A Mathematical Look at the 7 Liberal Arts as Humanities

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Abstract

What is "art" as opposed to "tech"? What is "liberal" (as opposed to what)? Where did the notion of "7 Liberal Arts" come from, and how are they related to human nature so as to merit being considered "humanities"? By historical accident, the mediæval scholastic curriculum was based on a minor work of the last period of Roman imperial decline before on-set of the Dark Ages. Nevertheless, there are mathematical viewpoints which can legitimately relate the arts of the Trivium (grammar, rhetoric, logic) to child development on the one hand and those of the Quadrivium (arithmetic, geometry, music, astronomy) to the development of harmonic analysis on the other.

1. "Art" vs. "Tech," and their Common Background in Classical Greece

What are the "arts," as opposed to the "tech" subjects? Until re-arranged by the 1994 earthquake, the C.S.U. Northridge campus presented a fairly stereotypical lay-out illustrating the common perception of this distinction: The music, drama, dance, and graphic arts depts. with their respective listening and viewing spaces were all on the (sunny) south side of campus with ready public access, whereas the engineering and computer science depts. were relegated to the (cold) north side with no public access roads. I ask students in my Math. Through the Liberal Arts course (Math 131) to look up "art" and "tech" in the *American Heritage Dictionary* and discover their true meanings (this dictionary featuring a special appendix showing whole Indo-European word families). Both turn out to mean a craft activity of putting things together in some sense, the Latin *ars* as joinery ("articulate," "carpentry," "harmony") and Greek *techne* as weavery ("textile," also Spanish *tejado* "roof" and *tejido* "fabric") — thus a non-distinction: a Greek *tekton* is still a carpenter! For similar ironic treatment of conservationist "liberals" vs. free-market "conservatives," see *Doonesbury* (e.g. Sunday strip for May 20, 2001).

It may be objected that carpenters and weavers are artisans or technicians, not artists, but that is more a question of the perceived social standing of such workers than of the intrinsic nature of their craft. Characteristically, technical universities (*T.H.*'s) split off from traditional liberal-arts-based universities in Germany at the turn of 18th to 19th centuries just at start of the Romantic era, much to the consternation of poet / civil servant Goethe. Now, at turn of 20th to 21st centuries, such once separate institutions are renaming themselves plain "universities" again yet requiring little or no liberal arts courses of their graduates. Is the distinction merely one of cachet, or is there a deeper catch?

The question of which arts are "fine" (exerting a refining influence on the humans exposed to them, as a precursor to wisdom) and which are merely ones with "finesse" (having a pleasing effect on the vain or foolish) was raised and quite emphatically answered by Plato: Music and gymnastics are recommended in Book 2 of *The Republic* as appropriate initial training for young minds and bodies, understanding by music all inspirations of the muses including "true" literature but warning against "false" (or what we would call today "fiction") [1, p. 220]; Book 3 discusses merits of various scales in the narrower sense of music [1, p. 246]; while Book 7 briefly touches on "1,2,3" (arithmetic), geometry, astronomy, and logic [1, pp. 385, 390, 393, and 406]. Grammar is not recognized as a separate topic in any of the Platonic dialogues, and rhetoric is ridiculed as flattery in both *Phædrus* (talking with Lysias) and *Gorgias* (talking with Callicles), Socrates in the latter comparing sophistry and rhetoric to cosmetic and confectionery [2, p. 817]. The order of argumentation seems not to have mattered much: "He works freely, and is not to be supposed to have arranged every part of the dialogue before he begins to write" (in Jowett's introduction to *Phædrus*, quoted in [2, p. 818]), for "dialectic was still a 'world not realized,' and he was continually conscious of using imperfect methods" of logical argumentation [2, p. 818]. The truth of

what was said was what mattered essentially to the philosopher seeking wisdom, not the pleasantness of *how* it was said.   Rhetoricians got in trouble with philosophers for being willing to argue on either side of a matter, depending on who was paying for it, much as lawyers do today.

