BRIDGES
Mathematical Connections
in Art, Music, and Science

168 Butterflies on a Polyhedron of Genus 3

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Abstract

In 1985 the mathematician H. S. M. Coxeter suggested that one could cover a torus in a regular way with 18 copies of the butterfly motif of M. C. Escher's Notebook Drawing Number 70. In this paper, we show how 168 copies of that motif can cover a surface of genus 3 — a 3-holed torus. In fact, we cover a polyhedron of genus 3 with such butterflies.

1 Introduction

The goal of this paper is to show how to cover a polyhedron of genus 3 with a pattern of 168 butterflies like those in M. C. Escher's Notebook Drawing Number 70. The polyhedron is shown below in Figure 1 and described in Section 2. We explain how to cover the polyhedron with the butterfly pattern in Section 3.

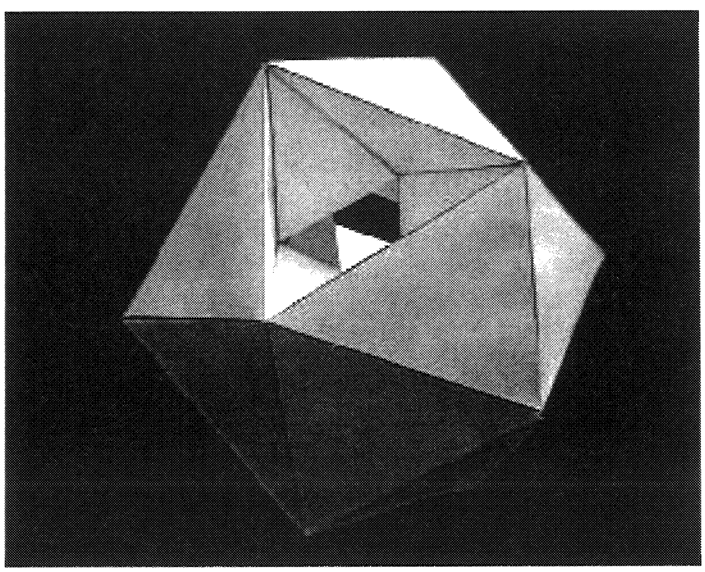

Figure 1: The polyhedron, consisting of 56 triangles.

Then in Section 4, we discuss connections between this decorated polyhedron, regular maps on a surface, the "realization" problem for polyhedra, and hyperbolic patterns. In the last section, we indicate directions for future work.

2 The Polyhedron

The polyhedron, shown in Figure 1, consists of 56 triangles, seven of which meet at each of 24 vertices. One simplified description of it is as a "thickened" skeleton of a tetrahedron, and thus it has genus 3. A better description is that it consists of two concentric, slightly twisted icosahedra with four holes connecting the inner to the outer icosahedron. If one starts with "aligned" concentric regular icosahedra and removes a corresponding face from each one, the hole that is left has the shape of a truncated triangular pyramid, which has trapezoidal lateral sides. To obtain a hole with triangular sides, the concentric icosahedra must be "twisted" with respect to each other. Vertices for an icosahedron can be chosen on the coordinate planes in two ways as follows: (1) as $(0, \pm\tau, \pm 1)$, $(\pm 1, 0, \pm\tau)$, and $(\pm\tau, \pm 1, 0)$, or as (2) $(0, \pm 1, \pm\tau)$, $(\pm 1, \pm\tau, 0)$, and $(\pm\tau, 0, \pm 1)$, where τ is the golden section (see page 163 of [1]). So we if we choose one set of vertices for the inner icosahedron and multiply the other set of coordinates by a factor of 2 for the outer icosahedron, we will obtain icosahedra that are twisted with respect to each other. However, if we remove a corresponding face from each one, the triangular sides of the "connecting holes" are not isosceles. To obtain isosceles triangles, we use the following (integer) coordinates for the icosahedra instead: for the inner icosahedron, we use $(0, \pm 2, \pm 1)$ $(\pm 1, 0, \pm 2)$, and $(\pm 2, \pm 1, 0)$, and for the outer icosahedron, we use $(0, \pm 2, \pm 4)$, $(\pm 2, \pm 4, 0)$, $(\pm 4, 0, \pm 2)$. The resulting icosahedra have 8 equilateral triangles and 12 isosceles triangles each, and the equilateral triangles are in corresponding positions. Now, if corresponding equilateral triangles are removed from the inner and outer icosahedra, the connecting hole has the shape of a non-regular antiprism with equilateral triangular bases and isosceles triangles as sides.

