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3D Visualization Models of the Regular Polytopes in Four and Higher Dimensions

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Abstract

This paper presents a tutorial review of the construction of all regular polytopes in spaces of all possible dimensions. It focusses on how to make instructive, 3-dimensional, physical visualization models for the polytopes of dimensions 4 through 6, using solid free-form fabrication technology.

1. Introduction

Polytope is a generalization of the terms in the sequence: point, segment, polygon, polyhedron ... [1]. Such a polytope is called regular, if all its elements (vertices, edges, faces, cells ...) are indistinguishable, i.e., if there exists a group of spatial transformations (rotations, mirroring) that will bring the polytope into coverage with itself. Through these symmetry operations, it must be possible to transform any particular element of the polytope into any other chosen element of the same kind.

In two dimensions, there exist infinitely many regular polygons; the first five are shown in Figure 1.

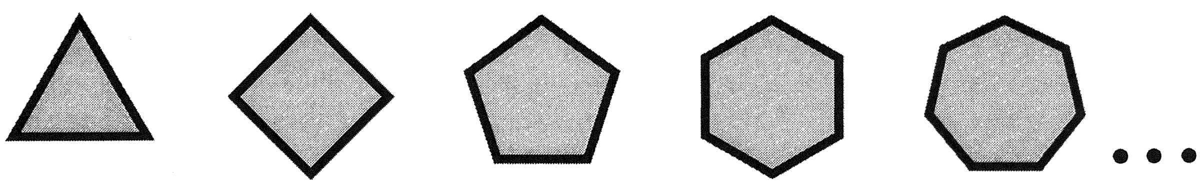

Figure 1: *The simplest regular 2D polygons.*

In three-dimensional space, there are just five regular polyhedra -- the Platonic solids, and they can readily be depicted by using shaded perspective renderings (Fig.2). As we contemplate higher dimensions and the regular polytopes that they may admit, it becomes progressively harder to understand the geometry of these objects. Projections down to two (printable) dimensions discard a fair amount of information, and people often have difficulties comprehending even 4D polytopes, when only shown pictures or 2D graphs. While a 3D model is still not the real thing, it helps considerably in promoting an understanding of the structure of these fascinating objects. Of course, there are many ways of projecting these higher-dimensional objects down to 3D, and many ways to physically realize these models. Many skillful enthusiasts have built such models from wooden dowels, folded paper (origami), or with construction kits such as the Zome™ tool [3]. The emergence of solid free-form fabrication (SFF) based on layered manufacturing offers intriguing new solutions. This paper tries to provide easy introductions to the higher-dimensional regular polytopes as well as to the possibilities and problems of model making with SFF. Some of these models also have the potential to be scaled up to yield dramatic, large-scale constructivist sculptures.

2. The Regular Polyhedra in 3 Dimensions

One can easily see that there cannot possibly exist more than the five regular 3D polyhedra shown in Figure 2 by contemplating their construction from regular polygons, – a proof that was already known to Euclid. We need at least 3 n-gons at a corner to create a solid angle. This works for triangles, squares, or pentagons. Three hexagons fit snugly together around a joint vertex in the plane without needing to bend out of this plane. We may try to fit more than three regular polygons around the same vertex; but this can be accomplished only for the triangle. Four squares already form a flat constellation, and three regular pentagons cannot be fit together at one corner without warping into a hyperbolic space. For equilateral triangles we also obtain useful solid angles by joining four and five triangles around a shared vertex; at six, the constellation becomes flat.

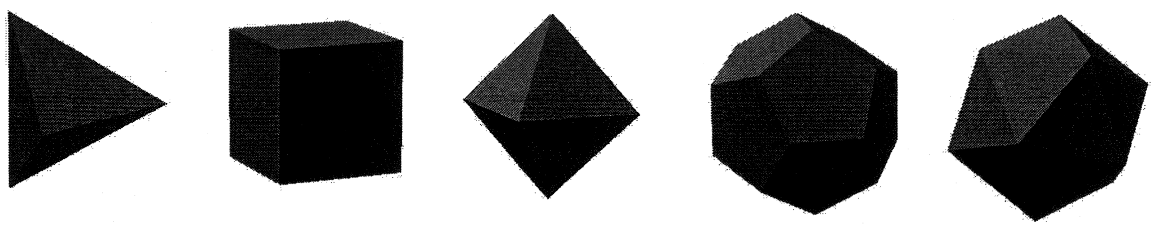

Figure 2: *The Platonic solids in 3D: Tetrahedron, Cube, Octahedron, Dodecahedron, Icosahedron.*

It is a little more work to show that all these five possible solid-angle constructions will indeed close into a "spherically" symmetrical object when the construction is repeated at every vertex. A corner of three triangles leaves an open base in the shape of an equilateral triangle with the same edge length as the starting faces; if we close this off with an equilateral triangle, we obtain the tetrahedron. Four triangles leave a square base. Two such square pyramids can be joined at their bases, creating four more corners with four triangles each, and overall forming an octahedron. Five triangles in a regular configuration around a joint vertex form a 5-sided pyramid with a regular pentagon as a base. If we were to stick two such pyramids together, we would not get a totally regular polyhedron, since the vertices at the joined bases are shared by only four triangles. To remedy that, we place a regular 5-sided anti-prism between the two pyramids, and thereby obtain the icosahedron. Joining 3 squares readily leads to the cube; because of all the right angles involved, it is easy to see that this polyhedron will close. The case of the pentagonal dodecahedron is somewhat harder. One can convince oneself most easily of its existence by seeing it as the dual of the icosahedron: place vertices at the centers of all triangular faces and connect nearest neighbors by turning all icosahedral edges through 90° (and adjusting their lengths).

