BRIDGES
Mathematical Connections
in Art, Music, and Science

How a Math Class Can be Two Places at Once: A Proposed Mathematical/Philosophical Collaboration

Eric S. Egge
Department of Mathematics
Gettysburg College
Gettysburg, PA 17325 USA
E-mail: eegge@gettysburg.edu

Steven Gimbel
Department of Philosophy
Gettysburg College
Gettysburg, PA 17325 USA
E-mail: sgimbel@gettysburg.edu

Abstract

The movement to integrate mathematics back into the general curriculum has largely focused on those students who tend to avoid mathematics and ignore those students who have chosen to major in mathematics. We propose a course designed for mathematics and philosophy majors, team-taught by a mathematician and a philosopher, to act as a model for bringing a more humanistic view of mathematics to mathematics majors and a more mathematical view of the humanities to philosophy majors.

1. The Need for a Broader Model of Mathematics for Math Majors

Sir Isaac Newton is reputed to have laughed only once in his life: when a student asked him of what practical use geometry could possibly be. Every teacher of mathematics has heard the complaint from a frustrated student that he should not have to endure a mathematics class because he will not need it in "real life." In part to cut this complaint off at the pass, many mathematics classes have become more focused on the practical applications of mathematical training, alienating it from its deeper, more philosophical side.

Over the past four years, discussions at *Bridges* conferences have included wonderfully creative and innovative ways to reintroduce mathematics into the general education curriculum for students who otherwise would not be excited about mathematical questions (e.g., see [3], [7] and [9]). Suggestions have also been made for restoring mathematics to its rightful place as one of the liberal arts (e.g., see [1], [4], and [11]). As necessary and effective as these moves are, they largely overlook an important constituency of students right under our noses: math majors.

We assume, often wrongly, that because a student has officially declared her intention to concentrate on mathematics as her major course of study that she is intrigued about the deep questions raised and sometimes answered by mathematical investigation. Sadly, she has too often never even heard of the problems that launched the discipline and its branches over the centuries.

In response to this, we are proposing a class, "From Zero to Infinity: Philosophical Revolutions in the History of Numbers," that will introduce the humanistic side of mathematics to an audience partially consisting of mathematics majors by concentrating on what is viewed by many students as the most basic aspect of mathematics, arithmetic. By rigorously developing a historical and philosophical understanding of the concept of number and the ultimate complications associated with the completeness of arithmetic, students will become aware of the interesting theoretical issues that underlie every aspect of mathematical study.

This is not to say that we do not also share in the interest of "main-streaming" math classes. The ultimate goal of this course is that with a single set of team-taught lectures by a mathematician and a philosopher, in addition to class discussions and group exercises, the course will satisfy requirements as both an upper level mathematics elective AND an upper level philosophy elective. In this way the class is to be comprised of roughly half mathematics majors and half philosophy majors. We aim above all for a coming together of the students of the most exact of the exact sciences and the most humanistic of the humanities that allows all to learn from all, to see alternative rational methodologies in action, and hopefully to synthesize both into a richer approach to intellectual matters.

2. Why This Collaboration?

With math anxiety all but recognized as a diagnosable psychological condition (see [2], [5], [8], and [12]), interdisciplinary mathematical pedagogy has traditionally focused upon the math-phobic student. This has had the wonderful effect of making professional mathematicians, whose natural interests lay in the beauty of mathematics, aware of the need to convey that grandeur to the wider population of students who see it least. But in our effort to celebrate the return of these prodigal sons and daughters from their travels in the arts, humanities, and social sciences, and as we broaden our mathematical course offerings, we must take care not to overlook their siblings who have made their home in the mathematics department. Too often mathematical humanism is seen as "dumbing down" mathematics or selling out to teach "baby math" instead of what it could be, i.e., the rigorous, exciting examination of the foundational issues and big questions that sit at the heart of the entire mathematical enterprise [10].

The problem of an overly utilitarian conception of mathematics by mathematics majors has several sources. With the physics department needing their students to have taken three semesters of calculus and one semester each of differential equations and linear algebra, with the psychology and/or education departments needing a statistics course different from that required of biology students, and with several sections of some version of math for poets needed by the institution to guarantee that everyone fulfills their quantitative reasoning distribution requirement, mathematics majors see that the majority of course offerings in their own department are applications-based instruction for other disciplines. In smaller departments especially, this leaves fewer slots than desired for teaching mathematics for mathematics' sake. When they see their professors' time largely taken up with these service courses, students cannot help but develop a picture of mathematics as just a tool for answering other people's questions.

