**BRIDGES**
**Mathematical Connections**
**in Art, Music, and Science**

# Transforming Some Infinite Polyhedra

Paul Gailiunas
25 Hedley Terrace, Gosforth
Newcastle, NE3 1DP, England
email: p-gailiunas@argonet.co.uk

### Abstract

A few infinite polyhedra divide space into congruent pieces. Any dual should have the same symmetries as the original polyhedron, so in these cases should also provide a congruent division of space. These conditions do not define a unique dual, and the polyhedra that satisfy them can be related to each other by continuous deformations, that generate finite space-filling polyhedra in the limit.

## Infinite Polyhedra

Infinite polyhedra were first discovered by H.S.M. Coxeter and J.F.Petrie in 1926[1], who considered the three regular cases: six squares at each vertex, ($4^6$), four hexagons at each vertex, ($6^4$), and six hexagons at each vertex, ($6^6$), (fig. 1). They have been called variously regular skew polyhedra, regular honeycombs, regular sponges and regular infinite polyhedra. From the figures it is clear that they can be derived from space-filling arrangements of cubes, truncated octahedra, and truncated tetrahedra and tetrahedra, respectively, by omitting certain faces. The same technique can be applied to other space filling arrangements of polyhedra to generate other infinite polyhedra, although this by no means exhausts all the possibilities[2], and in fact there is no known method that will generate all the infinite Archimedean polyhedra (i.e. with regular faces and vertex transitive).

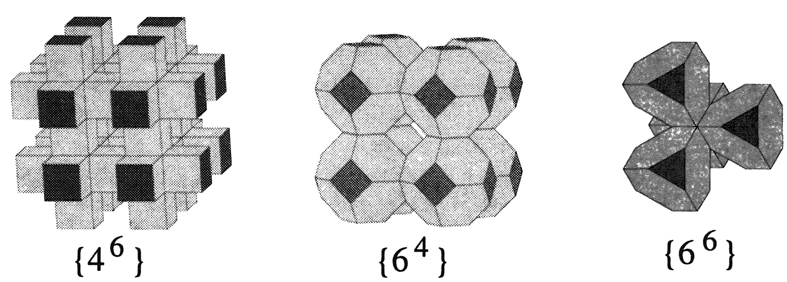



**Figure 1**: The three regular infinite polyhedra

## Duality

A pair of finite polyhedra are duals if there is a correspondence between the faces, edges and vertices of one and the vertices, edges and faces respectively of the other, such that faces with a common edge are dual to vertices connected by an edge. In the case of the regular (Platonic) polyhedra a dual can be constructed using the centroids of faces as vertices. By analogy it is not surprising that duals of the regular infinite polyhedra can be constructed by using the centres of the faces as vertices, and it can be seen that ($4^6$) and ($6^4$) form a pair of duals, and ($6^6$) is self-dual.

However this method fails in general since the centroids of all the faces at a vertex are not necessarily coplanar, even in some semi-regular (Archimedean) polyhedra, but the duals of finite Platonic and Archimedean polyhedra are unique (up to a scale factor), provided they are subject to the natural conditions that they have the same symmetry as the original polyhedron. The Archimedean polyhedra are isogonal (that is any vertex can be mapped onto any other by applying suitable symmetries of the polyhedron), and their duals are isohedral (the same applied to faces, rather than vertices). So in these cases, if symmetry is to be preserved, a vertex of the dual must lie on a line perpendicular to a face through its centre. Its precise position can be calculated by polar reciprocation in a sphere.

As with finite polyhedra, in general the centres of faces around a vertex of an infinite polyhedron are not coplanar, although there are several semi-regular infinite polyhedra (with faces that are regular polygons and all vertices the same) where they are. In fig.2, ($4^2 6^2$), for example, the four faces around a vertex have a plane of symmetry that does not pass through any of their centres, so joining each centroid with its mirror image produces a line perpendicular to the mirror plane. The two lines are therefore parallel, and define a plane that passes through all four centroids.

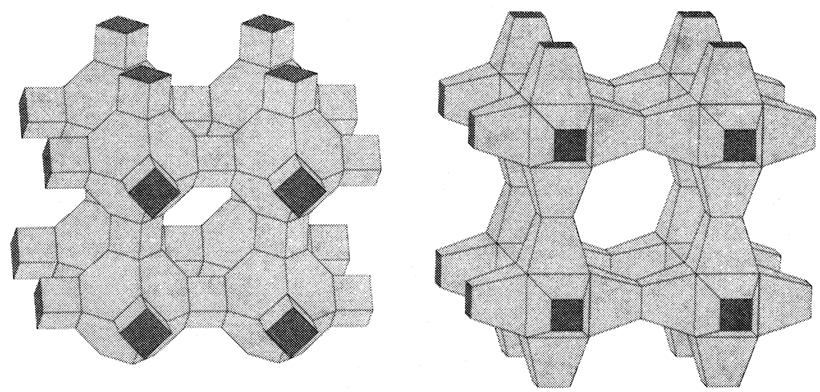
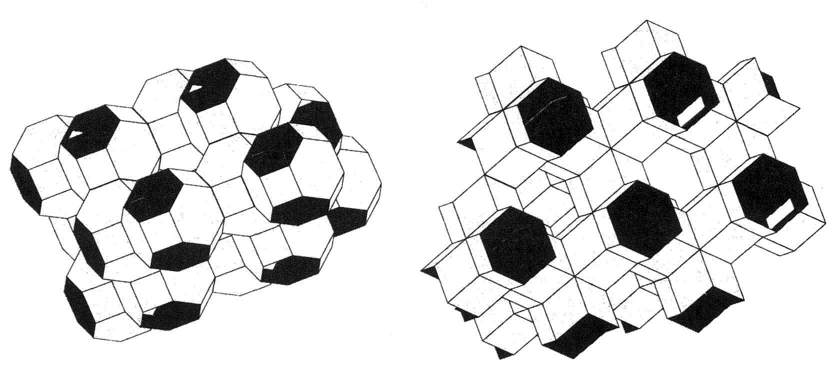