Table 1: Characteristics of the Platonic Solids [7]

	Tetrahedron	Cube	Octahedron	Dodecahedron	Icosahedron
# Vertices	4	8	6	20	12
# Edges	6	12	12	30	30
# Faces	4	6	8	12	20
Dihedral Angle	70.5°	90°	109.5°	116.5°	138.2°

3. Projections

In this paper, I can only show 2D pictures of the actual 3D Platonic solids. This is not a problem; our visual system is designed to capture two-dimensional images, and our brain then turns them into 3D mental models. We will use this same kind of "hyper-seeing" [2] to form mental models of higher-dimensional regular polytopes. The easiest way to do a projection to a lower-dimensional space is to simply set all unwanted higher dimensions to zero; this amounts to a set of parallel projections along the coordinate axes of these unwanted dimensions. However, for high-dimensional regular polytopes the "shadows" produced in this manner are neither very interesting, nor very informative; they are too "round" and show no internal structure (Fig.3a). It is much better to only project the edges of the polytopes; in this way one can readily see the front as well as the back of the polytope (Fig.3b). By adding perspective, one can introduce implicit depth cues: elements further away from the camera appear smaller (Fig.3c).

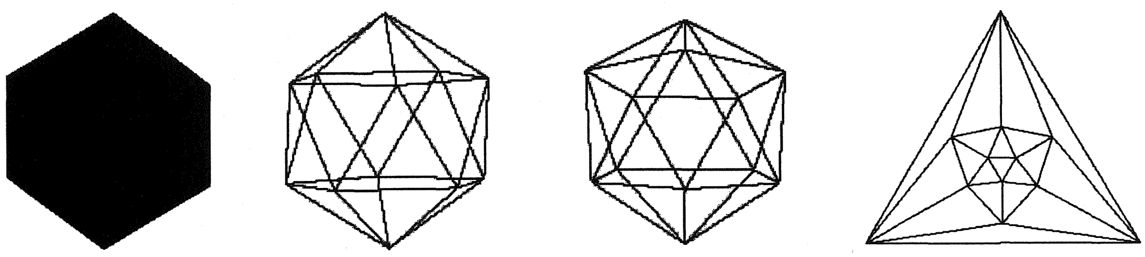

Figure 3: *Projections of a 3D icosahedron into 2D: (a) parallel shadow, (b) parallel wire frame projection, (c) perspective projection, (d) extreme perspective projection from a camera close to one face center.*

The direction of the projection is another important aspect to consider. Depending on the viewing or projection direction, the resulting figure can have more or less symmetry and more or less – desirable or undesirable – coincidences. Since we are trying to create visualization models of highly symmetrical objects, it is often desirable to maintain as much symmetry as possible. Figure 4 gives a survey of the relevant view axis alignments that can be used to bring out particular aspects of the polytopes. The *vertex-first* projections, and the projections that look perpendicularly onto one of the highest-dimensional polytope facets (*face-first* in 3D, *cell-first* in 4D), are particularly well-suited to maintain a maximum of symmetry.

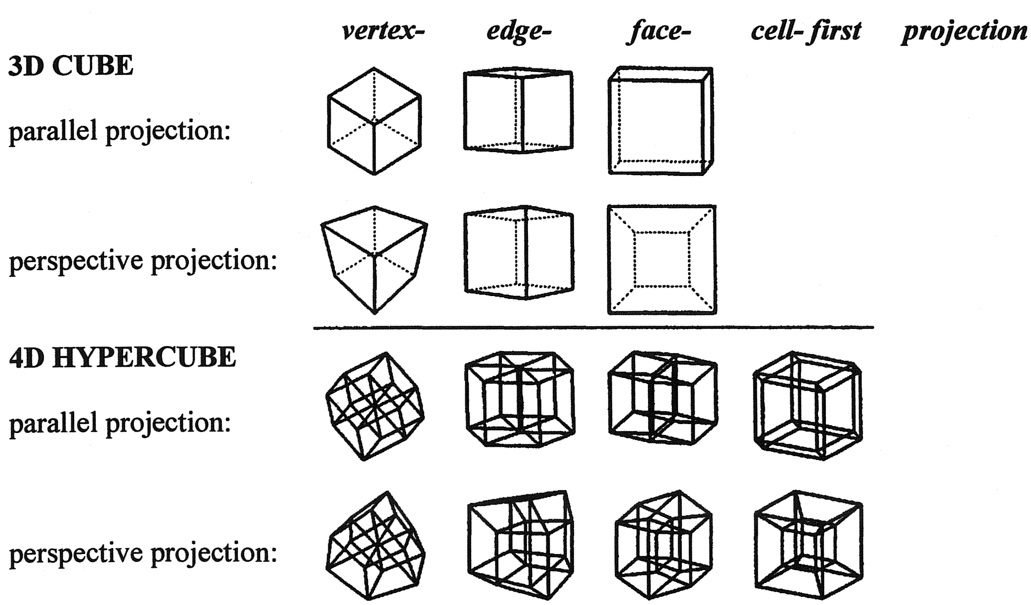

Figure 4: *Varying the direction of projection to emphasize different aspects of a polytope.*

Figure 3c shows a face-first perspective view of an icosahedron. If we move the camera close enough, the projection of the front face can become so large that all the other elements of the polytope appear inside the projection of this front face (Fig.3d); this is particularly useful to highlight the cells of which a particular 4D polytope is composed (Figs.6a,b,c and 7a,b).

Another objective of these special alignments is to reduce the complexity of the result; this is particularly important for the two "monster" objects the 120-Cell and the 600-Cell in 4D space (Figs.7b,c). By using parallel projection with an appropriate alignment (Fig.7c), about half the vertices, edges, faces, and cells can be hidden. This makes it simpler to build physical models of these complex polytopes.

