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Tonal Optimization in Music Playing

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Abstract

Music performance and enjoyment involve stylish interpretation, intonation, rhythm and specific technique associated with various instruments. While artistic interpretation is subjective – different audiences may enjoy different styles – the precision of the pitch of a certain tone, hence the harmony of a chord, is a physics phenomenon and can be determined mathematically. The enjoyment of music performances is certainly affected by whether or not the tones are generated correctly. This article attempts to explore objective criteria that can be measured mathematically for "better" or "best" playing of music in terms of harmonic chords and tonal relations in the melody. This measurement can be used in determining a tuning system of an instrument to "optimize" the playing of a certain piece of music. With the advancement of ever so sophisticated electronic digital keyboards and computers as a new generation of performing and recording instruments, one can predetermine the best tuning system for a specific piece of music. Some segments of classic music will be used as examples.

1. Introduction to Harmony

The pitch of a given musical tone is determined by the frequency of vibration of the sound wave that produces it. For example, the frequency of A_4 is widely accepted as being 440 Hz. When two or more pitches are generated simultaneously or in a rhythmic sequence, they form a chord or melody. Composers and musicians use different types of chords and melodies to express various emotions and to describe characters. Consonant (harmonic) chords or broken chords are usually adopted in pleasant themes, while agitating disturbance is often displayed with a series of dissonant chords.

Why are only certain chords perceived by humans as pleasant? Pythagoras (580 BC – 500 BC) was credited as the first to find the simple whole number ratio of consonant pitches. Take chords with two different pitches as examples. The ratio of the frequencies of the two pitches that form a perfect fifth is a simple 3 to 2 (the frequency of the higher pitch to that of the lower pitch), while that of a major third is 5 to 4. A special ratio of 2 to 1 is for two pitches in an octave, which for some unexplainable reason are perceived by human ears as the "same."

With the help of mathematics and physic, the relation can be better understood by, for example, the vibration of a string with length l of a string instrument. This vibration can be expressed as a Fourier series (For simplicity, we assume only initial displacement):

$$u(x,t)=\sum_{n=1}^{\infty} b_n\left[\sin \tfrac{n\pi}{l}(x+ct)+\sin \tfrac{n\pi}{l}(x-ct)\right]$$

The traveling wave speed $c=\sqrt{\frac{T}{\rho A}}$, where T is the tensile force exerted on the string, ρ is the density and A is the cross-sectional area of the string. The frequency of each term in the Fourier series is

$$f_n = \frac{nc}{2l} = \frac{n}{2l}\sqrt{\frac{T}{\rho A}}, \qquad n = 1,2,\cdots.$$

The frequency of the leading term, f_1, is the frequency of the pitch of that string. Every string player can easily identify with this formula. When a string sounds flat, you turn the peg to tighten the string, i.e., to increase the tension T. The lower strings are thicker, i.e., with larger cross-sectional area A, so that the pitches are lower. Aluminum strings have lower density than that of steel strings; therefore they require less tension to reach the same pitch, so they can be made thicker and softer, which is believed to make the sound mellower.

The Fourier series reveals that the sound of a vibrating string consists of a series of different pitches whose frequencies have the ratio relations $1:2:3:4:5:6:7:8\cdots$. The first and second pitches form an octave, the second and third form a perfect fifth, the third and fourth form a perfect fourth, the third and the fifth form a major sixth, the fourth and fifth form a major third, the firth and the sixth form a minor third. The number 7 or greater are considered too large for simple whole number ratios, except the ratio of 8:5, which is the inversion of major third and is called the minor sixth.

One might wonder why we do not hear all the different pitches from the same vibrating string. The reason is that the amplitude of each pitch, b_n, decreases quickly as the inverse of the square of n, and the first two dominant pitches are already in an octave relation. String players often use a technique to play only the second, third or fourth harmony by placing a finger lightly at the position of one half, one third, or one fourth of the string length. For example, placing a finger at the position of one third of the length of a string with A_4 base pitch without pressing all the way down to the fingerboard will suppress the first and second harmony but allow the entire string to vibrate. This will produce the pitch of the third harmony, E_5. Theoretically, one can produce all the pitches in the Fourier series. However, except for the first few, most of them are either too weak, or of too high frequencies to be heard by human ears.

