch the rules. On one modern house we found a "shout for freedom" from the straight-line-and-circle Euclidean dictum in the form of waves that incorporate a color-reversal aspect. (See Figure 10.) There are also recent instances of vertical, rather than horizontal, friezes and of the incorporation of some space filling patterns, indicating that the art of Pirgí is not static, but a living, developing art form. We leave it to the reader to spot three places above where the old rules are being relaxed. As we left Pirgí we discovered a delightful and unique statue, the "Aphrodite of Pirgí," standing in front of the regional health services.

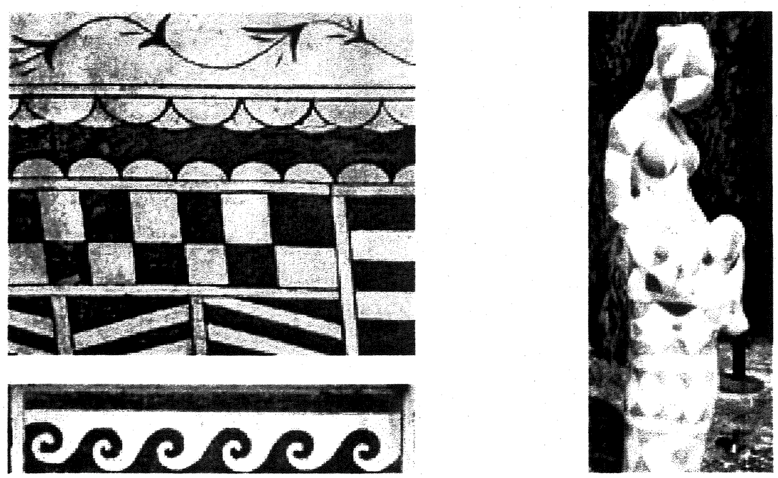


Figure 11: *Frieze waves and "Aphrodite of Pírgi."*

Conclusions

With their color-reversing friezes, the artists of Pirgí have produced a new decorative art form that had not been recognized until now. The format of this art can be understood only by adding the seventeen group color-reversing analysis described above to the standard seven group analysis. Not only does this new tool clarify the nature of the activities and intentions of the Pirgí artists, but also it makes clear which frieze types are most likely and unlikely to appear in this art form. Among the seventeen color-reversing frieze types, there are five that contain only one symmetry action (in addition to the always present r), and only one of these low-symmetry friezes appear at Pirgí. The only other mixed frieze type that fails to appear (Type 6´) possesses only three symmetry actions (besides r) and furthermore its only non-color-reversing action is a glide-reflection, a very weak symmetry action. There is a marked preference in Pirgí toward groups that contain vertical mirror images, either m_v or m_v'. About half of all the twenty-four groups have neither of these two actions and were very rarely used. Simply put: for the artists of Pirgí, symmetry appeals and more symmetry appeals more.

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