Projecting a high-dimensional polytope down to three or even two dimensions, obviously makes some dramatic changes to its geometry. It also creates a large number of overlaps and possible coincidences that may make it difficult to understand the connectivity of the original object. Thus it is sometimes more important to maintain some of the key topological features by choosing a transformation that keeps independent edges and vertices separated as much as possible. For this purpose, oblique projections such as the Cavalier projection in 3D are most useful; elements parallel to the projection plane/space maintain their exact geometry without any distortion, and elements that are perpendicular to this plane/space are folded down at a predefined angle so that they also can be seen in an un-foreshortened view. We will use such projections to visualize the Hypercubes (Fig.9c, 10c).

4. Edge Treatments

In the simplest case, and in several figures in this paper, edges and their projections are simply rendered as line segments. In 3D edge models, we need to give the edges some finite thickness for physical strength. This gives us an opportunity to use their cross sections to impart additional visual cues about the structure of the polytope. Rather than using just round cylinders, the edges could have prismatic cross sections that indicate the orientations of the associated faces, or they could even have flanges attached that cover the outermost few percent of each face connected to that edge.

Leonardo DaVinci constructed beautiful wood frame models for some of the Platonic and Archimedean solids, in which properly mitred furring strips represented the borders of each of the faces (Fig.5a). Such compound edges become particularly informative for the projections of higher-dimensional polytopes where more than two faces join at each edge (Fig.6b, 7b, 9a, 10b); however, creating appropriate clean 3D solid models is not trivial.

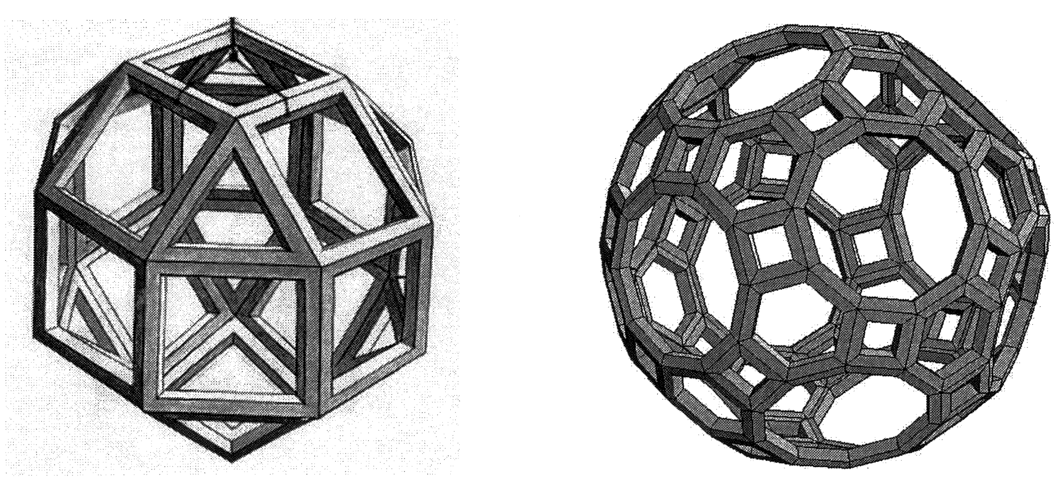

Figure 5: *Frame representations of semi-regular polyhedra, (a) by Leonardo DaVinci, (b) by G. Hart [4].*

Handcrafting such nicely mitred models of polytopes with hundreds of edges is a rather daunting task. Even making a clean CAD model on a computer has its challenges. Creating the geometry for a single properly flanged and mitred edge can readily be done on most commercial CAD tools. However these tools often lack convenient primitives to replicate this geometry hundreds of times with suitable rotations and translations and then to merge them into a single clean boundary representation. This is something that should be done procedurally and with high accuracy, because for many applications, the resulting description must be a perfectly "watertight." The whole task requires many different computational steps, which are rarely all found on one and the same CAD tool.

George Hart describes an elegant computational approach that relies on an efficient and robust program to compute the convex hull of a set of points [4]. Repeated application of this program first creates polyhedral balls at the vertices with flat faces perpendicular to all the directions of the incoming edges that join in this vertex. In a second phase, each edge beam is formed as convex hull around the two – possibly irregular – facets on the vertex polyhedra at either end of this edge. A drawback of this approach, when applied to projections of complex 4D polytopes that have a rather irregularly angled set of edges joining at each vertex (Fig.7b), is that the edges do not have the nice prismatic shapes displayed in DaVinci's models.

Another approach to making a computer model uses a prismatic sweep with the desired edge cross section to extrude each edge beam. The difficulty is that these beams must then be mitred in often rather complicated ways to properly join together at each vertex (Fig.9b, 10b). Conceptually one would like to join them with a Boolean union operation of the un-mitred, perpendicularly cut beams and then fill in possible interstitial voids. Not many CAD systems have robust enough software to do this operation cleanly and to return a flawless, properly oriented boundary representation without any self-intersections. Most layered-SFF machines demand such clean boundary representations, and may produce surprising results if these conditions are violated.

However, through some experimentation we found that the rules can be bent within some limits. In particular, we had good success in building "balls-and-sticks" models, if we observed a few heuristics. These models are easy to generate, even for complex models with many edges joining at each vertex. We simply place a ball (a polyhedral approximation to a sphere) at each vertex, with a procedurally assigned diameter that may be a function of the distance of that vertex from the camera in the high-dimensional space. The edges then are cylinders or cones around the ideal line-edges, with diameters at either end that match the diameters assigned to the vertex balls. The truncated cones or cylinders must be closed surfaces with matching end-discs. We found that the slicing software of Stratasys' FDM machine [11] and of Z-Corporation's 3D-Printer [12] will form an acceptable de-facto Boolean union operation, when a collection of such balls and sticks is sent to them. Such models can easily be designed in our custom-built SLIDE environment [10] or in Mathematica [8] or in Matlab [5]. However, this simple approach no longer worked when we used concave edge shapes which resulted from sweeps of more complex cross sections for edges with several flanges on them. Some students have written their own special-purpose C-programs to generate polygonal surface descriptions of more complicated edge frames.