2.   The Fall of Rome, and Transitions to Middle Ages and Renaissance

Imagine, then, how Plato would feel to find rhetoric elevated to the status of a divine art by another writer at the close of the Roman era!   It could only have been the work of a lawyer (with considerable independent means), and that is exactly what it was: The allegorical fantasy *De Nuptiis Philologiæ et Mercurii* or *The Marriage of Philology and Mercury*, by the North African Neo-Platonist Martianus Capella, "of whom we should care to know little, if the intrinsic merit of [his] work were the only thing concerned, but of whom we cannot know enough, because of the influence that [it has] exerted upon the subsequent course of human affairs [3, p. 16]."

"In the pages of any other author, writing at the beginning of the fifth century, it would have been strange to find no mention of Christianity, but it is not strange in Martianus. Prolix and self-satisfied, he seems to have cared less for the historical events that were going on about him than even the Christian writers of those times; and indeed, although the Roman Empire, as we have been accustomed to consider it, was toppling to ruins all about him, it is a fact that neither he nor other writers of the time appear to have grasped or appreciated the mighty phenomenon. His only care is to amuse himself and his readers with neo-Platonic mythology and allegory; and facts and events have little interest for him except as symbols. The empirical details which he was compelled to include in his work upon the several liberal arts were too tiresome to occupy much of his attention, and it was fortunate, perhaps, for succeeding generations, that Capella preferred to cull them directly from Varro as to grammar, dialectic, geometry and astronomy, from Aquila Romanus upon rhetoric, from Solinus and Pliny on geometry and geography, and from Aristides Quintilianus on music [3, pp. 16-17]."

Written in nine books, the first two giving the framework (the only original part of Capella's effort) and latter seven more or less cut-and-pasted from the late Roman authors cited above, the *Marriage* allegory begins in Book 1 by imagining Mercury, as god of communication, having "first sighed in vain for Sophie [wisdom], Mantice [prophecy], and Psyche [soul]," being "advised by Virtue to address himself to Apollo," who in turn suggests "Philology, the most learned daughter of the family of the ancients ... who knows the mysteries of the lower world as well as the will of Jupiter, the depths of the ocean as well as the kingdom of the stars; in a word, she is encyclopedic knowledge [3, p. 20]." Mercury accepts the suggestion.

The bride appears in Book 2, expressing initial fear of union with so august a mate, "but after a long calculation, she recognizes, by the numbers which form her name and that of her betrothed, that this marriage is very suitable for her. ... Athanasia [immortality], the daughter of Apotheosis [exaltation], afterwards appears to accompany Philology in the heavens. But she has first, by command of Athanasia, to get rid of a burden which inordinately enlarges her breast. She resigns herself, though with regret, to surrender a great quantity of books, which are gathered up by young maidens, ... [and then] ascends to heaven in a sedan chair [3, p. 21]."

There is no complete translation of the work in English (Cole offering an extract from Book 4 on Dialectic [in 3, pp. 22-28]), but the pertinent bit of gematrical name-reckoning may be found in Barry [4, p. 194]: Capella takes the Greek form of groom Mercury's Egyptian* name *Thouth* (Θωυθ) whose letters total 9+800+400+9 = 1218 (associating the first, second, and third sets of nine letters in Greek alphabet with numbers 1 to 9, 10 to 90, and 100 to 900, including obsolete Ϝ = 6, ϟ = 90, and ϡ = 900), the Greek form of bride Philology's name *Philologia* (Φιλολογια) whose letters total 500+10+30+70+30+70+3+1+1 = 724, reduces them by "rule of 9" to 1+2+1+8 = 12 → 1+2 = 3 and 7+2+4 = 13 → 1+3 = 4 (appropriately odd and even for male and female), whose sum 3+4 = 7 is the propitious marriage sign, and motivation for the canonizing of 7 liberal arts as bridesmaids. *[Capella could not have used Greek name Hermes ('Ερμης) since its letter sum 353 is congruent to 2 — unsuitably feminine for a groom — when reduced by mod 9 rule.]

