A Transformation Sketchbook

John Sharp,
20 The Glebe
Watford, Herts England, WD25 0LR
E-mail: sliceforms@compuserve.com

Abstract
Looking back through my notebooks and the ideas I have used for teaching, I found that I have many ideas which in some way transform a starting image using mathematical concepts. I have many hand-drawn or computer sketches which have not reached finished work. Bridges has provided an incentive to work on them again. This paper is a summary of ideas I have used.

Transformation in art and nature

It can be argued that all art is transformation, either a concept in the artist's mind or by a device or technique. Albrecht Dürer is a prime example of an artist who used geometrical ideas to explore transformations. It is not surprising that he did this since he was as much a mathematician as an artist and wrote a book of geometrical techniques for artists [2]. One of his most interesting ideas was to use grids to transform faces and this has been an inspiration for a great deal of my work.
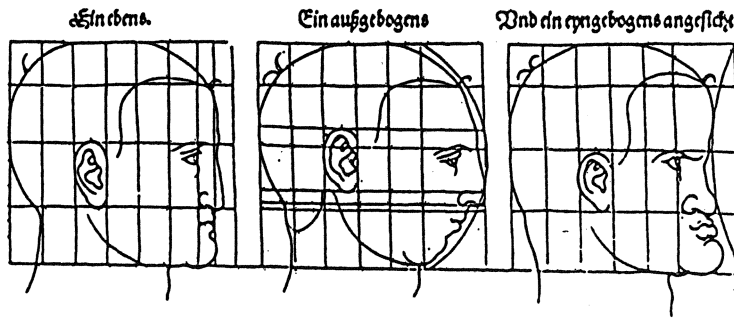

Figure 1, Durer heads

Piero della Francesca's *Prospettiva Pingendi* also contains studies of heads. I have explored a wide range of mathematical techniques, using standard and "artistically" modified methods on heads, using both Piero's and Dürer's heads. There are many common mathematical transformations such as rotation and reflection that I will take for granted, but the following is an example of rotating a profile of one of Piero's heads.
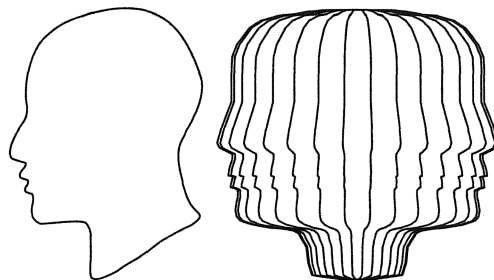

Figure 2, Profile rotations

Science is not as isolated from art as C P Snow would have us believe in his "Two Cultures" lecture. The biologist D'Arcy Wentworth Thompson [3] developed Dürer's ideas to study the relationship of the shape of a wide range of living organisms using non-linear grids. His work has been found in a number of libraries of twentieth century artists [4].

Art would be much poorer without the most common use of a mathematical transform using perspective to bring the three dimensional world into two. The perspective grid has been a common tool for artists. My interest in a special form of perspective, Anamorphosis, is a subject in itself. In [5] I have described the incorrect use of such grids and the problems it causes. I will not go into the subject here, other than to point out that the first book on techniques for creating it was written by the artist/mathematician François Niceron who was a pupil of Mersenne [13 and 14].

The examples in figure 3 show how a simple motif can be transformed quite simply into a wide range of designs by the use of grids. For teaching mathematics, this is an excellent way to introduce functions, number sequences and many other mathematical relationships. It also works well with the non-mathematical artist.
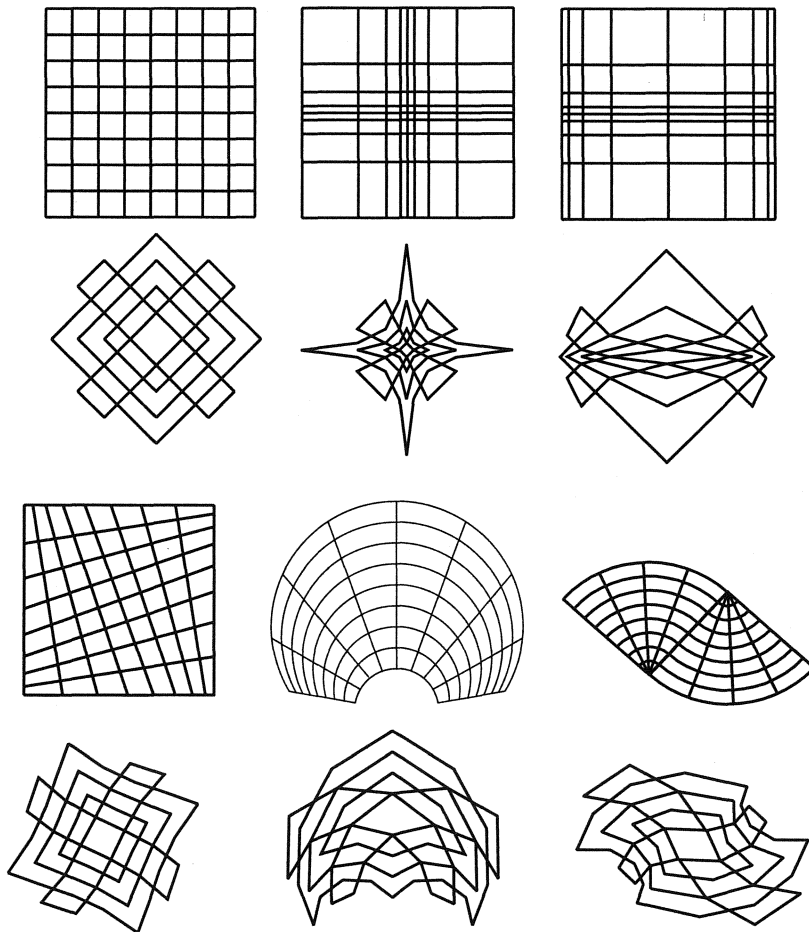