**Figure 2:** ($4^2 6^2$) In this example, which can be constructed from a space-filling arrangement of semi-regular polyhedra and cubes, space is not divided into congruent halves. The missing polyhedra are rhombi-truncated cuboctahedra. It is possible to construct a dual that has vertices at the centroids of the faces.

Fig. 3, (4 6 4 6), is based on the space filling arrangement of truncated octahedra, as is ($6^4$), but this time alternate hexagons are omitted, rather than squares. It has the symmetry of the diamond structure, like ($6^6$). In this case diagonals of both types of face pass through the common vertex, forming a network of infinite lines, so again the centres of the faces are coplanar. They define the vertices of a dual that can be derived from a packing of rhombic dodecahedra by omitting half the faces.



(4 6 4 6)

Dual (4 6 4 6), derived from space-filling rhombic dodecahedra by omitting half the faces.

**Figure 3.**

Fig. 4, (6 $4^3$), is not so obvious. Like most of the examples so far it divides space into congruent halves, but the faces around a vertex are not symmetrical by reflection, and although diagonals of the hexagons and squares common to a pair of octagonal prisms (which are adjacent to four other squares) are on infinite lines passing through the vertices, the diagonals of the squares belonging to the rhombitruncated cuboctahedra/octagonal prisms are not collinear. However, the octagonal prisms are related by a rotation of half a turn around a diagonal of their common face, so the centres of the squares are coplanar with it.

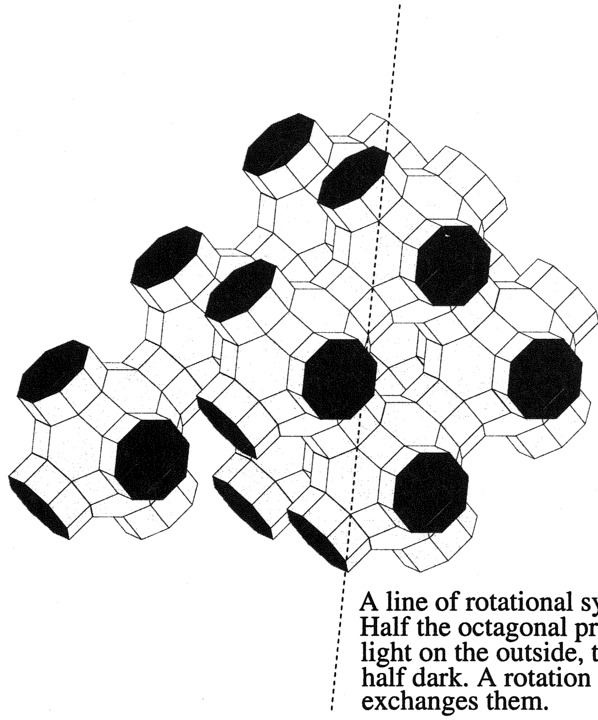



**Figure 4.** (6 $4^3$)

### Transforming Duals

Unlike finite cases the requirement for a polyhedron and its dual to have the same symmetry does not uniquely define the dual for infinite polyhedra, and polar reciprocation cannot usefully be applied. Infinite polyhedra such as those in the figures have planes that contain infinite sets of faces, edges and vertices, so that reciprocation would produce an infinite number of vertices at a point. Not only this but the infinite polyhedra considered here are periodic, and have translational symmetries that would not be preserved by polar reciprocation. There is no equivalent of polar reciprocation to decide between the possibilities for infinite polyhedra, although it is clear that dual vertices must lie on perpendicular lines through the centres of regular faces when these are lines of symmetry.

For the polyhedra described earlier it is not too difficult to consider all possible duals that retain the full symmetry of the original infinite polyhedron. Just as in the Archimedean polyhedra vertices of the dual must lie on perpendicular lines through the mid-points of the original faces. The mirror planes in fig.2, ($4^2 6^2$), mean that nothing surprising is produced by moving the dual vertices away from the centres of the faces, but the symmetries in the other examples allow the possibility of moving vertices such that corresponding face planes of the duals rotate about lines of symmetry. This ensures that the dual provides a congruent division of space. Of course the shape of the faces must change as this happens. In (6 $4^3$) there is only one type of rotation possible. The dual vertices corresponding to the square faces that are shared by octagonal prisms and rhombitruncated cuboctahedra (which are adjacent to two squares and two hexagons) can move on lines that are perpendicular to these faces, and join the centres of the prisms and the rhombitrunctaed cuboctahedra. When the vertices are at either of these extreme points (figs. 5 and 16) two edges of the dual face become collinear, so there is a triangle rather than a quadrangle, and, as some vertices coincide, space is cut into finite pieces, rather than congruent halves.

