BRIDGES
Mathematical Connections in Art, Music, and Science

Electrostatic Kaleidoscope

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Abstract

Electrostatic potentials created by static electric charge distributions are the core concept of charge related phenomenons. Plots of these curves and surfaces play an invaluable role in comprehending the underlying physical ideas. Generally, 2D plots of potentials are hardly considered; 3D surfaces are mostly ignored. The author has applied *Mathematica* and improved the situation -- he has also stumbled across the artistic features of the plots, bridging the gap between the art of science and abstract art. Animation of one such set of plots is considered to create figures resembling collages formed by a kaleidoscope.

1. Introduction

Electrostatics is a well established branch of physics -- however, comprehension of some of its abstract concepts relies on mental visualization of quantities such as scalar potentials and vector fields. Visualizing the potentials could be challenging; this by itself contributes to the challenge of the understanding the related concepts -- leave alone the associated vector fields. It is not a common practice to display the potentials, the majority of the standard texts [1-2] conveniently have ignored them. In a few cases, 2D contour plots are displayed[3], seldomly 3D plots of the contours are considered.

Since the aforementioned texts have been published, computer technology has progressed tremendously. Along with technological advancements, powerful software programs capable of performing symbolic and graphic scientific computation have been developed. By adapting one such program, *Mathematica* [4], the author by way of examples revisited a few basic cases.

The 2D and 3D potential curves and surfaces created by charged particles possess somewhat artistic features. Compatible with the theme of the conference the author has discussed specific examples that mathematically are easy to follow and artistically are pleasing to view. It is the objective of this article to run a bridge across the art of physics, mathematics and the abstract art.

Scientifically speaking, a charge distorts the homogeneity of the space. To study the shape of the distortion and the interaction between the charges, the electric potential is introduced. The electric potential at point $\vec{r}$ from a set of scattered discrete point charges q_i , $i = 1, 2, 3, \ldots n$ positioned at $\vec{r}_i$ is $V(r) = \sum_{i=1}^{n} V(|\vec{r} - \vec{r}_i|)$, where $V(|\vec{r} - \vec{r}_i|) = \frac{kq_i}{|\vec{r}-\vec{r}_i|}$ and $|\vec{r} - \vec{r}_i|$ is the distance between $\vec{r}$ and $\vec{r}_i$

with k being a constant. By applying this mathematical function to various situations in the following sections a few fundamental cases are discussed.

2. Case study I

To begin, we position a point charge at the origin, and for the sake of simplicity we set the value of kq to unity. The related potential is $v1(r) = \frac{1}{r}$. In 2D cartesian coordinate system, $v1(x, y) = \frac{1}{\sqrt{x^2+y^2}}$. As one may predict the equipotentials, i.e. the points in the space with the same potentials, should be unevenly spread out concentric circles about the charge. To confirm this, we apply *Mathematica's* ContourPlot. The left figure of Figure 1 is the display of this function. Although useful, it is not customary to display the 3D contour plot of the potential. *Mathematica*'s Plot3D function easily plots the function. This is shown to the right of Figure 1.

```
Block[{V1 = 1/Sqrt[x^2 + y^2], $DisplayFunction = Identity},
  g1 = ContourPlot[V1, {x, -1, 1}, {y, -1, 1},
    Frame -> False, PlotPoints -> 50, ContourStyle -> GrayLevel[0.5]];
  g2 = Plot3D[V1, {x, -1, 1}, {y, -1, 1}, Lighting -> False, Background ->
     GrayLevel[0.6], Mesh -> False, Axes -> None, Boxed -> False, PlotPoints -> 50]];
Show[GraphicsArray[{g1, g2}]];
```