2. Problems in Tuning

However, this simple ratio relation between harmonic pitches causes problems in scales. Take a keyboard instrument with white and black keys (in chromatic scales) as an example. If one tuned all fifths in a perfect 3:2 ratio, the ratio of the frequencies of C_8 to C_1 would be $(3/2)^{12} \approx 129.75$. This is because

$$\frac{f_{C_8}}{f_{C_1}} = \frac{f_{C_8}}{f_{F_7}}\frac{f_{F_7}}{f_{B_6^b}}\frac{f_{B_6^b}}{f_{E_6^b}}\frac{f_{E_6^b}}{f_{G_5^\#}}\frac{f_{G_5^\#}}{f_{C_5^\#}}\frac{f_{C_5^\#}}{f_{F_4^\#}}\frac{f_{F_4^\#}}{f_{B_3}}\frac{f_{B_3}}{f_{E_3}}\frac{f_{E_3}}{f_{A_2}}\frac{f_{A_2}}{f_{D_2}}\frac{f_{D_2}}{f_{G_1}}\frac{f_{G_1}}{f_{C_1}} = \left(\frac{3}{2}\right)^{12}.$$

Mean while, if all the octaves were tuned perfectly with a ratio of 1:2, the frequency ratio of C_8 to C_1 would be

$$\frac{f_{C_8}}{f_{C_1}} = \frac{f_{C_8}}{f_{C_7}}\frac{f_{C_7}}{f_{C_6}}\frac{f_{C_6}}{f_{C_5}}\frac{f_{C_5}}{f_{C_4}}\frac{f_{C_4}}{f_{C_3}}\frac{f_{C_3}}{f_{C_2}}\frac{f_{C_2}}{f_{C_1}} = 2^7 = 128.$$

This means that if C_1 of the two pianos are tuned exactly the same, then C_8 will be off.

To resolve this discrepancy, Pythagoras suggested, in addition to tuning all octaves perfectly, keeping all fifth perfectly tuned, except the enharmonic fifth $G^\#$-E^b which was tuned to the ratio of $2^{18}/3^{11} \approx 1.4798$ so that all the octaves can be kept perfectly tuned. Pythagoras based on his choice on that perfect fifths have the next simplest ratio of 3:2 after the simplest ratio of 2:1 for an octave. However, this enharmonic fifth is not the usual $3/2 = 1.5$. The two frequencies are too close to each other. In musical term this chord is said to be "flat," since the higher note sounds flat comparing to the base note. When a piece of music is played on an instrument tuned with Pythagorean system, as long as the enharmonic fifth $G^{\#}$-E^{b} is not used, all the octaves and fifths will sound perfectly. However, there are other chords that are considered to be harmonic. For example, the frequency ratio of the major third interval C-E is

$$\frac{f_{E_1}}{f_{C_1}}=\frac{f_{G_1}}{f_{C_1}}\frac{f_{D_2}}{f_{G_1}}\frac{f_{A_2}}{f_{D_2}}\frac{f_{E_3}}{f_{A_2}}\frac{f_{E_2}}{f_{E_3}}\frac{f_{E_1}}{f_{E_2}}=\frac{3}{2}\frac{3}{2}\frac{3}{2}\frac{3}{2}\frac{1}{2}\frac{1}{2}=\frac{81}{64}.$$

This is not the simple ratio $5/4=80/64$ for a major third chord. The ratio of 81/64 makes this major third chord "sharp," since the top note E_1 sounds sharp comparing to the base note C_1. It can be similarly verified that other harmonic intervals, such as minor sixths, are also imperfect. Some of them are "sharp" and some are "flat." To accommodate all the harmonic intervals, various tuning systems were proposed and adopted. One of the objects of this article is to quantify the comparison. We will discuss the difference between the different tuning systems in the following section.