Figure 3, Grid transformations of a motif

The same motif is used in all transformations. This motif is the first object of the second row in figure 3 and has been created by using straight lines which join intersections of the regularly spaced grid above it. The grid is then transformed and the new versions created by joining the corresponding intersections of the transformed grid. So it is not a smooth transformation, since the changed motif consists of a series of line segments. This makes it easy and quick to draw by hand once the grid has been created. The two orthogonal grids at the right of the first row are the result of applying a sine wave transform to the grid spacing. The second row below them shows the resulting motif changes. The grid at the left of the third row (with result below) is created by making the spacings along each side using divisions of 2, 3,, 4, 5, 6, 7, 8 and 9 units. The circular grids use radial lines with sets of concentric arcs. In the central grid on the third row, the radial lines emanate from a point which is not concentric with the arcs. In the one on the right two quadrants are used. Many more possibilities are to be found in [5], but even the few here show how rich a method this can be.

Duality and polarisation

The mathematics of projective geometry arose out of the study of perspective. The foundations were laid by Desargues whose famous theorem on perspective triangles is at the very heart of perspective construction. Unfortunately, firstly he had to create a whole new vocabulary which no one understood and secondly, Fermat and Descartes turned geometry into algebra and it was not until the early nineteenth century that it blossomed again. With the concept of duality and polarity, a whole new world opened up. Projective geometry goes in and out of fashion, and although it is used extensively in computer graphics to manipulate perspective, it does not seem to have been used as an end in itself. Perhaps this is because the transformation of points into lines and vice versa requires a computer to give accurate results in a reasonable time. The concept of duality between points and lines is a logical one, but practically it can be achieved by using a conic and using a pole and polar construction. Duality and polarity (sometimes called reciprocation) is described in books on projective geometry (with [11] having a clear description for the general reader). The construction of pole and polar is shown in figure 4, with a description of the procedure on the next page. It applies to any conic , not just the ellipse shown. I tend to use a circle in artistic transformations, because it is symmetrical.
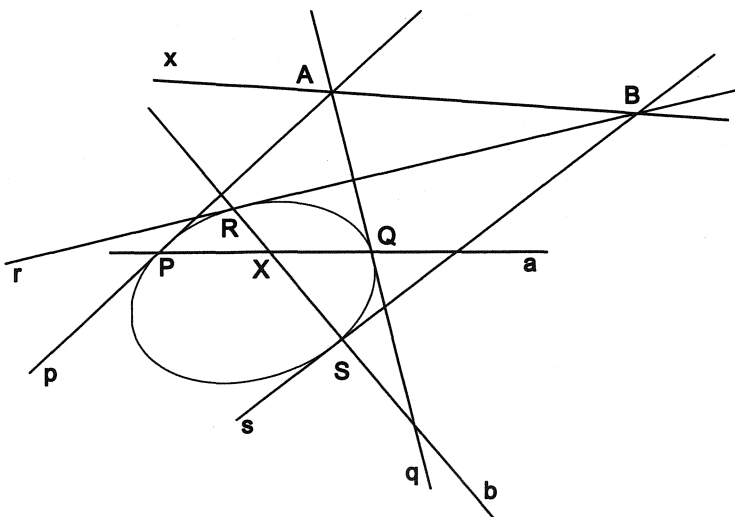

Figure 4, Construction of pole and polar

To find the polar line of point X

1. Choose any two lines (a and b) through X which intersect the conic in real points.
2. Intersect these lines and the conic in points P, Q (for a) and R, S (for b).
3. Draw the tangents p, q, r, s at P, Q, R and S.