Particularly attractive models result, if they are assembled from individual *open tile frames* representing the 2D faces of the polytopes. This gives us the opportunities to make multi color models even with the FDM machine (Fig.6b). As an assignment in a modeling class, we made some rhomboid tiles with edges compatible with the Polymorf™ system [9], so that we could build various models using the patented edge connectors of this system (Fig.9a).

5. The Regular Polytopes in 4 Dimensions

A 1D segment is bounded by two points. The regular 2D polygons are composed of symmetrical closed chains of 1D segments of equal lengths. The Platonic solids, as discussed in Section 2, are then formed from regular closed shells of these regular 2D n-gon tiles.

By inductive analogy, the regular polytopes in 4 dimensions are composed from a shell of regular Platonic 3D cells. If we try to fit three or more Platonic polyhedra around a shared edge, we find that for some cases the sum of the dihedral angles is less than 360°. Thus there is an empty wedge of space that is not "filled" by the assembled polyhedra. We can forcefully close this wedge-shaped gap by "bending" – or rather "folding" – the assembly into 4D space and thereby creating a convex 4D corner. This may be the start of a 4D polytope shell. If we are lucky, the process, when repeated at every "free" edge that is not already covered by the specified number of 3D tiles, will lead to symmetrical closure of the shell, so that every cell is equivalent, and every created "corner" looks the same. It turns out, that six regular 4D polytopes can be constructed in this manner (Table 2).

Table 2: Characteristics of the Regular Polytopes in 4D

	Simplex	Tesseract	16-Cell	24-Cell	120-Cell	600-Cell
# Vertices	5	16	8	24	600	120
# Edges	10	32	24	96	1200	720
# Faces	10	24	32	96	720	1200
# Cells	5	8	16	24	120	600
Dihedral Angle	75.5°	90°	120°	120°	144°	164.5°

Let's start with the tetrahedron. This 3D solid has a dihedral angle of 70.5°. Thus we can fit 3, 4, or 5 of them around a shared edge without exceeding the available 360°. If we fit only three tetrahedra around each edge, we obtain the 4D simplex with 5 cells (Fig.6a). If we use four tetrahedra instead, we obtain the cross polytope with 16 cells (Fig.6c). If we compose 5 tetrahedra around each edge, we obtain a beautiful "monster" with 600 cells (Fig.7c); such a large number of cells are required to close this object, because the gap left with five tetrahedra is only 7.3°, and the resulting "curvature" in 4D is thus rather weak.

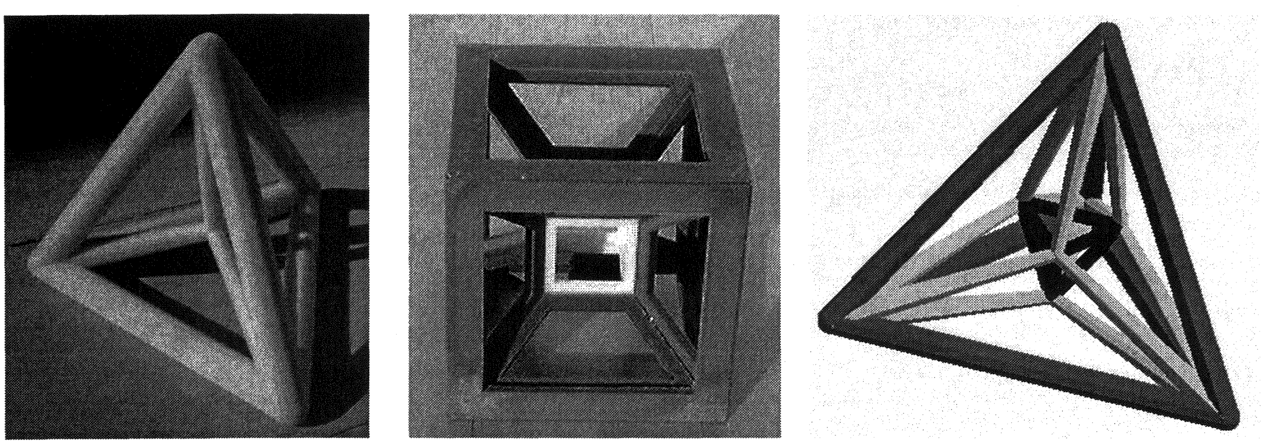

Figure 6: *Models of 4D polytopes: (a) Simplex (FDM), (b) Tesseract (ZCorp.), (c) Cross Polytope (SLIDE).*

If we start with cubes, we can use three around an edge, and if we force closure, we will obtain the 4D hypercube, or tesseract, with a total of 8 cells (Fig.6b). Four cubes around an edge add up precisely to 360° and will produce no "bending" of space; they are thus useless to form a 4D polytope. Three octahedra around an edge will close into a regular structure with 24 cells (Fig.7a). Since the dihedral angle is 109.5°, four octahedra exceed 360° and could only be accommodated in a hyperbolic space. Three pentagonal dodecahedra also fit around an edge without exceeding 360°. The small empty wedge of 10.2° leads again to relatively weak curvature, and the resulting 4D polytope has 120 dodecahedral cells (Fig.7b).