As example of how the 7 arts were viewed by mediæval scholastics, 12th cent. abbess Herrad von Landsberg [5, frontispiece] displays them thus in her *Garden of Delights*: We find (clock-
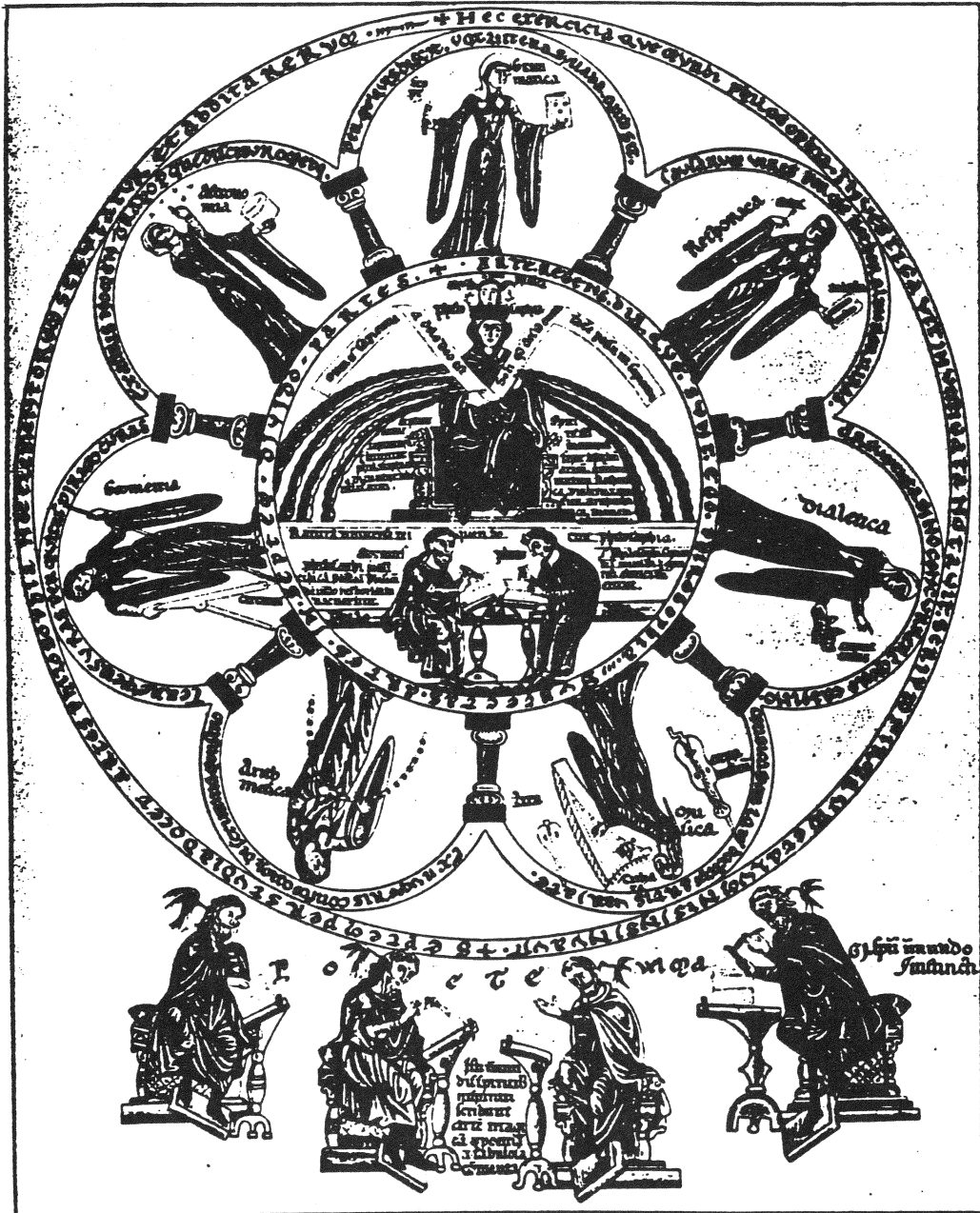
Herrad of Landsberg