The final polyhedron, shown in Figure 1, is formed by removing four corresponding equilateral triangles from the inner and outer icosahedra, and connecting the resulting holes with a "cycle" of six isosceles triangles each. The four equilateral triangles are chosen alternately from the eight of each icosahedron, so that none share vertices. To sum up, the resulting polyhedron is made up of four large and four small equilateral triangles, and 48 isosceles triangles (12 each on the inner and outer icosahedra, and $24 = 4 \times 6$ to form the connecting holes). Figures 2 and 3 show nets that form the inner and outer icosahedra — four copies of each net are required to make the polyhedron of Figure 1. The central triangles of each net are

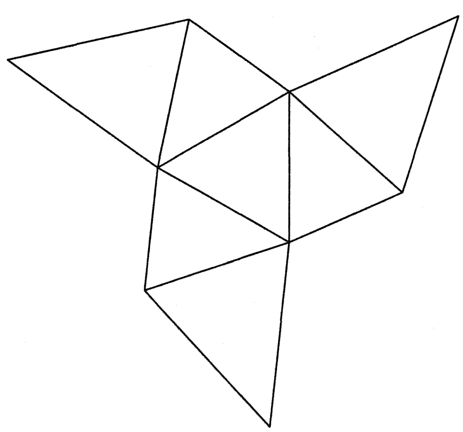

Figure 2: The net for the inner icosahedron.

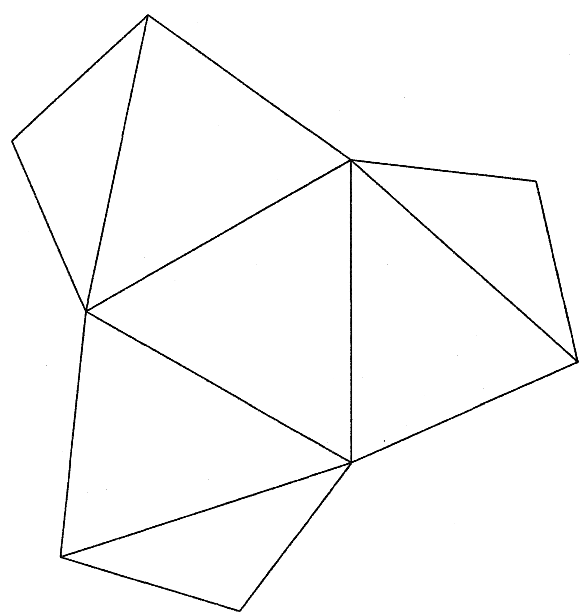

Figure 3: The net for the outer icosahedron.

equilateral; the outermost triangles of each net form the connecting holes. The polyhedron of Figure 1 has a slight twist to it, and so its symmetry group is the tetrahedral group (of rotation symmetries) and not the full group of symmetries of the regular tetrahedron. We can obtain a mirror image of our polyhedron (with the opposite twist) by interchanging the x and y coordinates that we used above. Thus, our polyhedron comes in two enantiomorphous forms.

To construct a model of the polyhedron, make four copies each of the nets in Figures 2 and 3 (they are the right size relative to each other) and put them together (you might want to use the "zoom" feature of the photocopier to get larger nets). Figures 5 and 6 show larger nets for the same polyhedron with the butterfly pattern on it. In Figure 2 the inner edges are "valley" folds and the outer edges are "ridge" folds; all the edges of Figure 3 are "ridge" folds. This also applies to Figures 5 and 6 (the edges connect left front wing tip points). Tom Gettys has useful suggestions for constructing polyhedra out of paper on his web site [5].