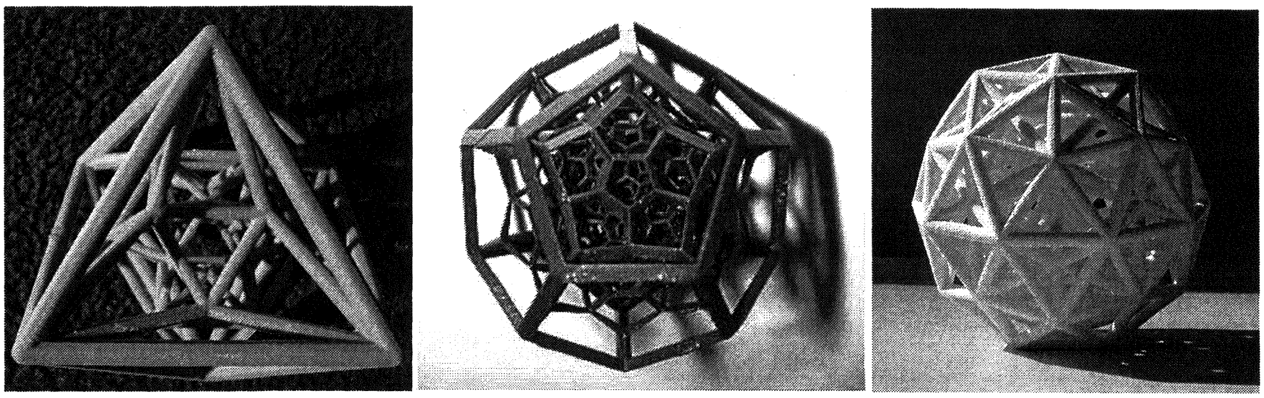

Figure 7: *Models of regular 4D polytopes: (a) 24-Cell (FDM), (b) 120-Cell (wax), (c) 600-Cell (ZCorp.).*

With the icosahedron we are out of luck. Its dihedral angle is 138.2°; thus three of them would already fill an angle of more then 360°, and therefore we cannot use this solid to make a regular 4D polytope. It is again rather intriguing, that all these six possible 4D polytopes do indeed exist. A proof of their existence is beyond the scope of this paper; interested people should consult Coxeter [1].

6. Higher Dimensions

To find all possible regular polytopes in 5 dimensions, we can reapply the procedure used to find all five Platonic solids and all six 4-dimensional regular polytopes. First we need to determine the "dihedral" angles between adjacent cells, i.e., the angle through which one cell has to be rotated around the shared face in order to be brought on top of its neighbor. If we then can fit three or more of these 4D cells around a joint face without exceeding a total of 360° for the sum of the dihedral angles, then a convex corner will form in 5D space, and there is hope that by repeating this procedure on all the other faces that are not already shared with the chosen number of neighbors, a closed regular 5D polytope will form.

Table 2 shows the dihedral angles for all six 4D polytopes. Only two of them have angles less than 120° and can thus be used to form convex 5D corners. The four larger ones are already too "round" to be useful to make positively curved corners in the next higher dimension – just like the heptagon in 2D and the icosahedron in 3D. Thus we can either pack three or four 4D simplices around a face – and indeed a 5D simplex or a 5D cross polytope will result, respectively. Alternatively, we can pack three tesseracts around each face to form the 5D measure polytope.

Table 3: Characteristics of the Regular Polytopes in >4D

	Simplex Series	Measure Polytopes	Cross Polytopes
# Vertices	$d+1$	2^d	$2d$
# Facets of dim M	$\binom{d+1}{m+1}$	$2^{d-m}\binom{d}{m}$	$2^{m+1}\binom{d}{m+1}$
# Hyper faces	$d+1$	$2d$	2^d
Dihedral angle	$\arccos(d^{-1})$	90°	$\pi-2\arcsin(d^{-0.5})$

This exact process can then be repeated for spaces of six and higher dimensions. All measure polytopes have dihedral angles of 90°, and grouping three *d*-dimensional hypercubes around a (*d*-2)-dimensional cell, will form the measure polytope in *d*+1 dimensions. Moreover, grouping three simplices will form the next higher simplex, and grouping four simplices will form the cross polytope in (*d*+1)-dimensional space. These cross polytopes have dihedral angles of 120° or more for spaces of more than three dimensions (Table 4), and they thus cannot be used to form new regular polytopes. Thus, for spaces of more than four dimensions, there are exactly three continuing series of regular polytopes: the *simplex series*, the series of *cross polytopes*, and the *measure polytopes* or *hypercubes*. There are no other special cases as there are in three and in four dimensions. We will discuss these three series in turn, and not only describe how these polytopes can be constructed most easily, but also what might be the most preferable methods of projection for viewing them.

7. Measure Polytopes

The series of n-dimensional hypercubes are readily constructed by an inductive process of extrusion. We start with a unit-length line segment representing the "measure polytope" in one dimension. By sweeping it perpendicular to itself through one unit distance, we extrude a unit square. Sweeping this square in turn parallel to itself through one unit distance in a direction that is perpendicular to the two dimensions that it already occupies will extrude a unit cube. The cube in turn is extruded along a fourth dimension into a hypercube, or "tesseract" (Fig.8). In direct analogy, every *d*+1-dimensional measure polytope is obtained by extruding the *d*-dimensional measure polytope through a unit length perpendicular to its space. In this sweep, every (*d*-1)-dimensional "face" becomes a new *d*-dimensional "cell." In addition, the starting position of the *d*-dimensional polytope becomes the "bottom" cell of the new measure polytope, and the end position of the sweep becomes its "top" cell; of course, all of these 2*d*+2 cells are equivalent.