Hortus Deliciarum

(The Seven Liberal Arts as
Garden of Delights)

wise from top) Grammatica, Rethorica, Dialectica, Musica, Arithmetica, Geometria, and Astronomia. They are still refined ladies, but no longer bridesmaids of Philology (love of the word), grouped instead about the central figure of Philosophia (love of wisdom). Enjoying the protection of her inner sanctum we find Socrates and Plato (still revered, though the aberrant Greek spellings warn us that such authors were known only in summarized Latin translations). At bottom, beyond the pale, we find figures of *Poete vel Magi* (Poets or Magicians), namely Magica, Poetria, and Fabulosa Commenta (fiction), inspired by dubious Edward-Gorey-like birds at their ears.

As example of the surprisingly far reach of Capella's influence, it should be noted that a passage in Book 8 "contains a very clear statement of the heliocentric system of astronomy. It has been supposed that Copernicus, who quotes Capella, may have received from this work some hints toward his own new system [6, p. 249]." This mercurial wedding may thus have formed the fateful link not only from Classical Antiquity to Middle Ages, but also from Middle Ages to Renaissance (though the astronomical credit should go to his source in Varro).

3. The Trivium as Skills of the First Three Years of Childhood

When I began teaching Math 131 at CSUN in 1984, I was faced with choice between math history texts presuming calculus or "poets' math" texts presuming near innumeracy. 7 years later, at my students' request, I began writing a text for their needs [7], aimed primarily at English and history and journalism majors, with the intent of using story-telling devices to hook their interest and pique their curiosity. Math, like all else done by humans, is — must be! — an expression of our humanity, so we study problem-solving techniques not for sake of the solutions but to get a feel for the differing situations and characters of the peoples who did them, their times and places. I decided to base this course literally on the 7 liberal arts, since the takers were what the university classified as liberal arts students. But only the 4 upper subjects of the Quadrivium (arithmetic, geometry, music and astronomy) seemed evidently math-related; of the 3 lower subjects of the Trivium (grammar, rhetoric, and logic) only logic sounded "mathy" — what to do with the other 2?

I was reading at the time with an elderly friend, who had fallen and greeted me at her door sporting a pair of crutches, saying proudly "I've been studying the cats and dogs in the neighborhood, and this is how they walk ... ." Now I had prided myself in "walking" (as I thought) with my fingers on the desk during idle study hall hours in high school, but used a diagonally paired gait that is actually a "trot," as I later learned. This near blind woman on crutches was showing me true 4-footed walking (left rear, left front, right rear, right front), which turns out to be of same pattern as human infant crawling. The dictionary revealed that *grammata* are, in fact, foot-prints (cognate with "tracks" and "graphs"). The Trivium subjects grammar, rhetoric, and logic are walking, talking, and thinking — the skills of first 3 years of childhood. They are truly "humanities," for they make us fully human; only after their mastery does individual memory begin.

• As a true grammar schooling, we may consider tracks left by human steps or other motions:

(1)	11	L L L L L L		
(2)	1g	L Γ L Γ L Γ		
(3)	m1	V V V V V V		
(4)	12	N N N N N N		
(5)	mg	V Λ V Λ V Λ		
(6)	1m	D D D D D D		
(7)	mm	H H H H H H		

The various forms of 1-dimensional locomotion leave their traces as 7 frieze groups, 1-footed hopping being type (1), 2-footed walking type (2), crab-wise side-shuffle (3), snake-like side-winding (4), two-to-tango (5), bunny-hop (6), and hambo (7), to use dance analogies. [2nd col. gives international crystallographic symbols.] Walking can also be used to illustrate the four Galois axioms for a group: It is associative [doesn't matter whether one steps L(RL) or (LR)L], has inverse elements (back-steps), identity (standing still), and closure ("Wherever you go, there you are").