3 The Butterfly Pattern

In 1948, M. C. Escher created his Notebook Drawing Number 70, a pattern of butterflies based on the regular tessellation $\{3, 6\}$ of the Euclidean plane by equilateral triangles [7, pages 114 and 172]. Each equilateral triangle contains three butterflies. We show how to slightly distort these "butterfly triangles" to fit into each of the triangles of the nets in Figures 2 and 3 — the resulting polyhedron is shown in Figure 4.

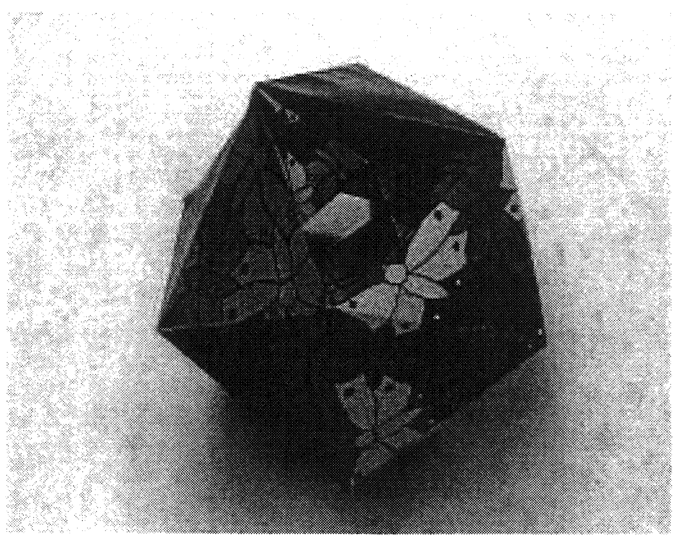

Figure 4: The polyhedron, covered with the butterfly pattern.

In 1977, Schattschneider and Walker [8] used 20 butterfly triangles to cover an icosahedron; I used 4 and 8 butterfly triangles to cover a regular tetrahedron and a regular octahedron respectively (unpublished). This did not require any distortion of Escher's pattern, since the polyhedra have equilateral triangular faces. At Bridges 2001, Carlo Séquin exhibited a sphere covered with 60 butterflies — the icosahedral pattern blown up onto its circumscribing sphere. The tetrahedron, octahedron, icosahedron, and sphere all have genus 0 (i.e. they are surfaces with no holes).

At the 1985 Escher Congress, H. S. M. Coxeter suggested that 18 copies of the butterfly motif could be used to cover an ordinary 1-holed torus (a surface of genus 1) in a regular way [2, page 24], however he didn't suggest using a polyhedral torus. Coving such a torus with the butterflies would involve distorting them in some way since a flat torus cannot be embedded in Euclidean 3-space. Similarly, we must distort the butterfly triangles to place them on the polyhedron of Section 2. However, since our triangles are isosceles (at worst), it is possible to transform Escher's butterfly triangles to our triangles by using only (differential) scaling transformations. To linearly map an equilateral triangle onto an arbitrary triangle also requires the use of a shearing transformation. Each of the isosceles triangles in Figures 2 and 3 except the three outermost ones of Figure 3 have lateral-side to base ratios of $\sqrt{6}/2 \approx 1.225$. The three outermost triangles of Figure 3 have half that ratio, $\sqrt{6}/4$. So except for these outermost triangles, there is not much distortion required to place the butterfly pattern on the isosceles triangles, and no shearing is required.