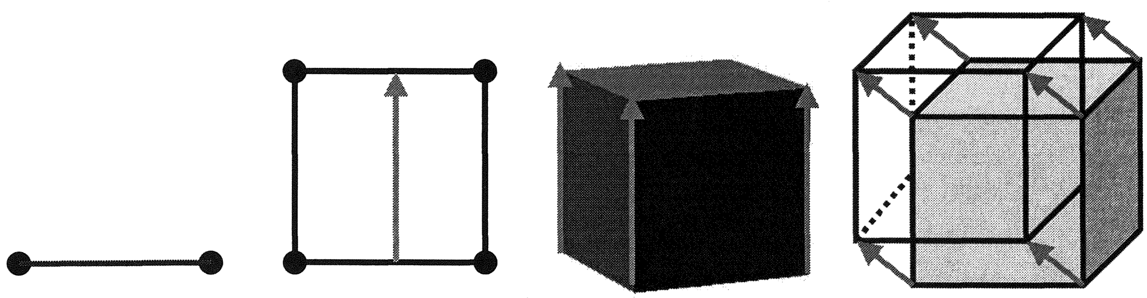

Figure 8: *The series of measure polytopes, D=1...4.*

Figure 9 shows three possibilities of how to project and visualize the 4D hypercube: a) vertex-first parallel projection assembled from individual quadrilateral frames; b) cell-first perspective view with prismatic beams; c) Cavalier-like oblique projection with beam diameter acting as a depth-cue in 4-space.

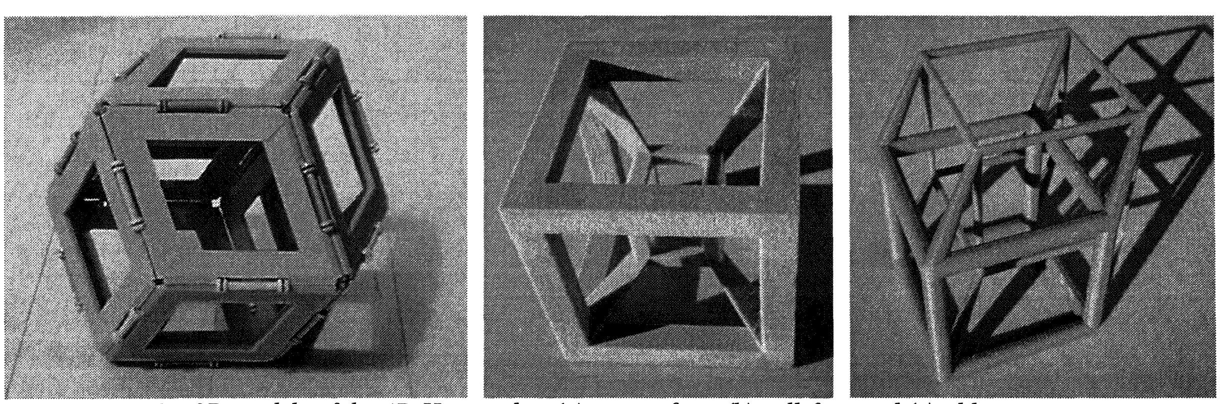

Figure 9: *3D models of the 4D Hypercube: (a) vertex-first, (b) cell-first and (c) oblique projection.*

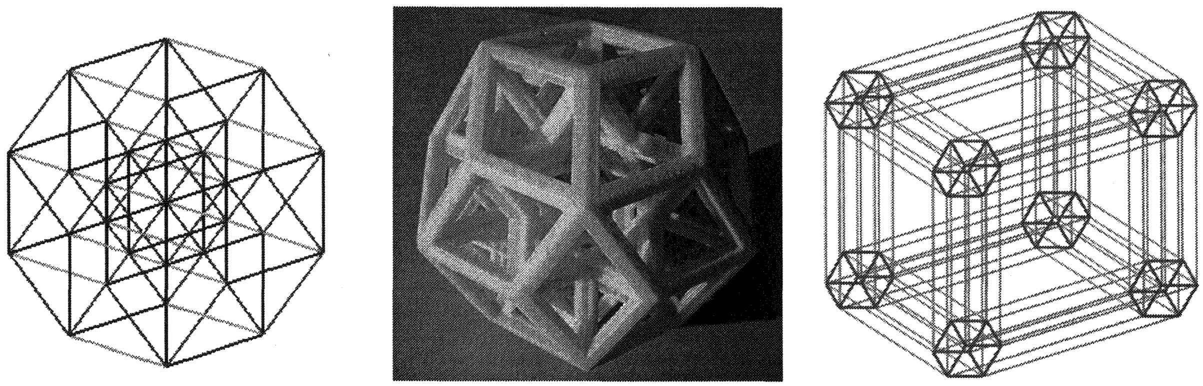

Figure 10: *Parallel projections of measure polytopes: (a) 5D vertex-first to 2D, (b) 6D vertex-first to 3D (FDM), (c) 6D oblique to 2D.*

The oblique parallel projections best convey the notion of the repeated extrusions by which the measure polytope series can be formed. So we use this approach also for the 5D and 6D hypercubes (Fig.10). Preferably we should choose the oblique projection axes so that they lie in directions that are as much separated from each other as possible. Figure 10a is a simple wire-frame drawing in two dimensions with the five axes being placed at uniformly spaced directions that lie 72° apart. This also represents a vertex-first parallel projection. Figure 10b shows a fully symmetrical vertex first projection from 6D to 3D, forming a 6th order zonohedron in which the edge bundles lie parallel to the six space diagonals in an icosahedron. Because of the high symmetry of this construction, there are a lot of edge intersections. Finally, Figure 10c uses oblique parallel projections from 6D to 2D. Three of these steps use a much stronger fore-shortening factor; thus the result looks like an ordinary wire frame of a cube, traced out with a small, compact "wire cube brush."