3. Quantification of Off-pitches in Tuning Systems

As mentioned above, no tuning system can keep every harmonic interval in perfect simple whole number ratio. But how much off is each system comparing to others? The terminology of "cents" is thus introduced to measure this error. The frequency ratio of an octave is divided into 1200 equal ratios, each is represented by a cent, *i.e.*, two pitches are one cent apart if the ratio of their frequencies satisfies

$$\frac{f_1}{f_2}=2^{\pm\frac{1}{1200}}.$$

A pitch with frequency f_1 is n cents higher ($n>0$) or lower ($n<0$) than the pitch with frequency f_2 if

$$\frac{f_1}{f_2}=2^{\frac{n}{1200}},\text{ or } n=1200\log_2\left(\frac{f_1}{f_2}\right)=-1200_2\left(\frac{f_2}{f_1}\right).$$

For example, the note E of the major third C–E with a simple whole number ratio of 5:4 is $1200\log_2(5/4)\approx 386.3$ cents higher than C. However, in the Pythagorean tuning system, C-E has the ratio of 64/81. The top note E therefore is $1200\log_2(81/64)\approx 407.8$ cents higher than C. This is 21.5 cents higher (sharp) than that in an ideal major third chord. Therefore the major third C-E is 21.5 cents sharp in the Pythagorean tuning system.

How many cents an interval is "off" can be calculated with the following formula:

$$\delta=1200[\log_2(\text{actual ratio})-\log_2(\text{simple whole number ratio})]\equiv 1200\log_2\left(\frac{\text{actual ratio}}{\text{simple whole number ratio}}\right).$$

As mentioned earlier, the ratio of the sacrificed enharmonic fifth $G^{\#}$-E^{b} in the Pythagorean system is $2^{18}/3^{11}\approx 1.4798$. With this formula, we calculated the error for $G^{\#}$-E^{b} to be

$$\delta=1200\log_2\frac{2^{18}/3^{11}}{3/2}=1200\log_2\frac{2^{19}}{3^{12}}\approx -23.46\text{ cents},$$

i.e., this enharmonic is about 23.5 cents flat. Thus the enharmonic major thirds, for example $G^{\#}$-E^{b}, which involve the fifth $G^{\#}$-E^{b} in the tuning relation, are 2 cents flat, instead of 21.5 cents sharp. One can use this formula to calculate the errors in cents of other harmonic chords in the Pythagorean tuning system.

There have been other attempts on tuning systems to compromise the different harmonic intervals. The so called "Just Intonation" makes the three diatonic major thirds and three of the diatonic major sixths perfect by sacrificing the fifth D-A by 21.5 cents flat. The "mean-tone temperament" system makes all (except enharmonic) major thirds perfect. The perfect fifths and major sixths are all tolerable with 5.4