The broader culture also plays a role in over-emphasizing a utilitarian view of mathematics. With contemporary undergraduates adopting an excessively pragmatic attitude towards higher education — college as vo-tech job training — many place perceived marketability issues over depth and well-roundedness in making choices about course enrollment. Believing that the marketing or actuarial firms they hope to work for would look askance at such foundational exploration, they eschew the historical and philosophical aspects of mathematics as fluffy indulgences with no cash value in this new era of high-technology.

Further, the assessment methods and tools of the standard math class discourage thinking about the deeper aspects of the subjects under consideration. Students' grades are seen as the degree of success in a class and these hinge primarily on exam performance. There are few questions instructors despise more than "Is this going to be on the test?", but students quickly learn that what is expected of them is not intellectual curiosity, but test-taking ability. We may not always teach to the test, but because of the nature of our standard assessment tools students are trained to learn to it.

Additionally, younger faculty, who would generally bring the greatest excitement to the classroom about deep mathematical issues, seem to be receiving little exposure in graduate schools to history and foundations. With increasing specialization and the need to get graduate students through and out onto the job market, graduate requirements and offerings in history and *Grundlagen* seem to be disappearing. With concerns about tenure never far from the surface, the time and risk newer instructors would take self-educating after the fact are often not worth taking. The pending retirement of a large cohort of seasoned teachers threatens to take away from the community much of the institutional memory of the history of mathematics.

Fortunately, there is a reservoir of this knowledge that may be tapped, a reservoir located in one of the last places one would think to look, buried deep in the heart of the humanities. Analytic philosophy traces its roots to the intellectual revolution caused by the meta-mathematical exploration of the second half of the 19^{th} and early part of the 20^{th} centuries. The discovery of non-Euclidean geometry and subsequent relative consistency proofs, the axiomatic project of David Hilbert, and the developments of proof theory by Alfred Tarski and other members of the Polish school are all essential parts of the development of contemporary philosophy by and through such figures as Bertrand Russell, Gottlob Frege, and Ludwig Wittgenstein. Indeed, Kurt Gödel himself was a participating member of the Vienna Circle, one of the birthplaces of analytic philosophy. Philosophers trained in the analytic school will generally have a working knowledge of the paradoxes of set theory, the foundations of geometry, and the history of calculus. As a result, many mathematics departments will find curricular redundancy in the philosophy department's logic course and the mathematics department's abstract mathematics course.

If the knee-jerk academic instinct for turf protection can be set aside, this intellectual overlap can result in a win-win situation for the mathematician and the philosopher. The mathematics professor gets to put aside the calculus textbook briefly and in a classroom setting pursue the sort of big questions that may have led her into mathematics in the first place. The analytic philosopher finds a classroom less intimidated by the technical notions underlying the sort of philosophy she does professionally. The mathematician can feel comfortable being philosophical in public and the philosopher has license to be more technical, without either worrying that she is talking about things she is not trained in, each providing a safety net for the other. The result is that students see one central question, "What is a number?", approached through both philosophical and mathematical methodologies, not merely side by side, but actively intertwined, the mathematical proofs and counter-examples acting as premises in larger philosophical arguments that lead to new and interesting questions in mathematics. The divides we place as disciplinary boundaries begin to show their artificiality and students begin to step away from overspecialization and into a world appreciative of intellectual breadth.

3. The Class

At the heart of the class are the questions: the ontological question "What is a number?", the epistemological question, "What is the nature of mathematical truth, especially the truths of arithmetic?", and the historical question, "What was the impetus for the introduction of new types of number?" These questions are put to the students from the very beginning.

We begin by asking our students how many know what the number two is. Not wanting to look like an idiot in front of everyone, most will claim such knowledge. When asked what two is, students will provide (a) examples of two-membered sets, inevitably two fingers, and (b) the numeral "2" drawn largely on a piece of notebook paper. When they see that what they have shown are fingers (an instance) and a numeral (a representation) and not two itself, the seemingly elementary question looms as a deeper more interesting inquiry. We now take a slightly different line by asking them if they know any true statements about two. When someone suggests a simple truth of addition involving two, we ask, "What is it that makes this sentence true?"