8. Simplex Series

The wire frame of a simplex in d dimensions consists of $d+1$ vertices that are all mutually connected and lie at unit distance from each other. Again, an inductive procedure can be used to construct them. Starting with the unit line segment in one dimension, we find its center of gravity (COG), i.e., its midpoint, and on its perpendicular bisector we search for the point that is a unit distance away from both end points; this forms an equilateral triangle. On the normal above the COG of this triangle, we find the 4th point of a tetrahedron with all unit length edges. Then, "above" (in the fourth dimension) the COG of the tetrahedron, we find the last corner of the 5-cell – and so on – for all simplices in higher dimensions. Simplexes are self-dual, i.e., they have the same count of vertices as the (d-1)-dimensional tiles that form the shell of these objects.

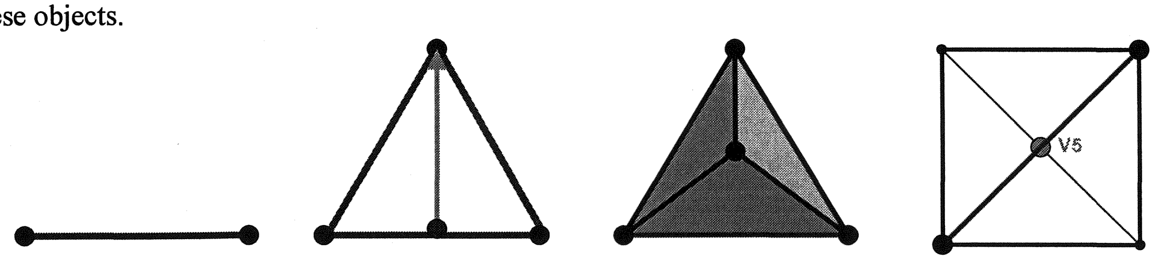

Figure 11: *The simplex series, D=1...4.*

Since the essence of these simplices is to be symmetrical clusters of points that are all equidistant from one another, we should look for projections that preserve as much of that symmetry as possible. The difficulty starts with the 5-Cell: we cannot find a way to place 5 points in a completely regular manner onto the surface of a 3D sphere. An appealing compromise solution was shown in Fig 6a, where the 5th vertex was placed into the center of a regular tetrahedron. For the model of the 5D-simplex we could simply add another internal vertex to the tetrahedron (Fig.12a). But we can do better: the 6 vertices can be placed in a completely symmetrical manner at the corners of an octahedron. However, this has the drawback that the three pairs of space diagonals now intersect in the center, which might imply the existence of a 7th vertex at that point. To avoid this degeneracy, we displace all six vertices in such a way that the three mutually perpendicular space diagonals are offset from the center by an appropriate amount. In the resulting model (Fig.12b) we have also highlighted three of its tetrahedral cells -- (and conversely assumed that the other 12 tetrahedra that touch the surface of the outer octahedron are all transparent. Since the 7 vertices of the 6D simplex cannot be placed regularly onto a sphere, we can also use this same constellation to place six of the vertices and then put the 7th vertex in its center (Fig.12c).

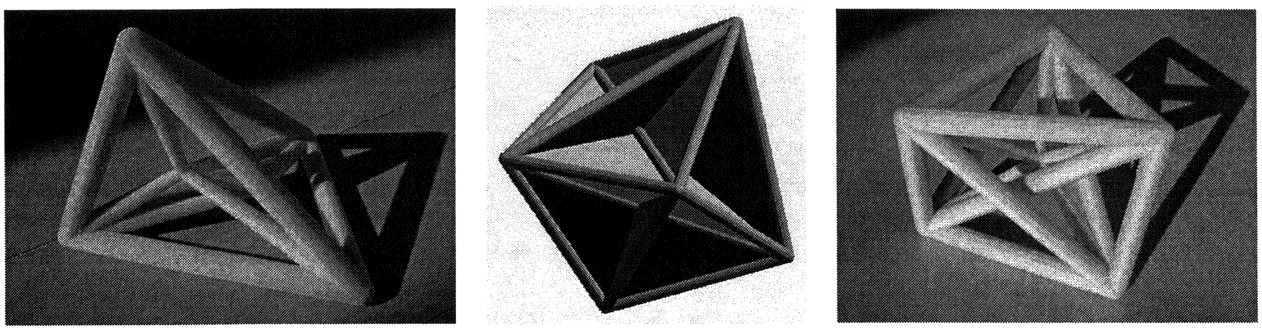

Figure 12: *3D SFF models of simplices; (a) 5D, (b) 5D with three internal tetrahedra, (c) 6D.*

Eight vertices could again be located in a symmetrical manner at the vertices of a cube, but avoiding all edge intersections becomes ever more difficult First we have to offset the four space diagonals so that they don't pass through the centroid. But we also need additional vertex offsets to eliminate the pair-wise intersections of the edges that fall onto the face diagonals of the cube.

9. Cross Polytopes.

The cross polytopes are the duals of the measure polytopes in the same *d*-dimensional space; each vertex in one corresponds to a (*d*-1)-dimensional cell in the other, and vice versa. A cross polytope can be constructed directly by placing vertices onto all the coordinate half-axes, a distance sqrt(0.5) away from the origin and then connecting all pairs of vertices that do not lie on the same coordinate axis.