•   But in order to walk successfully one must be *coordinated*, able to combine the motor skills of forward motion with rhythmic exchange of weight between two feet, which is what makes it so difficult for toddlers.   Mathematically, all frieze patterns are in fact Cartesian products of two groups, the infinite group of integers under addition and 2-element group of exchanges between mirror image feet. Thus the 1-dimensional-seeming line of footprints ꝺꝺ ꝺꝺ ꝺꝺ ꝺꝺ ꝺꝺ ꝺꝺ can be viewed e.g. as corresponding to 2-dimensional group elements (-2,I), (-1,M), (0,I), (1,M), (2,I), (3,M) etc., convening the left foot to be identity I and right foot its mirror image M, coordinated with e-ven and odd numbers of steps forward or backward, respectively.   When the line becomes a 1-dim. graph in a 2-dim. plane, its Cartesian equation $Ax + By = 1$ is conventionally interpreted as (x,y) "footsteps' — i.e. points — along a linear "road" with $1/A$ and $1/B$ its x- and y-intercepts. This is appropriate, historically, since René Descartes had been a footsoldier, fighting as mercenary in Thirty Years War, before the fateful night of his dream (Nov. 10, 1619 — cf. [8]) when a voice spoke to him out of a whirlwind and admonished him to change his ways.   What is little realized, due to current neglect of projective geometry, is that the same equation could be viewed not only as all points (x,y) lying on line [A,B] but as all lines [A,B] passing through point (x,y), as illustrated by pair of graphs below showing dual interpretations of $1 \cdot x + 2 \cdot y = 1$ vs. $A \cdot 1 + B \cdot 2 = 1$:
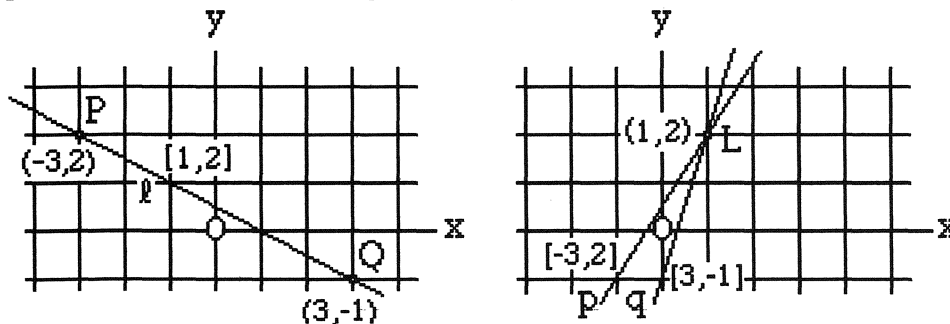
In one case we see footsteps from P to Q along ℓ (gesture of march), in the other a rotation from p to q about L (gesture of whirlwind).   The rhetorical or linguistic analogue, I suggest, is the placement of adjectives before or after nouns in Germanic vs. Romance languages (*red house* or *rotes Haus* vs. *maison rouge* or *casa roja*). Moreover, the deeper connection to rhetoric — its reason for being reviled by Plato and what the whirlwind was presumably admonishing Descartes against, arguing either side of a trial or shooting men on either side of a war, depending on who is paying — is, I suggest, redeemed by this projective interpretation: both views are indeed legitimate!

•   "What was peculiar to Descartes … was the speculative interest which made it impossible for him to rest in mere experience, whether of things spiritual or of things secular … . In both cases it is his aim to go back to the beginning, to retrace the unconscious process by which the world of experience was built up, to discover the hidden logic that connects the different parts of the structure of belief, to substitute a reasoned system, all [of] whose elements are interdependent, for an unreasoned congeries of opinions. Hence his first step involves reflection, doubt and abstraction [9, p. 414]." Doubt — tentative denial — is thus the first step in learning to think for oneself, and that is just what the 2-going-on-3-year-old does: say "No" even to its favorite dessert, just to see what effect this has on mommy and daddy. The negation of a proposition p can be written many ways, one of which is by the symbol ¬, for example as ¬p ("not p"). George Spencer-Brown suggested in the 1960's [10] that this symbol be written around the letter in question, and that juxtaposition of letters be agreed to represent logical "or." Earlier logicians such as Sheffer had known that only a single explicit symbol was needed to represent all of symbolic logic, and