Before applying the scaling transformations, it was necessary to obtain only the parts of the butterflies contained within one of Escher's equilateral triangles. This was done by using the computer graphics technique of clipping. This process guaranteed that the butterfly outlines would match up along the edges of the polyhedron. Since there are (parts of) three complete butterflies in each triangle, there are $3 \times 56 = 168$ butterflies on the polyhedron. The resulting nets (with tabs) are shown in Figures 5 and 6, and can be used to construct the patterned polyhedron shown in Figure 4. Note that the butterflies in the center of the inner net should be facing inward toward the center of the polyhedron (the origin), since that is the direction of the polygon's exterior at that point. If these nets are used, the resulting polyhedron will be about 6 inches in diameter (the nets can be blown up to obtain a larger polyhedron).

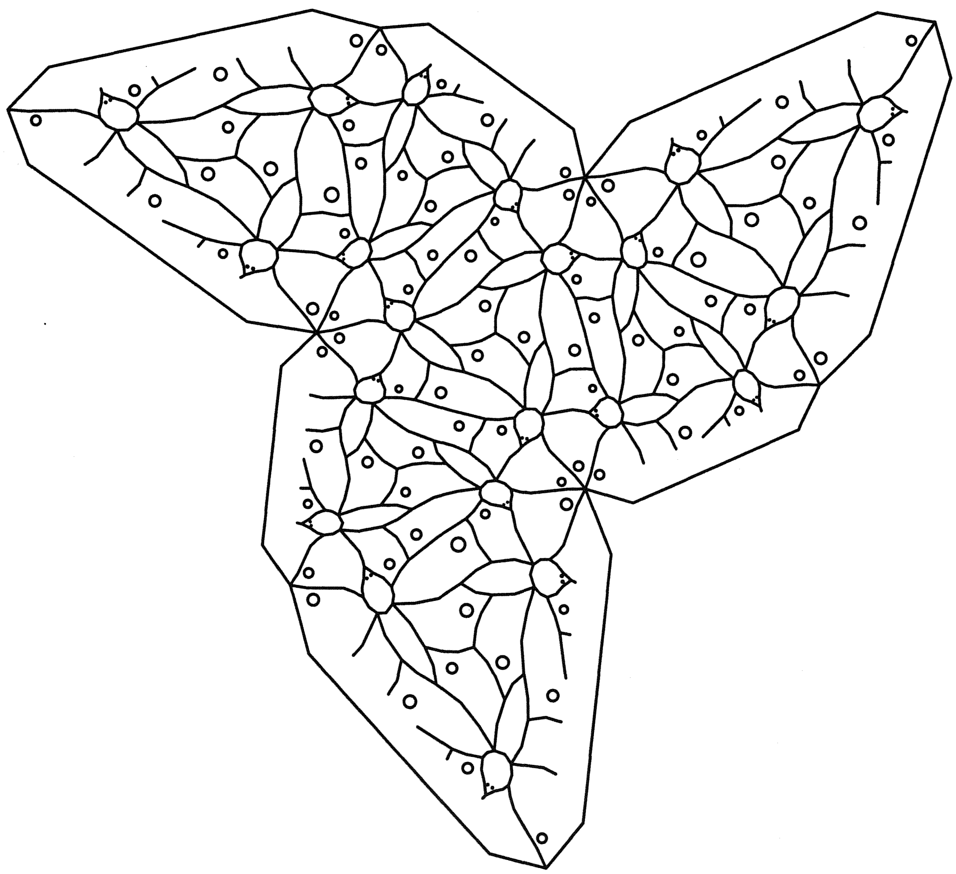

Figure 5: The net for the inner icosahedron pattern.

It is possible to color the butterflies with 8 colors in such a way that the 12 rotation symmetries of the polyhedron are also "color symmetries" of the pattern — that is, each rotation exactly permutes the colors of the butterflies. Thus there will be 21 butterflies of each color, as shown in Figure 4.