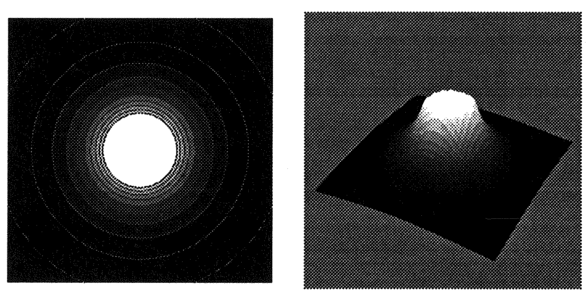

Figure 1: *The light-gray concentric circles of the left figure are the equipotentials of a positive point charge; its 3D plot is displayed to its right*

3. Case study II

a) The investigation of displaying the equipotentials now is extended to two-points. First we consider a *symmetrical* case created by two identical positive charges. In a two-dimensional coordinate system, the charges are positioned at (a,0) and (-a,0). According to the general format of the potential function given in the introduction the corresponding potential is $V21(x, y) = \frac{kq_1}{\sqrt{(x-a)^2+y^2}} + \frac{kq_2}{\sqrt{(x+a)^2+y^2}}$. As before, we set $kq_1 = kq_2 = 1$ and conveniently we choose $a = 0.5$ units. The 2D and 3D contour plots of equipotentials are displayed in Figure 2.

```
Block[{V2 = 1/√((x-0.5)²+y²) + 1/√((x+0.5)²+y²), $DisplayFunction = Identity},
 g1 = ContourPlot[V2, {x, -1, 1}, {y, -1, 1},
   Frame → False, PlotPoints → 50, ContourStyle → GrayLevel[0.7]];
 g2 = Plot3D[V2, {x, -1, 1}, {y, -1, 1}, Lighting → False,
   Background → GrayLevel[0.6], Mesh → False,
   Axes → None, Boxed → False, PlotPoints → 50]];
Show[GraphicsArray[{g1, g2}]];
```

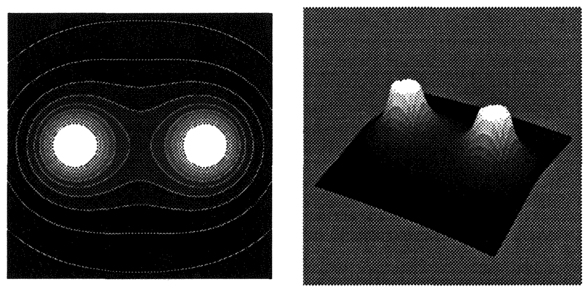

Figure 2: *The light-gray curves of the left figure are the equipotentials of a pair of identical positive point charges; its 3D plot is displayed to its right*

b) By changing the sign of one of the charges we study the potential created by an electric dipole. This requires $kq_1 = -kq_2 = 1$; the corresponding potential is $V22(x, y) = \frac{kq_1}{\sqrt{(x-a)^2+y^2}} + \frac{kq_2}{\sqrt{(x+a)^2+y^2}}$. As in the previous case, the charges are positioned at (a,0) and (-a,0) with $a = 0.5$ units. Its 2D and 3D equipotentials are shown in Figure 3.

```
Block[{V22 = 1/√((x-0.5)²+y²) - 1/√((x+0.5)²+y²), $DisplayFunction = Identity},
 g1 = ContourPlot[V22, {x, -1, 1}, {y, -1, 1},
   PlotPoints → 50, Frame → False, ContourStyle → GrayLevel[0.3]];
 g2 = Plot3D[V22, {x, -1, 1}, {y, -1, 1}, Lighting → False,
   Background → GrayLevel[0.3], Axes → None,
   Boxed → False, PlotPoints → 50, Mesh → False]];
Show[GraphicsArray[{g1, g2}]];
```

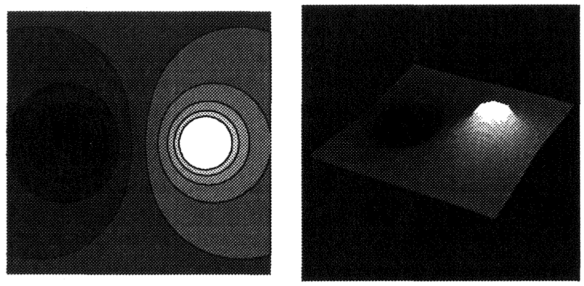

Figure 3: *The dark-gray curves of the left figure are the equipotentials of a dipole; its 3D plot is displayed to its right*