4. Intersect tangents p, q in A and r, s in B.
5. Join A and B to give the polar line x.

To find the pole of line x

1. Choose any two points A and B on line x which have real tangents with the conic.
2. Draw the pair of tangents from each point, p, q (from A) and r, s (from B).
3. Draw the tangent points of these lines P, Q, R and S for p, q, r, s.
4. Join points P, Q in line a and R,S in line b.
4. Intersect a and b to give the pole X.

Even a simple polarisation of a curve in a circle as shown in figure 5 can yield forms which are aesthetically superior to the curves from which they originate. This is a simple rose curve polarised with respect to a circle.
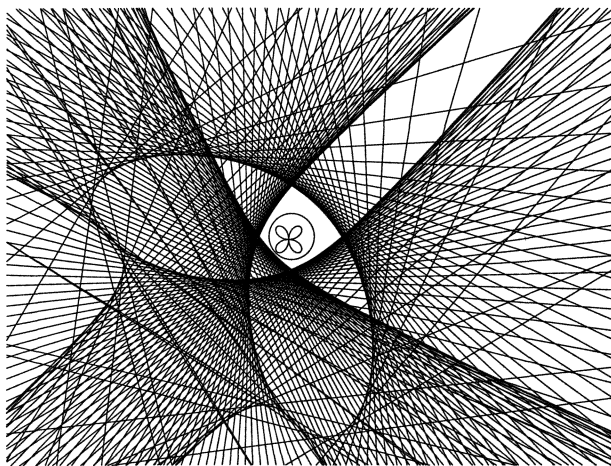

Figure 5, polarisation of a rose curve

Another area which has not been explored through duality/polarisation is fractals. Figure 6 shows polarisation of two levels a Von Koch Snowflake, using a circle centred on the centre of the fractal. Because of the symmetry of the circle and the fractal, the result is the same (except for scale) whatever the size of the circle in this case. The labelling identifies the corresponding pole and polar lines. To follow what is going on, move round the original curve (A to L) bearing in mind the following examples. When you move along the line AB, the polar equivalent is rotating through the marked angle between a and b. When you rotate to change direction at point B, the polar equivalent is moving along line b.
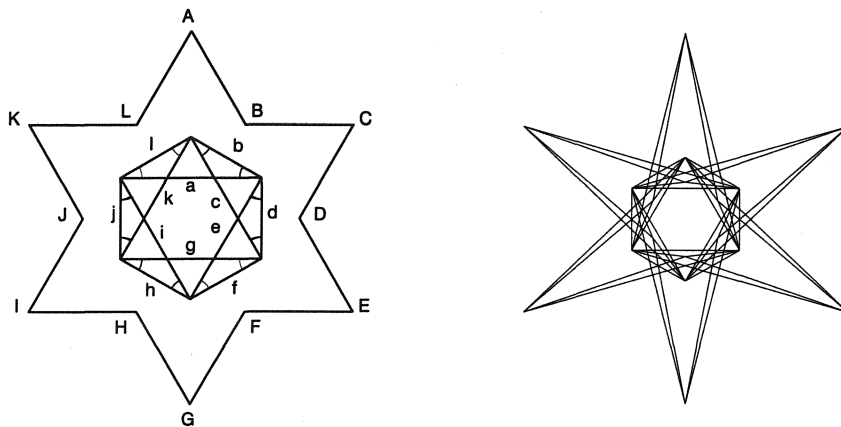

Figure 6, Koch polarisation

By changing the centre of polarisation, or by using different conics for the polarisation to give results which destroy the central symmetry, this transformation offers many possibilities for exploration. I have tried to work with non-mathematical curves and have not been too successful in finding shapes which are suitable. The smooth flowing curves of a letter form offer interesting possibilities and I have on my list to explore the whole alphabet. This is the letter S which takes on a very three dimensional form.
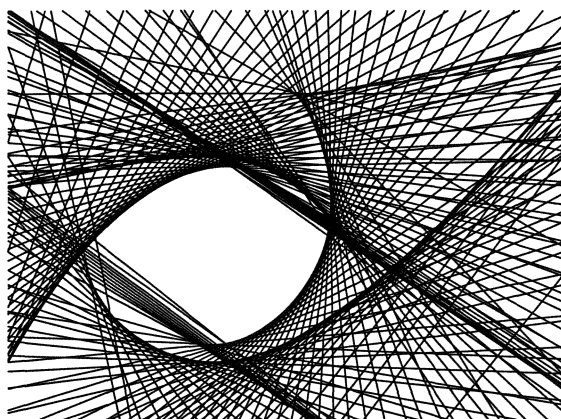
Figure 7, polarisation of letter S

The letter is the outline of the Times Roman S, which is placed outside a large circle above and to the right.