		Pythagorean		Just Intonation		Mean-Tone		Equal Temper	
		Ratio	±cents	Ratio	±cents	Ratio	±cents	Ratio	±cents
P5 / P4									
Diatonic	F – C	1.5000	0	1.5000	0	1.4953	–5.4	1.4983	–1.95
	C – G	1.5000	0	1.5000	0	1.4953	–5.4	1.4983	–1.95
	G – D	1.5000	0	1.5000	0	1.4953	–5.4	1.4983	–1.95
	D – A	1.5000	0	1.4815	–21.5	1.4953	–5.4	1.4983	–1.95
	A – E	1.5000	0	1.5000	0	1.4953	–5.4	1.4983	–1.95
	E – B	1.5000	0	1.5000	0	1.4953	–5.4	1.4983	–1.95
Chromatic	B – $F^{\#}$	1.5000	0	1.5000	0	1.4953	–5.4	1.4983	–1.95
	$F^{\#}$ – $C^{\#}$	1.5000	0	1.5000	0	1.4953	–5.4	1.4983	–1.95
	$C^{\#}$ – $G^{\#}$	1.5000	0	1.5000	0	1.4953	–5.4	1.4983	–1.95
	E^{b} – B^{b}	1.5000	0	1.5000	0	1.4953	–5.4	1.4983	–1.95
	B^{b} – F	1.5000	0	1.5000	0	1.4953	–5.4	1.4983	–1.95
Enharmonic	$G^{\#}$ – E^{b}	1.4798	–23.5	1.4983	–1.95	1.5312	+35.7	1.4983	–1.95
M3 / m6									
Diatonic	F – A	1.2656	+21.5	1.2500	0	1.2500	0	1.2599	+13.7
	C – E	1.2656	+21.5	1.2500	0	1.2500	0	1.2599	+13.7
	G – B	1.2656	+21.5	1.2500	0	1.2500	0	1.2599	+13.7
Chromatic	D – $F^{\#}$	1.2656	+21.5	1.2500	0	1.2500	0	1.2599	+13.7
	A – $C^{\#}$	1.2656	+21.5	1.2656	+21.5	1.2500	0	1.2599	+13.7
	E – $G^{\#}$	1.2656	+21.5	1.2656	+21.5	1.2500	0	1.2599	+13.7
	E^{b} – G	1.2656	+21.5	1.2656	+21.5	1.2500	0	1.2599	+13.7
	B^{b} – D	1.2656	+21.5	1.2656	+21.5	1.2500	0	1.2599	+13.7
Enharmonic	B – E^{b}	1.2486	–1.95	1.2642	+19.6	1.2800	+41.1	1.2599	+13.7
	$F^{\#}$ – B^{b}	1.2486	–1.95	1.2642	+19.6	1.2800	+41.1	1.2599	+13.7
	$C^{\#}$ – F	1.2486	–1.95	1.2642	+19.6	1.2800	+41.1	1.2599	+13.7
	$G^{\#}$ – C	1.2486	–1.95	1.2642	+19.6	1.2800	+41.1	1.2599	+13.7
M6 / m3									
Diatonic	F – D	1.6875	+21.5	1.6875	+21.5	1.6719	+5.4	1.6818	+15.6
	C – A	1.6875	+21.5	1.6667	0	1.6719	+5.4	1.6818	+15.6
	G – E	1.6875	+21.5	1.6667	0	1.6719	+5.4	1.6818	+15.6
	D – B	1.6875	+21.5	1.6667	0	1.6719	+5.4	1.6818	+15.6
Chromatic	A – $F^{\#}$	1.6875	+21.5	1.6875	+21.5	1.6719	+5.4	1.6818	+15.6
	E – $C^{\#}$	1.6875	+21.5	1.6875	+21.5	1.6719	+5.4	1.6818	+15.6
	B – $G^{\#}$	1.6875	+21.5	1.6875	+21.5	1.6719	+5.4	1.6818	+15.6
	E^{b} – C	1.6875	+21.5	1.6875	+21.5	1.6719	+5.4	1.6818	+15.6
	B^{b} – G	1.6875	+21.5	1.6875	+21.5	1.6719	+5.4	1.6818	+15.6
Enharmonic	$F^{\#}$ – E^{b}	1.6648	–1.95	1.6856	+19.6	1.7120	+46.4	1.6818	+15.6
	$C^{\#}$ – B^{b}	1.6648	–1.95	1.6856	+19.6	1.7120	+46.4	1.6818	+15.6
	$G^{\#}$ – F	1.6648	–1.95	1.6856	+19.6	1.7120	+46.4	1.6818	+15.6

Table 1: Ratios *and errors of harmonic intervals of various tuning systems.*

cents flat. However, the enharmonic intervals are all badly off by 36 to 46 cents, which is almost a quarter of a whole step! The enharmonic fifth $G^{\#}$-E^{b} sounds so badly it is often referred to as "the wolf" tone.

To get rid of all those tuning headaches, most contemporary instruments are tuned in the "equal temperament" system, which has an equal ratio, $2^{\frac{1}{12}}$, of all half steps, *i.e.*, all half steps are 100 cents apart. Thus all the perfect fifths are of equal ratio, as well as the thirds and sixths. However, in order to spread out the error, none of the chords (except for the octaves, of course) is perfect. It spreads out the nearly 24 cents flat of enharmonic fifth $G^{\#}$-E^{b} in Pythagorean system throughout the 12 fifth intervals, therefore each fifth is about 2 cents flat. With the formula, we can calculate the major thirds to be about 14 cents sharp, since

$$\delta = 1200 \log_2 \frac{2^{\frac{4}{12}}}{5/4} \approx 13.7 .$$

Similarly, all the major sixths are 15.64 cents sharp. Since the octaves are always tuned perfectly, the perfect fourths, minor sixths, and minor thirds are inversions (by raising the bottom notes one octave higher) of perfect fifths, major thirds, and major sixths. They are thus 2 cents sharp, 13.7 cents flat and 15.6 cents flat respectively.

The summary of the ratios and errors of each of the four tuning systems is listed in Table 1. In the table the twelve notes, C, $C^{\#}$, D, E^{b}, E, F, $F^{\#}$, G, $G^{\#}$, A, B, and B^{b}, in a chromatic scale are used. Other accidentals are considered enharmonic. For example, $D^{\#}$ is considered the same note as E^{b} (Even though a violinist often plays them differently.).