4. Case study III

By choosing different charges, *asymmetrical* equipotentials are produced. For instance, we set $kq_1 = 1$ and $kq_2 = 4$. Without moving the charges from their previous positions, the corresponding potential becomes $V23[x, y] = \frac{1}{\sqrt{(x-0.5)^2+y^2}} + \frac{4}{\sqrt{(x+0.5)^2+y^2}}$. The contour and density, as well as a 3D profile of the 2D contour of the equipotentials are displayed in Figure 4.

```
Block[{V23 = 1/√((x-0.5)^2+y^2) + 4/√((x+0.5)^2+y^2), $DisplayFunction = Identity},
 g1 = ContourPlot[V23, {x, -1, 1}, {y, -1, 1},
   PlotPoints → 50, Frame → False, ContourStyle → GrayLevel[0.5]];
 g2 = DensityPlot[V23, {x, -1, 1}, {y, -1, 1},
   PlotPoints → 20, Frame → False, Mesh → False];
 g3 = Plot3D[V23, {x, -1, 1}, {y, -1, 1}, Lighting → False,
   Background → GrayLevel[0.6], Axes → None,
   Boxed → False, PlotPoints → 50, Mesh → False]];
Show[GraphicsArray[{g1, g2, g3}]];
```

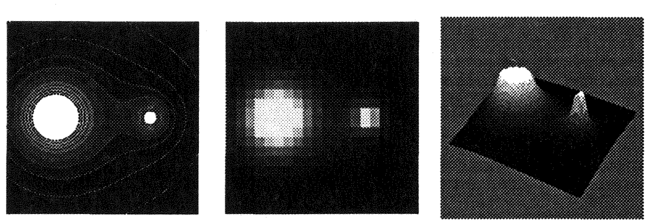

Figure 4: *The light-gray curves on the left are the equipotentials. The middle and the right figures are the tiling and 3D plots of the contour plots of the left figure, respectively*

5. Case study IV

a) In this study we consider four identical point charges and position them *symmetrically* about the origin at (a,0), (-a,0), (0,a) and (0,-a) with a = 0.5 units. To generate *symmetrical* equipotentials we choose equal charges, $kq_1 = kq_2 = kq_3 = kq_4 = 1$. The corresponding potential according to the general format given in the introduction is $V41[x, y] = \frac{1}{\sqrt{(x+0.5)^2+y^2}} + \frac{1}{\sqrt{(x-0.5)^2+y^2}} + \frac{1}{\sqrt{x^2+(y-0.5)^2}} + \frac{1}{\sqrt{x^2+(y+0.5)^2}}$. As in the case study III we plot its contour, density and 3D profile of the 2D equipotentials. These are shown in Figure 5.

```
Block[{V41 = 1/√((x+0.5)²+y²) + 1/√((x-0.5)²+y²) + 1/√(x²+(y-0.5)²) + 1/√(x²+(y+0.5)²),
  $DisplayFunction = Identity},
 g1 = ContourPlot[V41, {x, -1, 1}, {y, -1, 1}, PlotPoints → 50,
   Frame → False, ContourStyle → GrayLevel[0.5]];
 g2 = DensityPlot[V41, {x, -1, 1}, {y, -1, 1}, Mesh → False,
   PlotPoints → 20, Frame → False];
 g3 = Plot3D[V41, {x, -1, 1}, {y, -1, 1}, Lighting → False, Background →
    GrayLevel[0.6], PlotPoints → 50, Axes → None, Boxed → False, Mesh → False]];
Show[GraphicsArray[{g1, g2, g3}]];
```