Inversion

Inversion using a circle is a well known transformation of points to points. It is normally defined (see figure 8) by saying that a point P is inverted with respect to a circle, radius r and centre O, to give a point P_1, so that O,P and P_1 are in the same line and $OP.OP_1 = r^2$. Another definition is that P_1 is obtained by taking the polar of P with respect to the circle and then intersecting the line OP with the polar line. This is a broader definition which extends the possibilities, because the line can be drawn from any point, and any conic can be used instead of the circle. This is then known as a quadratic transformation.
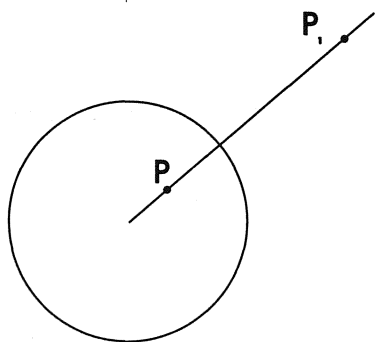
Figure 8, inversion of a point

The most interesting transformations are when the original image is made up of lines. Inversion turns lines into circles. Herman von Baravalle was the first to use these for creating early op-art as in the checkerboard inversion at the left of figure 9. Inverting the grids as in figure 3 also give interesting results. One such grid (as in the top row of figure 3) which has been converted into a checkerboard and inverted is shown at the right of figure 9.
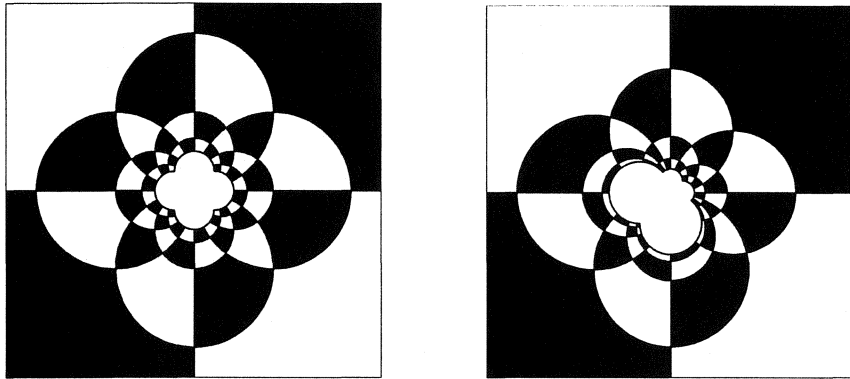
Figure 9

Inversion is also involved with certain map projections, for example the stereographic projection of a sphere onto a plane. Robert Dixon has some artistic work showing this in [12]. Robert Krawczyk has also used inversion in producing curving Spirolaterals [15]. My work in using inversion artistically with iterative methods is described at the end of this paper.

"Circles"

Another type of transformation is to think in a different way; to be able to see something not so much as what it is but what it might be. This has always been a function of the artist in society. Mathematicians are not always able to cope with this type of divergent thinking.

The duality concept of seeing curves as either sets of points (pointwise) or sets of line (linewise as tangents) can be extended. A number of curves can be defined as envelopes of other curves, especially circles. When a circle is drawn by computer it is really drawn as a polygon with a large number of sides. Taking this to extremes, it could be drawn crudely as a simple polygon such as a triangle or square. Thus designs based on circle envelopes are instantly capable of being turned in a number of new designs, by merely replacing each circle by a polygon. The results are so different, I often have difficulty in convincing people that this is all I have done, so perhaps the word "merely" is somewhat simplistic. Because a circle is infinitely symmetric about its centre, and a polygon is not, there are many possibilities for placing the polygon and additional rules need to be specified in order to determine the orientation of the polygon. These might be as simple as having all the polygons pointing the same way, either parallel to an axis, or towards a point, or the rotation of the polygon may be varied according to a rule.