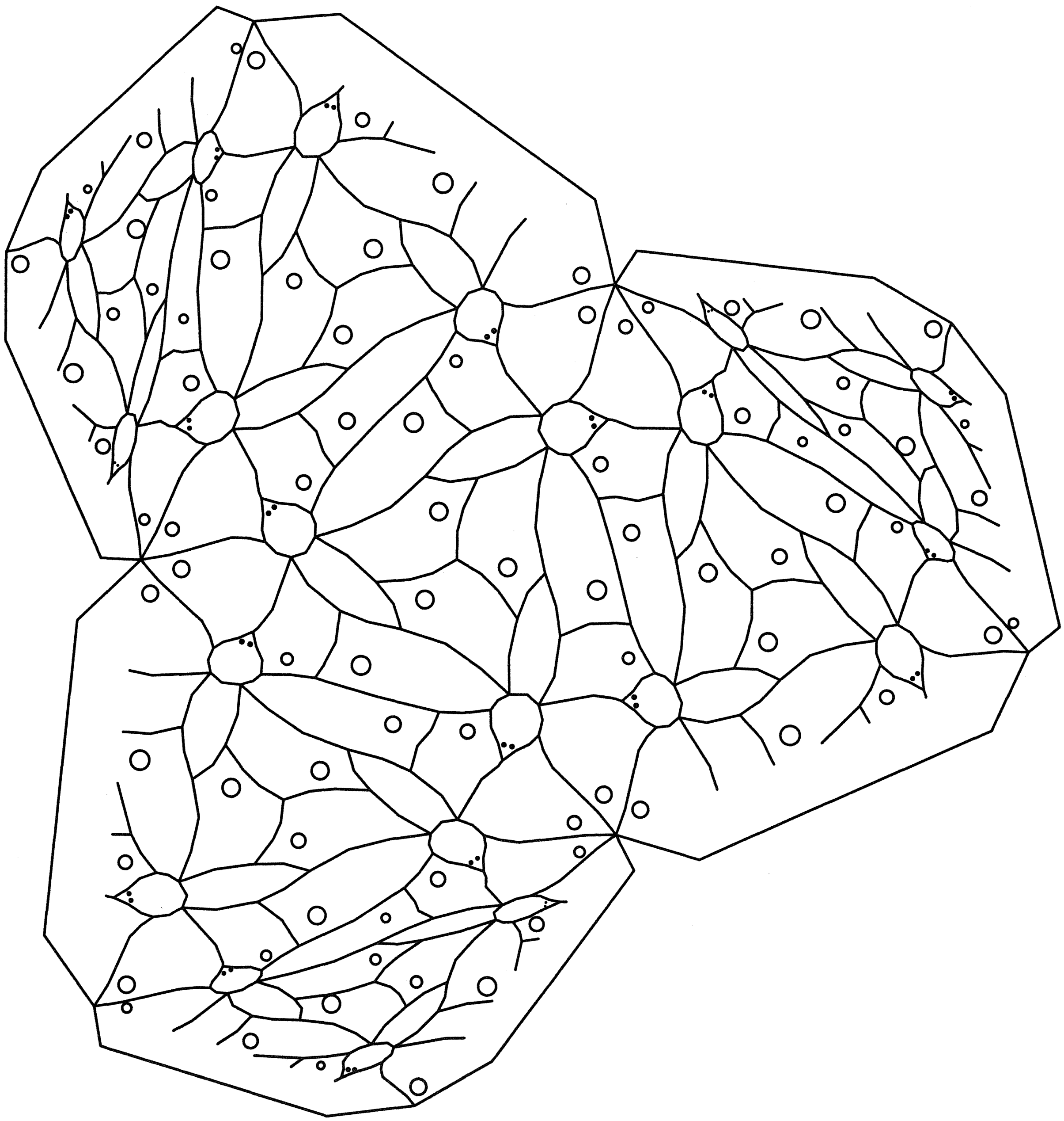

Figure 6: The net for the outer icosahedron pattern.