this is one convenient way to do it: If pq represents "p or q," then ¬(¬p or ¬q) is the same as "p and q" by DeMorgan's laws, so no separate symbol for "and" is needed; graphically, we can then draw it as p̄|q̄| in Spencer-Brown's forms. Similarly, since the implication p→q ("p implies q") has the same value as ¬p or q ("either not-p or else q"), it too requires no separate symbol but can be written simply as p̄|q in his forms. All of first-order logic can thus be written with this single explicit "No"; only when he introduces self-referential forms in Chapter 11 are there, alas, false statements which can be deduced as well as his very interesting true statements, since the laws of second-order logic cannot be derived from any finite set of assumptions.

## 4.   The Quadrivium as Repeated Discovery of Harmonic Analysis

For pedagogical purposes, my text [7] proceeds in the order Arithmetic, Geometry, Music, and Astronomy.   For human interest, I base the Arithmetic chapter on the Egyptian mythology of rivalry between Seth and Osiris: Osiris is healthy so long as he is whole, but Seth first attacks him rationally by suffocating him in a coffin made to indirect measure by shadow reckoning, and when Isis is about to revive him from that Seth literally fractionizes him, ripping him limb from limb; this attempts to address most students' fear of dealing with fractions in 4th grade. The Geometry chapter begins with the Pythagorean theorem, analyzing the right-angled triangles in circa 1°-per-entry order implicit in the marvellous Plimpton 322 tablet from 17th cent. BCE Babylonia, and concludes with the literal "Earth-measure" deduced by Eratosthenes from shadows at different latitudes (as successor to Seth rationality). The Music chapter contrasts the phenomena of arithmetic, geometric, and harmonic means encountered in judgments of center by senses of touch, hearing, and sight (using a monochord touching surface of a water bath dusted with lycopodium powder to visualize musical rationalities for the hearing-impaired students who come to Northridge as national center). The two Astronomy chapters, finally, first compare the Jupiter-and-Saturn-based calendars of Eurasia with Venus-and-eclipse-cycle-based calendars of Mesoamerica in treatment of temporal rhythms, then follow the long battle between the geo- and heliocentric models of spatial movements resolved post-modernly, as it were, by Newtonian recognition that *no* body is in center — *all* move (and inherently chaotically if three or more), the purpose being not so much to teach astronomy as to offer a model of conflict-resolution by standing back and seeing the larger picture.

What I would like to do here (anticipating the as-yet-unwritten second half of my CSUN text, but following my M.A. thesis [11]), is sketch how the Quadrivium subjects have, in effect, "reinvented the wheels" of harmonic analysis four times in reverse, from astronomy to arithmetic:

•   Claudius Ptolemæus took a lot of flak recently when it was noticed that some of the supposed observational data in Book 9 of his 2nd cent. CE *Almagest* were in fact tabular extrapolations, corresponding to times when the planets in question would have been slightly below the horizon and thus not directly observable. "So sue me!" a New Yorker would say. A fairer appraisal of Ptolemy's accomplishment can be found in Tricker's 1967 book [12, Introduction] which notes that "The achievement of Copernicus consisted essentially in transferring an annual component in the motion of all planets from them to the earth, thus replacing five independent movements by a single motion. However, ... there is no annual component to be directly discerned in the movement of the planets. There is, in fact, only one annual movement to be observed in the sky, and that is the apparent movement of the sun itself, or of the earth, according to the point of view. The annual components in the motions of the planets only become apparent after the harmonic analysis, provided by Ptolemy's theory, has shown them up." Mathematically, Ptolemy's epicycles and Kepler's elliptical orbits make identical predictions, when properly adjusted, despite their difference in geo- vs. heliocentric view; the epicycles are simply the workmanly way to carry out the elliptical orbits in an orrery-like model. And Brahe's simplified geocentric model makes the same inaccurate predictions as Copernicus' much-touted simplified heliocentric model; they, too, are mathematically equivalent.