An example is shown in figure 10. It begins with the construction of a cardioid as an envelope of circles and transforms it into a hexagon version.
1. The first diagram (top left) shows the standard construction for envelope of circles. Draw the bold circle and place a number of equally spaced points on it. Choose one of these points as the base point (where all the circles go though the same point).
2. With centre as each of the points on the bold circle, draw a circle whose radius is the distance to the base point. The envelope of these points is a cardioid. In this diagram, there are 12 such circles.
3. Now, as in the diagram at the top right, with one end on the base point, draw the diameters of these circles by rotating each diameter a fixed amount (in this case 12°), starting from the first (and smallest) as being vertical. Note that the initial line for this small circle is not shown here in order to make the diagram clearer, since there are many lines going through the base point.
4. Use these lines to draw the hexagon so that opposite vertices lie on each line. An example is shown in the diagram at the bottom left.
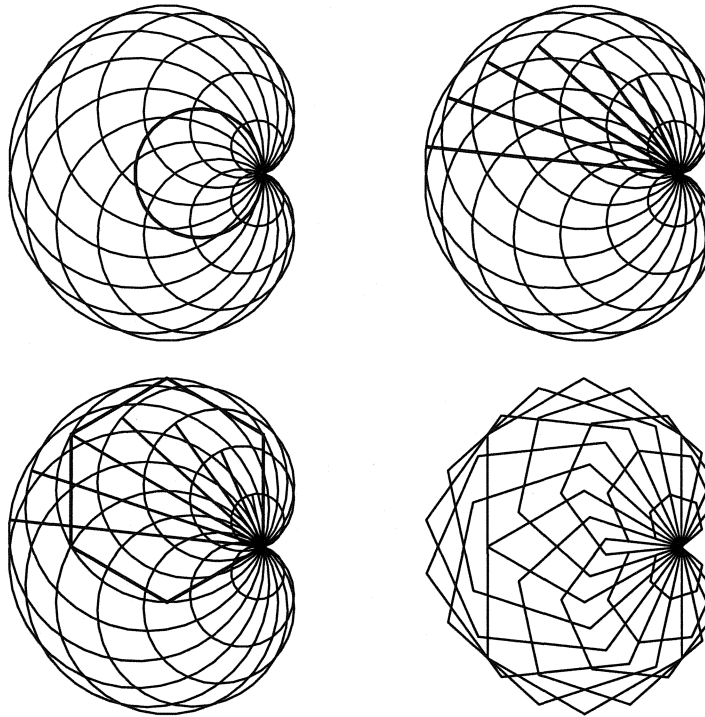5. Complete for each circle as shown in the diagram at the bottom right.

Figure 10, construction method

The examples in figure 11 show the cardioid as envelope of circles with some variations using different polygons (with 3, 4 and 6 sides). The circle version was used by Herman von Baravalle in simple geometric art in the 1940s, using shading as in the hexagon version on the right which is the same as the one constructed in figure 10, but with twice as many hexagons.
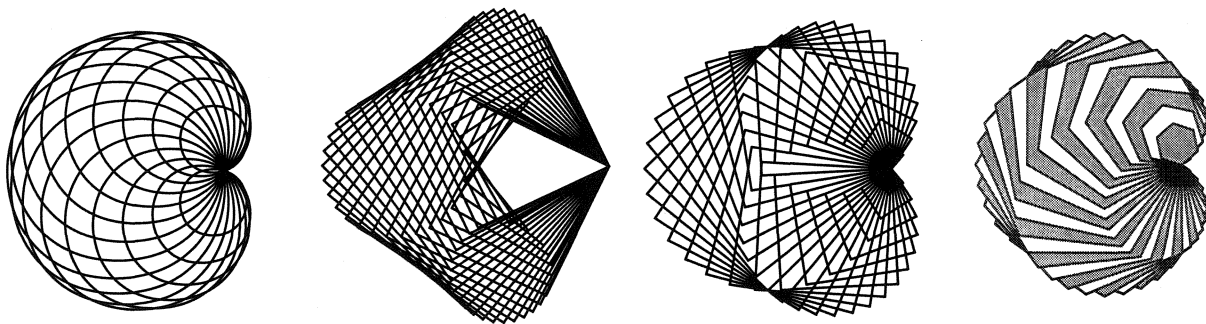

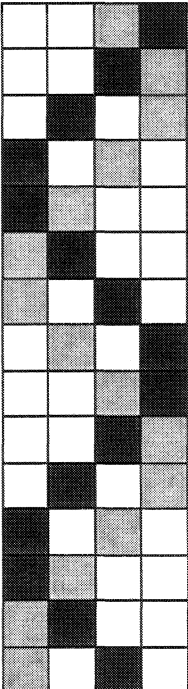Figure 11, examples of polygonal envelopes

Other methods for placing the polygons might be to have them all in the same direction, say all vertical or horizontal. All the examples shown here have a vertex of the polygon going through the base point. This need not be so. As well as these artistic effects, I will also pose the question as to what might the envelopes of these designs be. The locus of the vertices in the second design (based on triangles) in figure 11 might be a set of three circles.