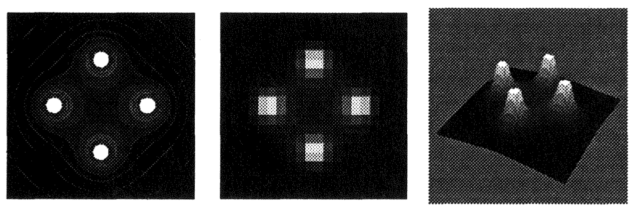

Figure 5: *The light-gray curves on the left are the equipotentials. The middle and the right figures are the tiling and 3D plots of the contour plots of the left figure, respectively*

b) In the following we have also shown the potential of two pairs of equal positive and negative charges, $kq_1 = kq_2 = -kq_3 = -kq_4 = 1$. To compare the effect of the negative charges v.s. the case study III, all four charges are left at their previous positions. The potential is $V42[x, y] = \frac{1}{\sqrt{(x+0.5)^2+y^2}} + \frac{1}{\sqrt{(x-0.5)^2+y^2}} - \frac{1}{\sqrt{x^2+(y-0.5)^2}} - \frac{1}{\sqrt{x^2+(y+0.5)^2}}$ and its various plots are shown in Figure 6.

```
Block[
 {V42 = 1/√((x + 0.5)² + y²) + 1/√((x - 0.5)² + y²) - 1/√(x² + (y - 0.5)²) - 1/√(x² + (y + 0.5)²),
  $DisplayFunction = Identity},
 g1 = ContourPlot[V42, {x, -1, 1}, {y, -1, 1}, PlotPoints → 50,
   Frame → False, ContourStyle → GrayLevel[0.5]];
 g2 = DensityPlot[V42, {x, -1, 1}, {y, -1, 1},
   PlotPoints → 20, Frame → False, Mesh → False];
 g3 = Plot3D[V42, {x, -1, 1}, {y, -1, 1}, Lighting → False, Background →
    GrayLevel[0.6], PlotPoints → 50, Axes → None, Boxed → False, Mesh → False]];
Show[GraphicsArray[{g1, g2, g3}]];
```

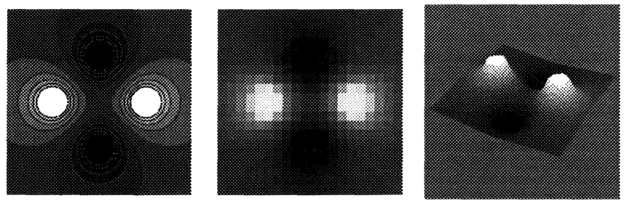

Figure 6: *The outlines of the lobes on the left are the equipotentials. The middle and the right figures are the tiling and 3D plots of the contour plots of the left figure, respectively*

6. Case study V

a) 2D and 3D regulated electrostatic kaleidoscopes

Thus far, the case studies graphically display the static relationships between the charge distributions and the potentials. *Mathematica,* however, is capable of animating a set of comparable figures, dynamically enforcing their graphical comparisons. For instance, in case study IV, by changing the charge of the fifth particle according to $kq_5 = n$ with $n = \{-3, 3\}$, generates seven comparable cases. The associated potential is V5n[x, y, n] = $\frac{1}{\sqrt{(x+0.7)^2+y^2}} + \frac{1}{\sqrt{(x-0.7)^2+y^2}} + \frac{1}{\sqrt{x^2+(y-0.7)^2}} + \frac{1}{\sqrt{x^2+(y+0.7)^2}} + \frac{n}{\sqrt{x^2+y^2}}$ and its 2D contour equipotentials are shown in Figure 7.

```
Block[{V5n = 1/√((x + 0.7)^2 + y^2) + 1/√((x - 0.7)^2 + y^2) + 1/√(x^2 + (y - 0.7)^2) +
      1/√(x^2 + (y + 0.7)^2) + n/√(x^2 + y^2), $DisplayFunction = Identity},
gn = Table[ContourPlot[V5n, {x, -1, 1}, {y, -1, 1}, PlotPoints → 50,
     Frame → False, ContourStyle → GrayLevel[0.5]], {n, -12, 12, 4}]];
Show[GraphicsArray[Table[gn]]];
```