4. Tonal Optimization

After the errors in different tuning systems are quantified, we can mathematically discuss the "best" for a specific piece of music. Many criteria for "best" tones can be proposed, we list some considerations in the following:

(1) All octaves must be in perfect 2:1 ratio, so that the ratio of any two pitches remains the same for all octaves. Thus the domain for the pitches can be restricted within one octave.
(2) Only the errors in consonant intervals P5 (therefore P4), M3 (m6) and M6 (m3) are considered to be the factors that affect the music performance or enjoyment.
(3) The value (time length) of each note is also taken into account. The longer the interval is played, the more important this interval is to the entire piece.
(4) The importance of different intervals may vary according to each specific piece of music.
(5) Even for intervals with same steps, one may be more important than others. For example, the M3 interval C-E is certainly of more prime importance in musics composed in C Major than D^{b}-B^{b}.

For the purpose of discussion, we consider a very simple least-square type error function

$$g = \sum_{i=1}^{n} \phi_i (\delta_i)^2 ,$$

where $\phi_i \geq 0$ are the weights that depend on the value of the notes, the importance of that particular interval or chords, or other factors. The optimal performance of a piece of music in the sense of "best" intonation can be considered as an optimization problem,

$$\min_{t \in S} g(t, m),$$

where S is the set of all feasible tuning systems, t is a particular tuning method to be chosen from S, m represents a particular piece of music to be performed with instrument(s) tuned in t. To simplify our examples and discussions, we consider the weight function only to be proportional to the value of the function, i.e., we consider all harmonic intervals (P5, P4, M3, m3, M6, m6) to be of equal importance. Thus $\phi_i = v_i$, where v_i is the time value of the notes of the interval. Certainly the time value depends on the variation of tempo during the performance. Again, we use a steady tempo in our examples.

To illustrate the application of the error function, we considered two keyboard music pieces: J. S. Bach's Saraband in *a* from the First French Suite and Brahms' Waltz in A^b, Op. 39, No. 15. To save time, we only used each piece's first theme. In counting the intervals, a chord with three or more notes is counted as two or more intervals, with the base note against each of the rest notes in the chord. This is obviously much too simplified. However, we only use this as illustrations that are easy to understand. The tally of all the occurring harmonic intervals and value of each note are listed in Table 2.

Frequency (Occurrence) of harmonic intervals		*J. S. Bach's Saraband in a*			*Brahms' Waltz in A^b*		
		1/8 note	¼ note	½ note	1/8 note	¼ note	3/8 note
P5 / P4	$C-G$					2	1
	$C^\#(D^b)-G^\#(A^b)$			1		2	
	$D-A$		1				
	$F-C$	1				1	2
	$G-D$	1	1				
	$G^\#(A^b)-E^b(D^\#)$					11	1
	$A-E$	2	2	1			
	$B^b(A^\#)-F$		1				
M3 / m6	$C^\#(D^b)-F$		1			2	
	$D-F^\#(G^b)$	1		1			
	$E^b(D^\#)-G$	2				1	
	$F-A$	2	1				
	$G^\#(A^b)-C$		1		8	6	3
	$A-C^\#(D^b)$	2		1			
	$B^b(A^\#)-D$	1	1				
M6 / m3	$C-A$		3				
	$C^\#(D^b)-B^b(A^\#)$					2	
	$D-B$					1	
	$E^b(D^\#)-C$				1	5	5
	$F-D$		2			1	
	$F^\#(G^b)-E^b(D^\#)$	1					
	$G-E$	1		1			
	$G^\#(A^b)-F$					6	
	$B^b(A^\#)-G$	2	2	1			

Table 2: *Tally of harmonic intervals and note values in the theme of J. S. Bach's Saraband in a from First French Suite, and Brahms' Waltz in A^b, Op. 39, No. 15.*

With the tally from Table 2 and the error (in cents) of each of the harmonic intervals in the four tuning systems from Table 1, we obtained the following results of the error function g or each of the two pieces of music.

	Pythagorean	Just intonation	Mean-tone	Equal Temperament
Bach	2548	1974	1279	1340
Brahms	3736	4099	15733	2242

From the result one may conclude that in general the equal temperament system works better in most cases. The reason is that the equal temperament spreads out the errors, while other systems may involve some of the chords that are off by 20 or more cents.