Campanology and Truchet Tilings

Transformations are usually straight forward: take an image and apply a function. This is not always appropriate and there are other transformations. Change ringing of bells [9, 10] is a permutation transformation, in sound. As Dorothy Sayers [8] says "England, alone in the world, has perfected the art of change ringing" and as an Englishman, I can see it extended visually. The group when displayed as a tiling, using tiling variants according to the number of bells in the change can be used to display the effect visually. There are many sophisticated changes, which do not always sound interesting to the non-enthusiast campanologist as I can testify having lived close to a church where many variants were practised.

I am sure this method has been used to create patterns of tiles, but unless you go around searching, or come across original documentation, then it is impossible to know the origin of a design when one sees it.

A simple plain hunt with four bells swaps the two outside pairs, then the centre pair and so on. In the numerical table, the full result at the left is pared down on the right to show how the path of a single bell follows a kind of sine wave. This is more apparent in a simple shaded tiling.

1 2 3 4				1 . . .			
2 1 4 3				. 1 . .			
2 4 1 3				. . 1 .			
4 2 3 1				. . . 1			
4 3 2 1				. . . 1			
3 4 1 2				. . 1 .			
3 1 4 2				. 1 . .			
1 3 2 4				1 . . .			
1 2 3 4				1 . . .			

Tilings have been created to display permutations for a long time, with Truchet's 1704 paper [1] being seminal in laying down a mathematical framework. Truchet's bi-coloured tiles are shown at the top of figure 12, with the above changes converted into a tiling below. The single coloured tiling at the left is based on the change as shown above. Varying the colouring of the tilings offers more possibilities. The centre variation in figure 12 is the result of identifying the first tiling and the one at the right takes this a stage further by also doing the same with the fourth tiling.
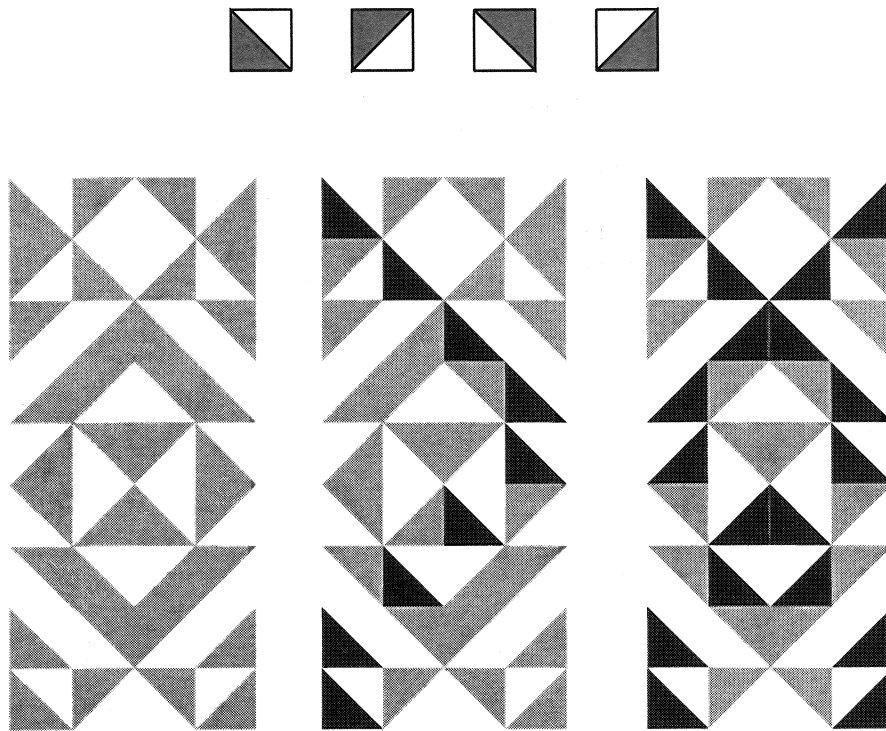
Figure 12, Truchet, changes tilings

Using more complex tiles, then patterns like the following arise.
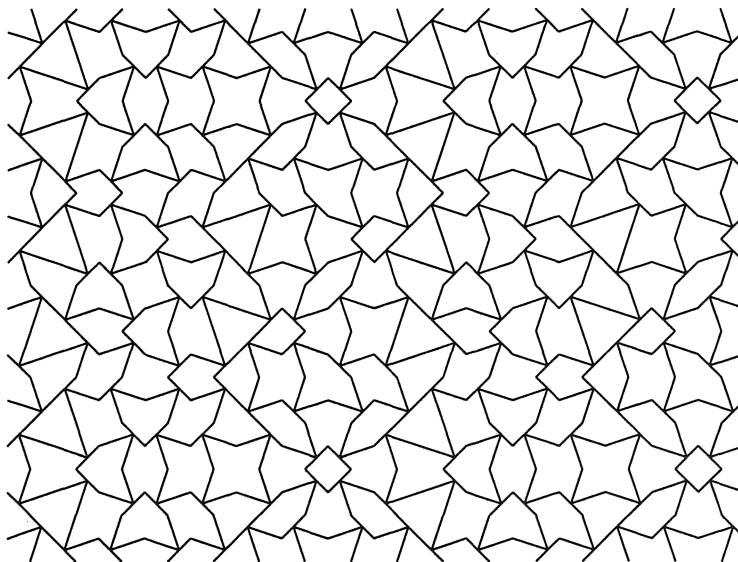

Figure 13, a puzzle tiling based on changes

This uses one motif, rotated in two sequences and the resulting changes added together as three strips. The changes occur across the page. I have deliberately not shown the motif, leaving it to the reader as a puzzle.