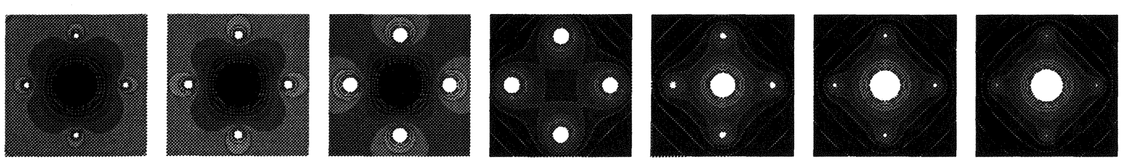

Figure 7: *Contour plots of equipotential curves of a five particle charge distribution. Curves are generated by changing the value of the charge of the center particle between n = {−3, 3}*

Double clicking one of these stationary frames will automatically activate the animation, interchanging the frames sequentially, giving an illusion of figures made by a kaleidoscope.

In the above code, by replacing ContourPlot function with DensityPlot a similar kaleidoscopic figures may also be formed, this is left for interested readers.
However, with *Mathematica* a 3D kaleidoscopic figures may also be formed. The following code generates a set of seven 3D profile equipotentials. These are shown in Figure 8.

```
Block[{V5n = 1/Sqrt[(x + 0.7)^2 + y^2] + 1/Sqrt[(x - 0.7)^2 + y^2] + 1/Sqrt[x^2 + (y - 0.7)^2] +
    1/Sqrt[x^2 + (y + 0.7)^2] + n/Sqrt[x^2 + y^2], $DisplayFunction = Identity},
gn = Table[Plot3D[V5n, {x, -1, 1}, {y, -1, 1}, Lighting -> False,
      Background -> GrayLevel[0.6], PlotPoints -> 50, Axes -> False,
      Boxed -> False, Mesh -> False], {n, -12, 12, 4}]];
Show[GraphicsArray[Table[gn]]];
```

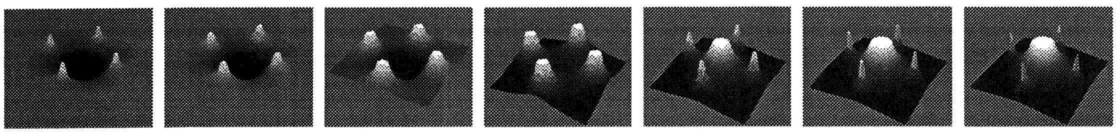

Figure 8: *3D profile of 2D equipotential contour plots of a five particle charge distribution*

Double clicking one of these stationary frames will automatically activate the animation, interchanging the frames sequentially, giving an illusion of figures made by a kaleidoscope.

b) 2D and 3D random electrostatic kaleidoscopes

The kaleidoscopic figures made with Figure 7 and 8 are regulated, meaning, animation interchanges cyclically the *same* seven figures. However, for example, by randomly positioning the fifth charge, and even by assigning a random value to the charge, one generates an unpredictable kaleidoscopic figures. The potential of one such case is $\mathrm{V5R}[x, y] = \frac{1}{\sqrt{(x+0.7)^2+y^2}} + \frac{1}{\sqrt{(x-0.7)^2+y^2}} + \frac{1}{\sqrt{x^2+(y-0.7)^2}} + \frac{1}{\sqrt{x^2+(y+0.7)^2}} + \frac{\mathrm{Random[]}}{\sqrt{x^2+\mathrm{Random[]}^2}}$. Figure 9 and 10 are two unpredictable contour and density plots of this potential.

```
Show[GraphicsArray[
  Table[ContourPlot[1/Sqrt[(x + 0.7)^2 + y^2] + 1/Sqrt[(x - 0.7)^2 + y^2] + 1/Sqrt[x^2 + (y - 0.7)^2] +
       1/Sqrt[x^2 + (y + 0.7)^2] + Random[Integer, {-2, 2}]/Sqrt[x^2 + Random[]^2], {x, -1, 1}, {y, -1, 1},
     PlotPoints -> 20, Frame -> False, DisplayFunction -> Identity], {n, 1, 7}],
  DisplayFunction -> $DisplayFunction]];
```