•   "Harmony," in the ancient worlds of both Greece and China, had meant the pleasing concatenation of tone-on-tone to form melodies.   In the course of the European Middle Ages, separate melodies began to be performed with increasing independence around a central voice which tried to hang on tenaciously to a *cantus firmus* (usually a borrowed bit of older Gregorian chant) and so

became known as the *tenor* voice; those singing "against" it above and below were known as *contratenor altus* and *bassus*, becoming eventually our modern *alto* and *bass* voices, to which a further *soprano* might be overlaid.   All were musically composed with a primary view to the pleasingness of what appears in written notation as their horizontal motions as melodic shapes in time.   This reached its height in the age of Palestrina, in late 16th cent. — the age of Galileo and Kepler, carrying out the implications of Copernicus' "revolutionary" views in astronomy.   Just as revolutionary a paradigm-shift was made in music by the early 18th cent. French composer Jean-Philippe Rameau (a contemporary of J. S. Bach), for this theorist and composer was the first to analyze music in his 1722 *Traité* [13] in what we consider its vertical aspect which everyone since then takes as meaning of "harmony."   He devised a system of figures which would indicate the intervals above a given bass line at which an accompanist should realize the implied harmonies in modern sense: If nothing is notated, a triad is completed by notes a 5th and 3rd above the bass note

Root Position   1st and 2nd Inversions
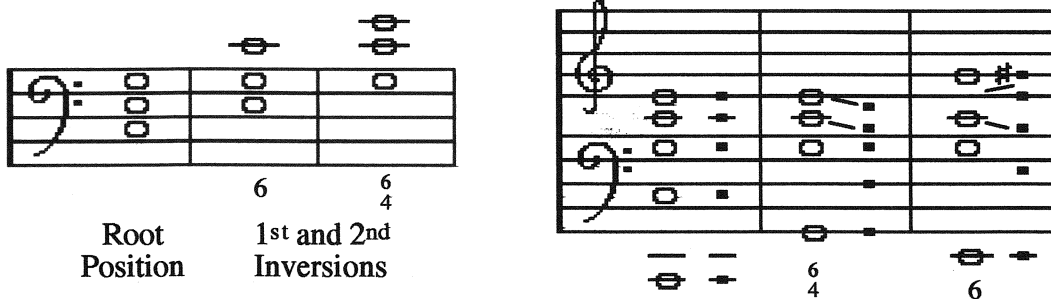
(for example, a G and an E above a given C); a "6" meant to use that interval instead of 5th (e.g. a C and G above an E); and a "$^6_4$" meant to replace both (E and C above a G), thus expressing the "inversions" of a triad in short-hand as at left above, leaving the performer free to improvise melodies in the various upper voices.   Only after-the-fact did Rameau become aware that an obscure French physicist of his era, Joseph Sauveur, had discovered in 1700-01 that a single monochord string was capable of vibrations in several modes at once (the Pythagorean tradition had always used separate monochords side-by-side to study intervals) resulting in overtones, partials, or harmonics.   They explain why a chord in root position is felt to be at rest, as at right above: all of the chord notes are in agreement with partials of the fundamental; a chord in $^6_4$ position is in slight dissonance with the partials of its fundamental, setting up expectation before a soloist's candenza; while a chord in 6 position is strongly dissonant to partials of its fundamental, hence an appropriate inversion to use before an evangelist's recitative in a passion setting.   They also explain an instrument's tone color, flutes sounding primarily a fundamental, oboes a fundamental plus octave or 1st and 2nd partials, and clarinets 1st and 3rd partials (no evens), as shown below.   This was proudly presented in 1726 2nd ed. of Rameau's work as proof that his title had been intuitively correct: *Traité de l'harmonie réduite à ses principes naturels*.   It was natural philosophy — it was physics! The last synthesis of his ideas was expressed in the *Génération harmonique* of 1737.