Repeated transformations and limit sets

Depending on the function used for the transformation, applying an iterative transformation may or may not yield a new image. For example, inversion and the dual/polarisation transforms described above are involutory (applying the transformation again reverts back to the original entity). In order to overcome this, different transformations can be combined. For space reasons, the following discussion is mainly limited to inversion.

If there are two centres of inversion, a single point when iteratively inverted about the two circles in succession gives rise to a circle. With more circles of inversion the result is similarly not artistically interesting. But, if the "chaos game" [7] is played, like the creation of the Sierpinski gasket from the three vertices of a triangle, then a whole new world opens. The transformation of each point in these images is randomly chosen, for example inversion takes place in one of a number of circles the variation of whose radii and position determine the image produced. The limit sets produced are a large class of images which have hardly been investigated.

Figure 14 shows the result of inversion limit sets of five circles, four equal ones centred on the points of a square, with another at its centre but larger.
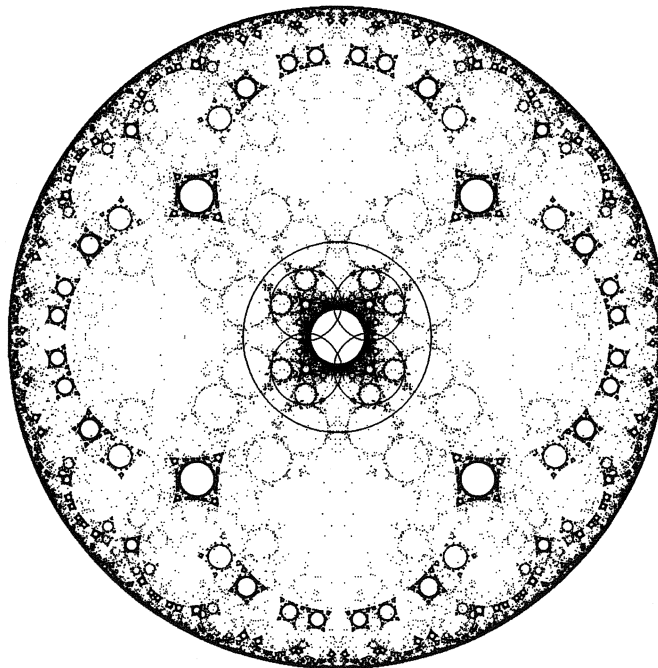

Figure 14, planar inversion limit set

This technique can also be used in three dimensions. Inversion in a sphere is similar to inversion in a circle. Figure 15 shows a stereo pair of an inversion limit set produced using a set of spheres of equal radii. Seven are in a plane packed around a central one, with a further pair above and below.
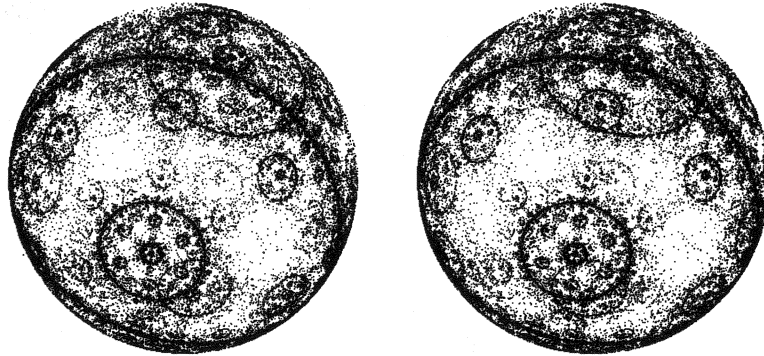
Figure 15 stereo pair of a spatial inversion limit set