4 Connections to Regular Maps and Hyperbolic Geometry

Informally, a *map* is a tessellation of a finite surface into a finite number of faces (the "countries"), edges (borders of countries), and vertices (points where three or more borders meet). We use $\{p, q; g\}$ to denote a *quasi-regular map* on a surface of genus g consisting of p-sided faces that always meet q at a vertex. This is just a combinatorial condition, which is in contrast to the use of the Schläfli symbol $\{p, q\}$ to specify regular tessellations in which the faces are (metrically) regular p-sided polygons. This concept is weaker than the term "regular map" as used by others [9], [11], and [12], where it is also required that the automorphism group be transitive on the vertices, edges, and faces.

In this paper, we will only be concerned with surfaces that can be embedded in Euclidean 3-space without self-intersections, and are therefore orientable. For such a surface, the *genus* g is the number of holes it has. For $g \geq 1$, a surface of genus g can be formed from a $4g$-sided polygon by labeling the edges $\overrightarrow{u_1}, \overrightarrow{v_1}, \overleftarrow{u_1}, \overleftarrow{v_1}, \overrightarrow{u_2}, \overrightarrow{v_2}, \overleftarrow{u_2}, \overleftarrow{v_2}, \ldots, \overrightarrow{u_g}, \overrightarrow{v_g}, \overleftarrow{u_g}, \overleftarrow{v_g}$ and identifying corresponding edges according to their arrows; the resulting surface is a g-holed torus. In fact every orientable surface is a sphere (genus 0) or such a torus [6, page 10]. The universal covering surface of the usual 1-holed torus is the Euclidean plane [1, page 381] (the surface of genus 0, the sphere, is its own universal covering surface); for $g > 1$, the universal covering surface is the hyperbolic plane.

The "realization problem" for the map $\{p, q; g\}$ is to find a polyhedron of genus g in Euclidean 3-space (preferably without self-intersections) whose polygonal faces are p-sided polygons meeting q at a vertex, and thus realize the map as a polyhedron. The polyhedron of Section 2 is a realization of the map $\{3, 7; 3\}$. Schulte and Wills gave a different, more combinatorially regular realization of that map in [9], based on Klein's quadric $x^3y + y^3z + z^3x = 0$ and shown in Figure 7 (from [13]). One can see that their polyhedron

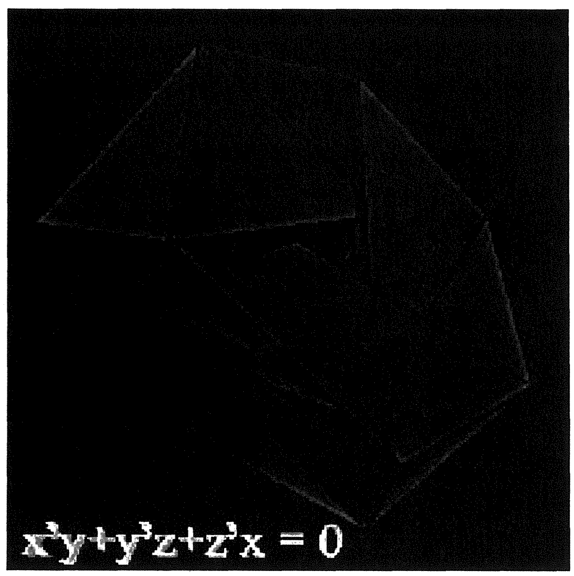

Figure 7: Schulte and Wills' realization of the map $\{p, q; g\}$.

is much more twisted than the one presented in Section 2. Possibly Schulte and Wills also knew of the polyhedron of Section 2. It is unusual to have two realizations of a map, since only a finite number of regular maps can be realized as polyhedra (it is an open question as to how many [11, page 232]).