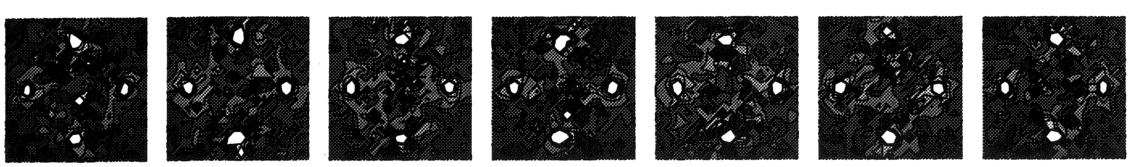

Figure 9: *Contour plots of equipotential curves of a five particle charge distribution. Curves are generated by randomly changing the position and the value of the fifth charge*

$$
\begin{aligned}
&\texttt{Show}\Big[\texttt{GraphicsArray}\Big[\\
&\quad \texttt{Table}\Big[\texttt{DensityPlot}\Big[\frac{1}{\sqrt{(x+0.7)^2+y^2}}+\frac{1}{\sqrt{(x-0.7)^2+y^2}}+\frac{1}{\sqrt{x^2+(y-0.7)^2}}+\\
&\quad\quad \frac{1}{\sqrt{x^2+(y+0.7)^2}}+\frac{\texttt{Random[Integer, \{-2, 2\}]}}{\sqrt{x^2+\texttt{Random[]}^2}},\ \{x,-1,1\},\ \{y,-1,1\},\\
&\quad\quad \texttt{PlotPoints}\to 20,\ \texttt{Frame}\to \texttt{False},\ \texttt{DisplayFunction}\to \texttt{Identity}\Big],\ \{n,1,7\}\Big],\\
&\quad \texttt{DisplayFunction}\to \texttt{\$DisplayFunction}\Big]\Big];
\end{aligned}
$$

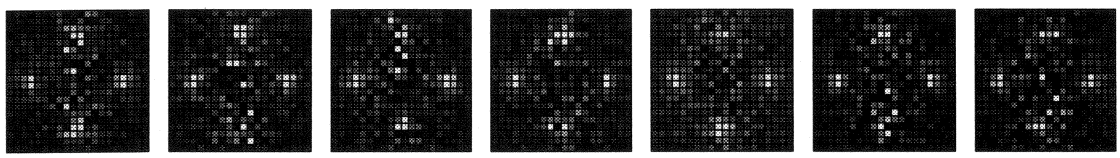

Figure 10: *These are density plots of Figure 9*

7. Summary and Conclusion

By way of examples it is shown that it is useful to display abstract mathematical functions to describe the distortion of charged space. Various fundamental examples have been considered. The described methods can readily be extended to study cases on a need basis. On the other hand, the plots possess artistic characters and can be viewed as art work. Reviewing these plots may intrigue a physicist to think about the implicit artistic features of the distorted space or an artist conversely may be fascinated about the way the nature works.

References

[1] D. Halliday, R. Resnick, J. Walker, *Fundamental of Physics,* 6th edition, John Wiley 2001.

[2] E. M. Purcell, *Electricity and Magnetism, Berkeley physics course* - volume 2, McGraw-Hill 1965.

[3] J. R. Reitz, F.J. Milford, *Foundations of Electromagnetic Theory*, Addison-Wesley, 1960.

[4] S. Wolfram, *Mathematica*, 4th edition, Cambridge 1999.