The initial response is that it is justified by simple experience. But this has several immediate and seemingly fatal flaws. When we consider arithmetic problems involving large numbers and non-integers, we clearly do not use experience and the extension to them using an inductive inference now opens a whole new set of problems. Further, the exactness of the arithmetic truths is not a perfect fit with the experimental error that accompanies observed results. Truths about two are not the result of simple experience. The stage is now set. Students have motivated for themselves the questions of the semester. We approach the task of answering these questions by dividing the class up into three parts.

The first part begins with the introduction of Plato's mathematical realism, and his concept of mathematical entities as forms. It is augmented with Russell's set-theoretic approach that defines integers as sets of sets. Numbers therefore do exist as mental entities and arithmetic is reduced to set theory, which is axiomatic and well-behaved, at least *prima facie*.

We then begin to consider the rational numbers, as they arise from measurement comparisons and the desire for closure under arithmetic operations, focusing our attention on their density. We raise questions about the applicability of the previously discussed philosophical conceptions of number and we examine the Pythagorean geometrical conception of number. The connection with measurement allows us to present John Stuart Mill's empirical approach to mathematical truths.

We then shift our attention to the irrational numbers and the difference between the reals and the rationals. The lecture moves students from the anti-Pythagorean proof of the incommensurability of the square root of two to Dedekind cuts as a method of creating the real line. We raise cardinality issues and then avoid them, putting them off for the promised upcoming discussion of infinity. We use the example of the continuum to motivate the Kantian approach to mathematical truth and the concept of numbers as synthetic *a priori* intuitions that are hardwired into the brain. Finally we use Dedekind to illustrate the move from Kantian classical intuitionism to Brouwer's constructivist intuitionism.

The next development we consider is the introduction of zero and negative numbers. This allows discussion of the introduction of the place-value system and its role in furthering arithmetic. Finally we introduce imaginary and trans-infinite numbers, and the first and major portion of the class concludes with students considering what each of the philosophical systems discussed would have to say about these mathematical entities.

The second part of the course turns the focus from the ontology of numbers to the epistemology of arithmetic. What is it that makes "1+1=2", or "2+2=4", a true sentence? We introduce non-standard arithmetics and ask what each of the philosophical systems we have discussed would make of this situation. We tilt the lectures to make the formalist approach seem the most promising.

With this lead in, we now derive the Gödel result using Nagel and Newman's wonderful little book, *Gödel's Proof* [6], which includes a straightforward discussion of the incompleteness result and an easy-to-follow derivation. When the result is clear, we consider the philosophical ramifications. We expect that this result will be opaque to undergraduates to varying degrees. We do not expect them to feel its full impact, but we find it is important to at least expose undergraduate mathematics and philosophy students to what is certainly one of the most important meta-mathematical results of the 20th century.

The third part of the class is the presentation of group projects. Students will pair up and choose a topic to investigate as a group over the course of the term. For mathematics students the topics will generally be groups of numbers that we do not cover in the first section of class, e.g., prime numbers, perfect numbers, Fibonacci numbers, and surreal numbers. They are expected to find an interesting aspect of the set and investigate historically important results and contemporary work related to those numbers. Philosophy students will choose a philosophical topic, e.g., the relation between Descartes' analytic geometry and his metaphysics and how Henri Poincaré's understanding of arithmetic was shaped by and different from that of Kant. Sometime during the last week of the course, the class will gather in the evening for a dinner together during which students give five to ten minute presentations on their topics. The discussion of the topics will serve as a chance for students to see how differently they now are able to understand and discuss the foundations of a field they had once taken for granted.

4. Conclusion

The overly utilitarian approach to mathematics taken by many mathematics majors is as much a problem as the avoidance of mathematics by the mass of non-mathematics majors. Indeed it is the same misperception by undergraduates of mathematics as merely algorithmic and not philosophically interesting that is at the root of both. We propose this course to help in overcoming both of these problems by focusing on mathematics and philosophy majors. We mean for this course to stand as a model. The mathematical/philosophical collaboration that is proposed could instantiate itself in many ways other than a historical investigation of number, e.g., the development of geometry, set theory, and logic all provide rich grounds for other such courses. Our purpose is to call attention to the need to remember the math majors when discussing the humanizing of mathematics.

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