$y = \sin x$ $y = \sin x + \sin 2x$ $y = \sin x + \tfrac{1}{2}\sin 3x$
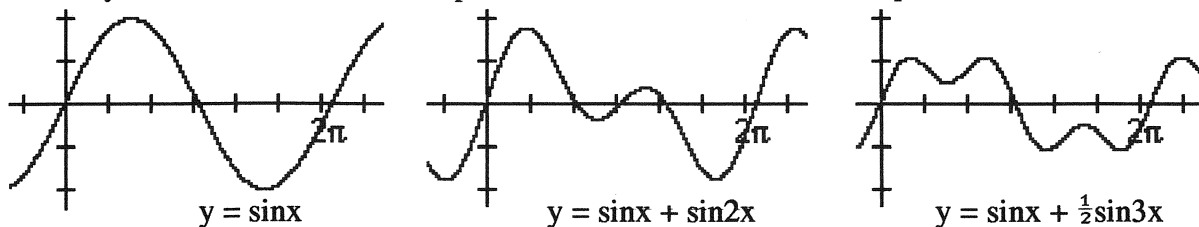
• The recognition that superposing circles upon circles and sine waves upon sine waves was essentially the same mathematical idea is due to Jean Baptiste Fourier.   Having been posted along with Napoleon's troups to Egypt as technical attaché during the expedition in first years of the 19th cent., he had greatly enjoyed the desert warmth; when subsequently posted to the chilly snow fields of the Haute Savoie he came to fear that he would die of cold, and began developing the mathematical theory of heat transfer that bears his name: Fourier analysis.   This basically says that any wave form of repetitive nature, no matter how complicated, can be expressed as infinite sum of trigono-

metric functions of multiples of a variable or angle with ever-diminishing coefficients — i.e. a fundamental signal with overlay of ever fainter higher partials; determining those coefficients (the extent of each partial's involvement) is the analysis, which Ptolemy had first performed for planets' annual motions and Rameau/Sauveur for different woodwinds' tone colors.

•   The late 19th cent. harbored hope that science and technology would provide solutions to all of the problems of supplying the world's practical needs; they rested on good physics; that rested on good mathematics; that rested on good logic; and that, in turn, was thought to rest on the abstract principles of set theory, which Gottlob Frege attempted to settle in his *Foundations of Arithmetic* until a young whipper-snapper named Bertrand Russell called his attention to problems with self-reference (which Russell himself never solved — they are intractable! — merely agreeing with Whitehead at outset of their *Principia* to restrict themselves to first-order problems). Meanwhile, a neo-Pythagorean named Georg Cantor was pushing investigations of the vibrating monochord to theoretical limits: What if it had infinitely many nodes — would it still admit a well-defined Fourier analysis? The surprising answer was "Yes and no": Yes, if countably infinite (like the sets of integers and rational numbers), and no if uncountably infinite or continuous (like the sets of irrational and real numbers), the distinction between the first two transfinite cardinal numbers which Cantor called $\aleph_0$ and $\aleph_1$ or C (the power of the continuum). He agonized over uncertainty whether there might be any intermediate cardinals, a question which has since also been answered with "Yes and no": There are valid arithmetics both with and without them — the so-called "continuum hypothesis," now recognized as having axiom status. A modern appreciation of Cantor's work, much misunderstood in his lifetime, has been given by Dauben [14].

If one now returns to the usual order of Quadrivium subjects, one recognizes that teaching the simple-looking "1,2,3" of arithmetic (as Plato put it in his *Republic*) in kindergarten already contains implicitly the wonders of geometry and geography (e.g. "round," "polar-flattened," and "pear-shaped" Earth as first three partials in its description as harmonically vibrating sphere as discovered during International Geophysical Year of 1959 [15]), as well as tone colors in music (at which French composers like Debussy and Ravel still excel), and what Kepler so fondly referred to as "the harmony of the spheres" in seeking to model the spacing of planets in the solar system (still a riddle, awaiting sufficient experience of systems around other suns). The Quadrivium offers a course plan for the human race, still so much in its infancy, to grow and grow, without limits to the imagination.

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