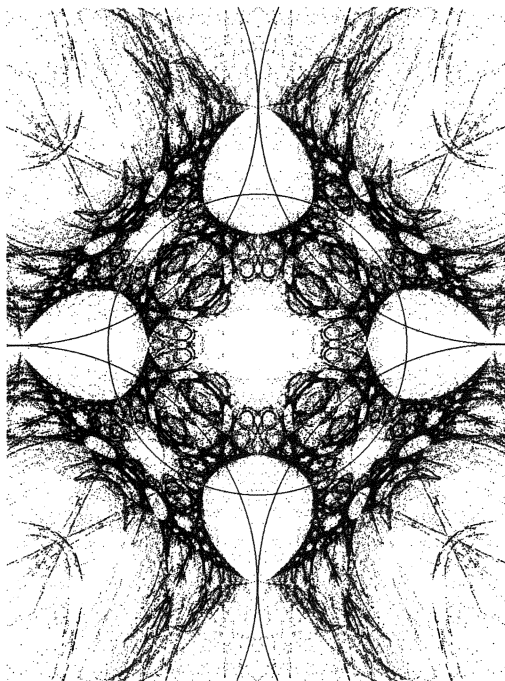Other transformations can also be used in this way as shown in figures 16 and 17.
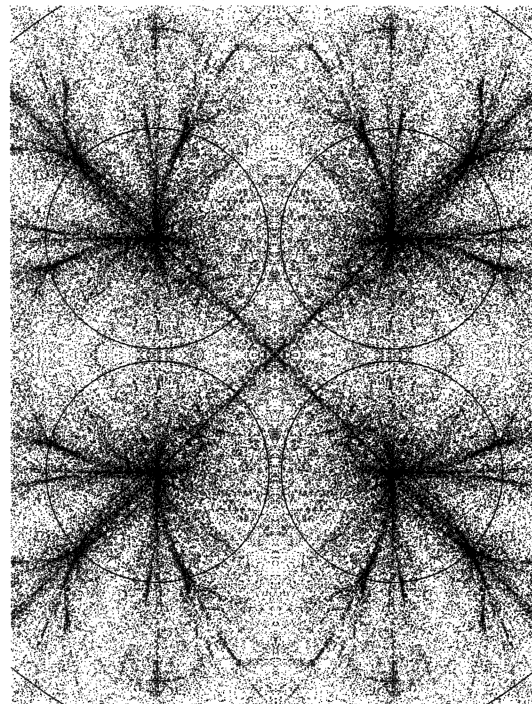

Figure 16

Figure 17

In inversion (figure 8), the points P and P_1 are separated harmonically by the points where the line OP meets the circle. Figure 16 is created using a transformation using a different harmonic division. This time the points P and P_1 are separated harmonically by the centre of the circle O and the point where the line OP meets the circle between P and P_1.

Inversion can be thought of a kind of reflection about the circle. Figure 17 uses the conventional meaning of reflection, about the circumference of a circle, so that, in an equivalent diagram to figure 8, the distance of P from the circle equals the distance of P_1 from the circle.

Conclusions

Transformations offer the artist a whole variety of possibilities for the creation of new images. The few ideas shown here are representative of the way it is possible to be creative using these techniques and they also suggest other routes which can only be fully shown in a presentation or *large* book.

References

1. Cyril Stanley Smith, *"The tiling patterns of Sebastien Truchet and the Topology of Structural Hierarchy"*, Leonardo **20** no 4, pp 373-385, 1987
2. Albrecht Dürer, *"Unterweyysung der Messung ..."*, translated by Walter Strauss as "The Painter's Manual" ....
3. D'Arcy Wentworth Thompson, *"On Growth and Form"*, CUP 1961
4. Martin Kemp, *"Spirals of life: D'Arcy Thompson and Theodore Cooke with Leonardo and Dürer in retrospect"*, Physis **32** 37-54, 1995
5. John Sharp, "Grid-Warps" part of the Explorer section of the UK Government's mathematical resource site at www.counton.org.
6. John Sharp, *"Problems with Holbein's Ambassadors and the Anamorphosis of the skull"*, Bridges Conference Proceedings 1998.
7. H O Peitgen, D Saupe, *"The science of fractal images"* Springer 1988
8. Dorothy L Sayers, *"The Nine Tailors"*
9. Wilfred Wilson, "Change ringing, The art and science of change ringing on church and hand bells.", Faber and Faber, London 1965
10. F J Budden, *"The Fascination of Groups"*, Cambridge University Press, 1972, chapter 24
11. H S M Coxeter and S L Greitzer, *"Geometry Revisted"*, MAA 1967, chapter 6.
12. Robert Dixon, *"Mathographics"*, Blackwell, Oxford 1987, reprinted by Dover
13. Jurgis Baltrusaitis, *"Anamorphic Art"*, Chadwyck-Healey, Cambridge 1976
14. Jean-Francois Niceron *"La Perspective curieuse"* Paris 1638
15. Robert J. Krawczyk, *"Curving Spirolaterals"*, Bridges Conference Proceedings 2001.