5. Method of Spectrum Shifting

We have already seen that different tuning systems give the error function g different values. However, one can create numerous different tuning systems, each being tolerable. To search for a best system that minimizes the error function g for specific piece of music is too big a topic to be included in this article. Nevertheless we would go a little further than just examine the existing tuning systems.

It is obvious that the same music transposed into different keys should have different values for the error function, since the errors (in cents) do not distribute uniformly for all the chords with the same interval, with the exception of the equal temperament system. Thus it is natural to consider redistributing those errors to minimize the error function. Again as an example for illustration, we consider a simple method – spectrum shifting, i.e., we shift the ordered error distribution list in haft steps on the chromatics scale bases. For example, in Pythagorean system, we can move the sacrificed enharmonic fifth $G^{\#}$-E^{b} one half-step up to A-E, or two half-steps up to B^{b}-F, etc. The error distributions in table 1 are referred as the spectrum of corresponding tuning system in "C". Thus each tuning system has 12 variations, they are in C, $C^{\#}$, D, E^{b}, E, F, $F^{\#}$, G, $G^{\#}$, A, B^{b} and B respectively. Table 3 lists the error distributions of just-intonation system with all 12 shifts of spectrum.

Spectrum		C	$C^{\#}$	D	E^{b}	E	F	$F^{\#}$	G	$G^{\#}$	A	B^{b}	B
P5/P4	C – G	0	0	0	0	–1.95	0	0	0	0	0	–21.5	0
	$C^{\#}$ – $G^{\#}$	0	0	0	0	0	–1.95	0	0	0	0	0	–21.5
	D – A	–21.5	0	0	0	0	0	–1.95	0	0	0	0	0
	E^{b} – B^{b}	0	–21.5	0	0	0	0	0	–1.95	0	0	0	0
	E – B	0	0	–21.5	0	0	0	0	0	–1.95	0	0	0
	F – C	0	0	0	–21.5	0	0	0	0	0	–1.95	0	0
	$F^{\#}$ – $C^{\#}$	0	0	0	0	–21.5	0	0	0	0	0	–1.95	0
	G – D	0	0	0	0	0	–21.5	0	0	0	0	0	–1.95
	$G^{\#}$ – E^{b}	–1.95	0	0	0	0	0	–21.5	0	0	0	0	0
	A – E	0	–1.95	0	0	0	0	0	–21.5	0	0	0	0
	B^{b} – F	0	0	–1.95	0	0	0	0	0	–21.5	0	0	0
	B – $F^{\#}$	0	0	0	–1.95	0	0	0	0	0	–21.5	0	0
M3/m6	C – E	0	+19.6	+21.5	+21.5	+19.6	0	+19.6	0	+21.5	+21.5	0	+19.6
	$C^{\#}$ – F	+19.6	0	+19.6	+21.5	+21.5	+19.6	0	+19.6	0	+21.5	+21.5	0
	D – $F^{\#}$	0	+19.6	0	+19.6	+21.5	+21.5	+19.6	0	+19.6	0	+21.5	+21.5
	E^{b} – G	+21.5	0	+19.6	0	+19.6	+21.5	+21.5	+19.6	0	+19.6	0	+21.5
	E – $G^{\#}$	+21.5	+21.5	0	+19.6	0	+19.6	+21.5	+21.5	+19.6	0	+19.6	0
	F – A	0	+21.5	+21.5	0	+19.6	0	+19.6	+21.5	+21.5	+19.6	0	+19.6
	$F^{\#}$ – B^{b}	+19.6	0	+21.5	+21.5	0	+19.6	0	+19.6	+21.5	+21.5	+19.6	0
	G – B	0	+19.6	0	+21.5	+21.5	0	+19.6	0	+19.6	+21.5	+21.5	+19.6
	$G^{\#}$ – C	+19.6	0	+19.6	0	+21.5	+21.5	0	+19.6	0	+19.6	+21.5	+21.5
	A – $C^{\#}$	+21.5	+19.6	0	+19.6	0	+21.5	+21.5	0	+19.6	0	+19.6	+21.5
	B^{b} – D	+21.5	+21.5	+19.6	0	+19.6	0	+21.5	+21.5	0	+19.6	0	+19.6
	B – E^{b}	+19.6	+21.5	+21.5	+19.6	0	+19.6	0	+21.5	+21.5	0	+19.6	0
M6/m3	C – A	0	+21.5	+21.5	+21.5	+19.6	0	+19.6	+21.5	+21.5	+21.5	0	+19.6
	$C^{\#}$ – B^{b}	+19.6	0	+21.5	+21.5	+21.5	+19.6	0	+19.6	+21.5	+21.5	+21.5	0
	D – B	0	+19.6	0	+21.5	+21.5	+21.5	+19.6	0	+19.6	+21.5	+21.5	+21.5
	E^{b} – C	+21.5	0	+19.6	0	+21.5	+21.5	+21.5	+19.6	0	+19.6	+21.5	+21.5
	E – $C^{\#}$	+21.5	+21.5	0	+19.6	0	+21.5	+21.5	+21.5	+19.6	0	+19.6	+21.5
	F – D	+21.5	+21.5	+21.5	0	+19.6	0	+21.5	+21.5	+21.5	+19.6	0	+19.6
	$F^{\#}$ – E^{b}	+19.6	+21.5	+21.5	+21.5	0	+19.6	0	+21.5	+21.5	+21.5	+19.6	0
	G – E	0	+19.6	+21.5	+21.5	+21.5	0	+19.6	0	+21.5	+21.5	+21.5	+19.6
	$G^{\#}$ – F	+19.6	0	+19.6	+21.5	+21.5	+21.5	0	+19.6	0	+21.5	+21.5	+21.5
	A – $F^{\#}$	+21.5	+19.6	0	+19.6	+21.5	+21.5	+21.5	0	+19.6	0	+21.5	+21.5
	B^{b} – G	+21.5	+21.5	+19.6	0	+19.6	+21.5	+21.5	+21.5	0	+19.6	0	+21.5
	B – $G^{\#}$	+21.5	+21.5	+21.5	+19.6	0	+19.6	+21.5	+21.5	+21.5	0	+19.6	0