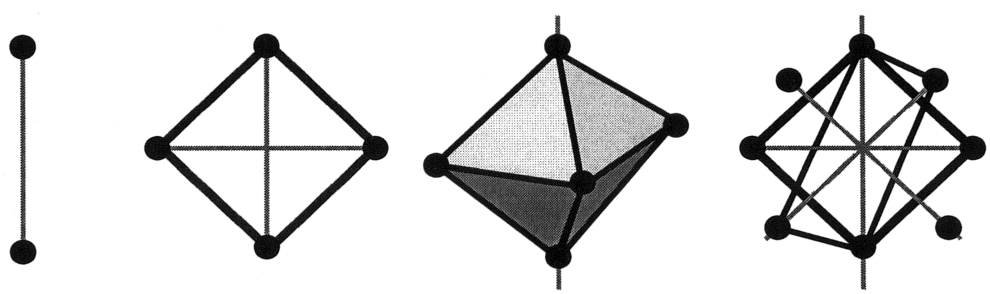

Figure 13: *The series of cross polytopes, D=1...4.*

Again we can try to maintain as much symmetry as possible among the cluster of projected points as we did for the simplices. Ideally, we might start by placing 2*d* points in a symmetrical a pattern on a sphere. For *d*=3, 4, and 6, this is rather straight forward, since we can choose three of the Platonic solids with the right number of vertices to guide us in point placement. The problem of mutually intersecting space diagonals falls away, since those point pairs need not be connected. Additional shifting of the vertex locations is necessary, if we want to avoid all edge intersections. For d=4, the intersections of the face diagonals of the cube can be avoided by giving the two interlocking tetrahedra slightly different sizes (Fig.14a).

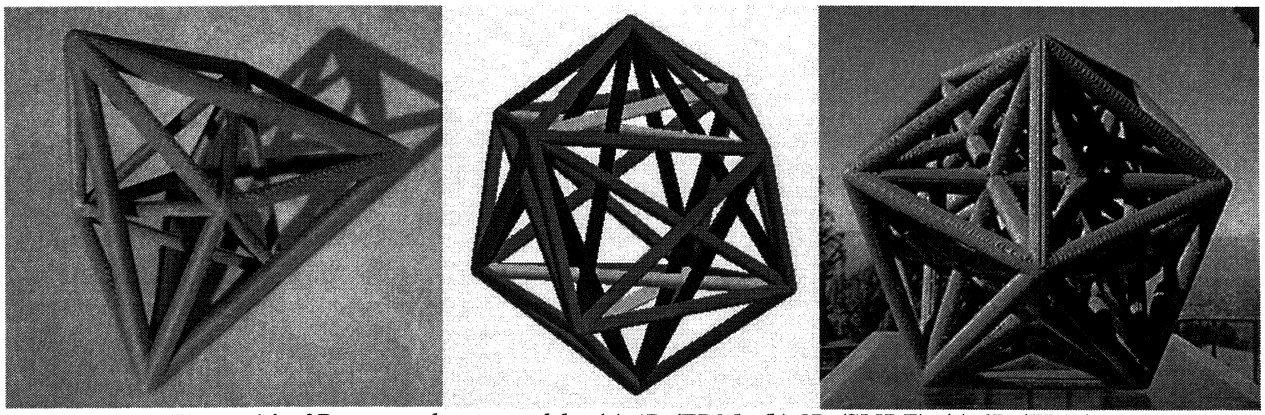

Figure 14: *3D cross polytope models: (a) 4D (FDM), (b) 5D (SLIDE), (c) 6D (FDM).*

To solve the case of 5 dimensions, we have to distribute 10 points as uniformly as possible on the sphere. Good approximations can be obtained, by placing two points at the two poles of a globe, and placing the other 8 points at the corners of a four-sided prism or anti-prism that symmetrically straddle the equatorial plane. In the first case we obtain a lot of edge intersections, on the prism faces as well as in the interior. Even when starting from the anti-prism, we need to further displace the vertices to avoid all inner edge intersections (Fig.14b). For d=6, we basically get an icosahedron with 30 additional inner edges. It is difficult to avoid all edge intersection. However, this object looks very good even with these intersections (Fig.14c), and it would be a pity to break its symmetry by shifting the vertices around.

10. Conclusions

While there are infinitely many regular n-gons in two dimensions, it comes as a surprise that there are only five Platonic solids in 3D space. When one then explores 4D space and finds that there are six regular polytopes, one expects an ever growing varieties as one moves to higher dimensions. But there is another surprise: from 5D upwards, there are only three regular polytope in each space. The reason is that except for the measure polytopes, which hold their dihedral angles at 90° regardless of their dimension, simplices and cross polytopes get gradually more round as we move up in dimensions. At d=4 we lose the cross polytope as a building block for higher-dimensional polytopes, since its dihedral angle has become 120°. Thus only hypercubes and simplices are available to construct regular higher-dimensional polytopes. Of the hypercubes it always takes three around each hyper-facet to form the proper corner. Of the simplices we can use either three or four around a shared hyper-facet. In the first case we obtain the next higher simplex, in the second case we obtain the cross polytope.

Table 4: Dihedral Angles of the Regular Polytopes in High Dimensions [1]

	D=2	D=3	D=4	D=5	D=6	D=7	D=8	D>>
Hypercubes	90°	90°	90°	90°	90°	90°	90°	90°
Simplex Series	60°	70.5°	75.5°	78.5°	80.4°	81.8°	82.8°	90°
Cross Polytopes	90°	109.5°	120°	126.9°	131.8°	135.6°	138.6°	180°

As Table 4 shows, the dihedral angle of the simplices moves closer and closer to 90°, and thus the angle-deficit to 360° when four of them are clustered becomes smaller and smaller. This implies less and less bending around that corner, and thus larger and larger objects with ever more cells. And indeed, the number of hyper-cells in the cross polytope grows exponentially (Table 3).

The math presented in this paper have been known for 40 years, but has been scattered in several publications, not all of which are easy to read. I have tried to bring some of the salient facts together in one brief document. By searching the web, enthusiasts of *regular polyhedra* and *polytopes* can find much more published material and many fascinating web sites with interactive models.

In this paper I describe how to add yet "another dimension" to these models: a sense of touch and physical reality. Well-chosen projections help the visualization and understanding of these fascinating objects. Projections into 3D-space as physical edge models are much more instructive than 2D projections onto paper or a computer screen. Hopefully the presented models have given you some new insights and enjoyment.

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