As remarked above, for $g > 1$, the universal covering surface of an orientable surface of genus g is the hyperbolic plane. In fact a regular hyperbolic $4g$-sided polygon, or $4g$-gon, of vertex angle $\pi/2g$ exactly covers such a surface of genus g [1, page 382]. So, if there is a map $\{p, q; g\}$ on that surface, and there is a repeating hyperbolic pattern based on the regular tessellation $\{p, q\}$ (by p-gons meeting q at a vertex), then a piece of that pattern within a $4g$-gon can be used to cover the surface. Figure 8 shows a hyperbolic pattern based on the $\{3, 7\}$ tessellation, with a triangle of butterflies centered in the bounding circle. Figure 9 shows the pattern of Figure 8 overlaid with a regular hyperbolic 12-gon that can be used to exactly cover a surface of genus 3. There are 56 "butterfly triangles", and therefore 168 butterflies, contained in the 12-gon. The

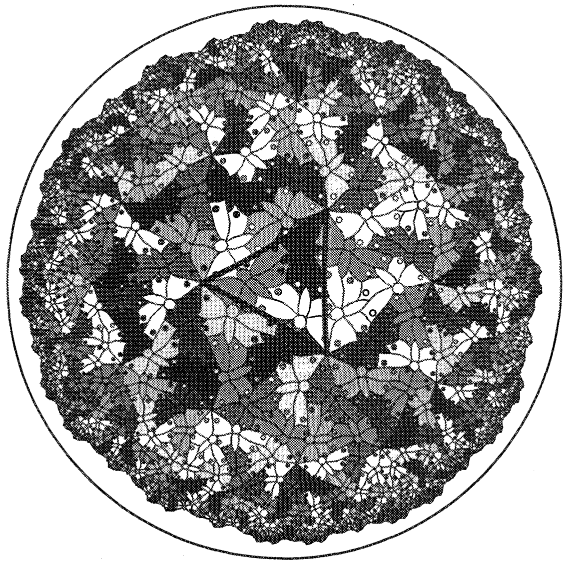

Figure 8: A hyperbolic pattern based on the regular tessellation $\{3, 7\}$ and Escher's motif of Notebook Drawing 70, with a centered butterfly triangle.

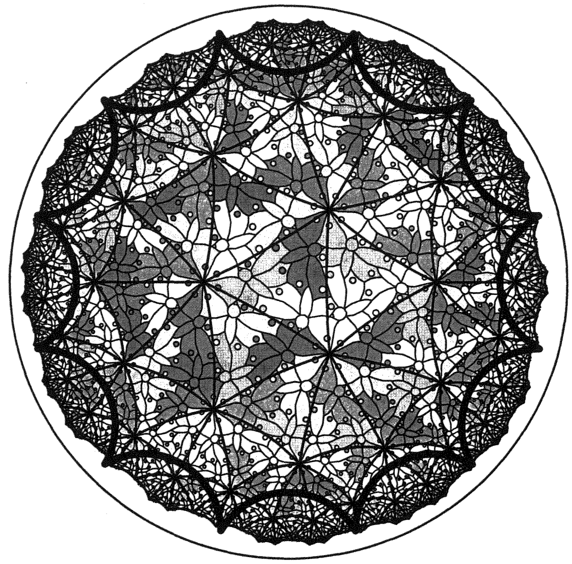

Figure 9: The pattern of Figure 8 overlaid with a 12-gon containing 56 butterfly triangles and consequently 168 butterflies.

same hyperbolic butterfly pattern is shown in [4, Figure 11], but with a vertex of butterfly triangles at the center of the bounding circle.

If the map $\{p, q; g\}$ can be realized as a polyhedron, then the polyhedron can be by covered by the repeating pattern, with a part of the repeating pattern within a p-gon covering each p-sided polygon of the polyhedron. This relationship between repeating hyperbolic patterns and polyhedral realizations of regular maps allows us to decorate such polyhedra in a regular way. This is how we came up with the pattern on the polyhedron in Figure 4. Aesthetically, it is desirable for the realized polyhedron to be as symmetric as possible, that there be as little distortion as possible in transferring the pattern to the polyhedron, and that the pattern on the polyhedron be colored symmetrically.

5 Conclusions and Future Work

The patterned polyhedron of this paper brings together concepts from four papers given at the 1985 Escher Congress in Rome. Both Coxeter [2] and Senechal [10] discussed the covering of 2-dimensional surfaces with Escher patterns. Wills exhibited polyhedron that solved realization problem for three regular maps [11], and Dunham presented hyperbolic patterns created from Euclidean Escher patterns [3].