Table 3: *Error distribution by spectrum shifting of Just Intonation system.*

We use this method to exam the Brahms' Waltz in A^b. The result is as following:

Spectrum Shift of Just Intonation											
C	$C^\#$	D	E^b	E	F	$F^\#$	G	$G^\#$	A	B^b	B
Value of Error Function g											
4099	212	3852	1733	4644	4489	3390	3813	443	4108	4852	4429

Since Brahms' Waltz is in A^b, the shifting of error spectrum to $A^b(G^\#)$, as expected, yields much better results. The error function shrank nearly 90% from the usual Just Intonation system (C-spectrum). However, it is interesting to notice that the lowest errors actually resulted from the spectrum in $C^\#$. This intricate aspect shows that the tonal optimization is not a trivial process, since so many variations of chords were involved in a composition. The key signature of a piece of music is a good indication what spectrum to use with a particular tuning system, but may not give the best result. A systematic mathematical study of the optimal tuning is necessary, and also is now practical with modern equipment and electronic gadgets.

6. Conclusion

We have discussed the classic tuning problem in music. The mathematical aspect of "best" tuning system (tonal optimization) was proposed. Simple illustrative examples were given to show the practicality and usefulness, though they may be considered a little too simplistic.
Further study of tonal optimization can go beyond the classic tuning systems. We can even consider a "continuous" optimization by allowing the frequencies to vary continuously within a certain domain. For example, we can propose:

$$\min_{\vec{x}\in S} g\{t(\vec{x}), m\},$$

where the tuning system t is a function of the twelve pitches of a chromatic scale, as denoted by $\vec{x}$. If we use cents as the unit of the pitches, the components of the twelve-dimensional vectors are $x_i = 1200\log_2 f_i$, where f_i is the frequency of the ith note in the chromatic scale. The domain S can be a sphere in a twelve dimensional real space: $\|\vec{x}-\vec{x}_0\| \le \varepsilon_0$, where $\vec{x}_0$ represents a reference chromatic scale, e.g., that of the equal temperament tuning system, and 5 to 15 cents may be chosen as a reasonable tolerance value for ε_0.

The further discussion of how to form the error function g is desired. Music performing artists and composers may have more say-sos in this aspect. How to assess the errors in a chord with multiple intervals, as well as the errors occurring incidentally between two intertwining themes, are subjected to human preferences. My over simplified examples are for illustration only, and should not be considered as an exclusion of artistry.

With more and more widely used electronic digital instruments and mixing devices, optimizing tuning system for each piece of music and each performance has become feasible and easier. Mathematical theories and methods will become an integrated part of music performing.

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