We have shown how to cover a polyhedron of genus 3 with Escher's butterfly pattern of Notebook Drawing 70 by using scaling transformations alone. It would also be interesting to cover Schulte and Wills' polyhedron of genus 3 [9] with Escher's pattern which would (perforce) also require shearing transformations. To my knowledge, no one has carried out the covering of a torus with 18 butterflies, as suggested by

Coxeter [2], let alone a polyhedral torus (which can be constructed from 18 triangles in a twisted form, or 36 triangles without twists — both $\{3, 6; 1\}$ maps).

According to Wills [12], there are no known realizations of combinatorially regular polyhedra of genus 2, so using butterflies to cover such a polyhedron awaits its discovery. However, a surface of genus 2 has been covered with Escher's fish pattern of *Circle Limit III* — the fish backbone lines describing a "semi-regular" map of triangles and quadrilaterals [2, pages 27–30 and the color plate on page 393]. This raises the possibility of generalizing Archimedean polyhedra to any genus and covering them with "semi-regular" patterns, such as that of *Circle Limit III.*

In addition to polyhedra of genus 3, realizations are known for combinatorially regular polyhedra of genus 5, 6, 7, 9, 17, 19, and 41 [12]. It would be interesting to cover them with repeating patterns, especially the polyhedra of low genus. As with the patterned polyhedron of this paper, it would be desirable for the polyhedra to have as many symmetries as possible, and for the symmetries to be color symmetries of the pattern. Thus, there are many more avenues to explore in the area of regularly patterned polyhedra.

References

[1] H. S. M. Coxeter, *Introduction to Geometry* 2nd Ed., John Wiley & Sons, New York, 1969.

[2] H. S. M. Coxeter, Coloured symmetry, in *M. C. Escher: Art and Science*, H. S. M. Coxeter *et al.*, editors, North-Holland, Amsterdam, 1986.

[3] D. Dunham, Creating hyperbolic Escher patterns, in *M. C. Escher: Art and Science*, H. S. M. Coxeter *et al.*, editors, North-Holland, Amsterdam, 1986.

[4] D. Dunham, Artistic Patterns in Hyperbolic Geometry *Bridges 1999 Conference Proceedings,* pp. 239–249, 1999.

[5] T. Gettys, *Polyhedral Model Building*, (web page) `http://home.teleport.com/~tpgettys/tech.shtml`

[6] W. S. Massey, *Algebraic Topology: An Introduction*, Harcourt, Brace & World, Inc., New York, 1967.

[7] D. Schattschneider, *Visions of Symmetry: Notebooks, Periodic Drawings, and Related Work of M. C. Escher* W. H. Freeman, New York, 1990.

[8] D. Schattschneider and W. Walker, *M. C. Escher Kaleidocycles* Ballantine, New York, 1977.

[9] E. Schulte and J. M. Wills, A polyhedral realization of Felix Klein's map $\{3, 7\}_8$ on a Riemann surface of genus 3, *J. London Math. Soc.* (2), **32** (1985), no. 3, 539–547.

[10] M. Senechal, Escher designs on surfaces, in *M. C. Escher: Art and Science*, H. S. M. Coxeter *et al.*, editors, North-Holland, Amsterdam, 1986.

[11] J. M. Wills, Polyhedra in the style of Leonardo, Dali and Escher, *M. C. Escher: Art and Science*, H. S. M. Coxeter *et al.*, editors, North-Holland, Amsterdam, 1986.

[12] J. M. Wills, Polyhedral manifolds with properties of the Platonic solids, *Discrete Geometry and Convexity,* 224–229, Ann. New York Acad. Sci., 440, New York Acad. Sci., New York, 1985.

[13] J. M. Wills, *Polyhedral Models of Felix Klein's Quadric*, (web page) `http://www.math.uni-siegen.de/wills/klein/`