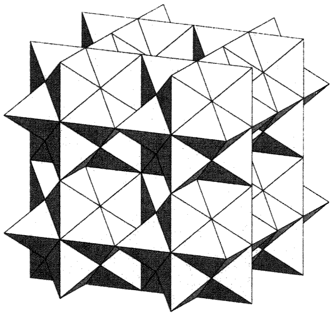


Figure 5.

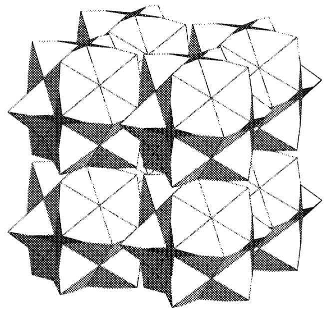


Figure 6.

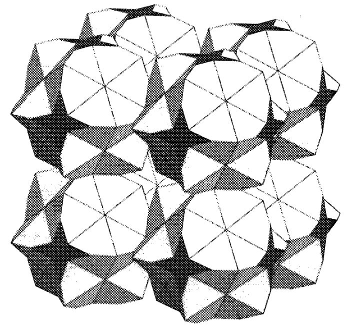


Figure 7.

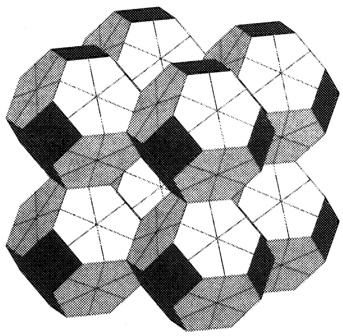


Figure 8.

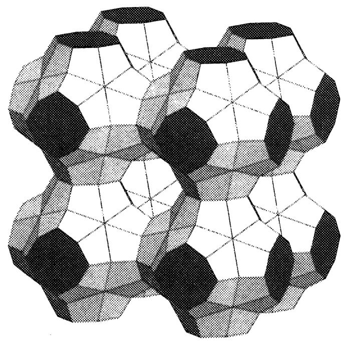


Figure 9.

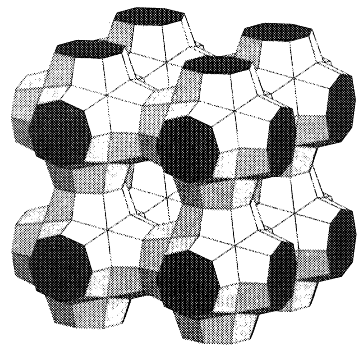


Figure 10.

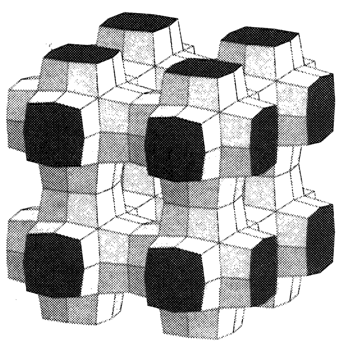


Figure 11.

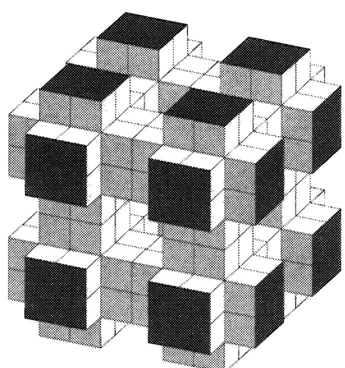


Figure 12.

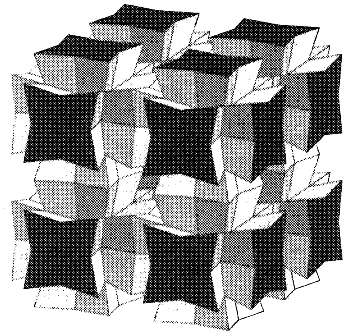


Figure 13.

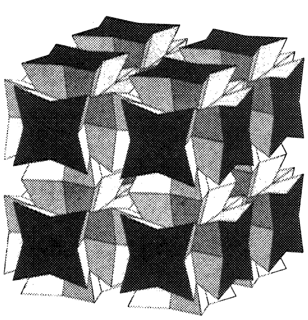


Figure 14.

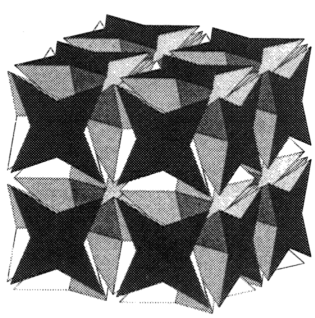


Figure 15.

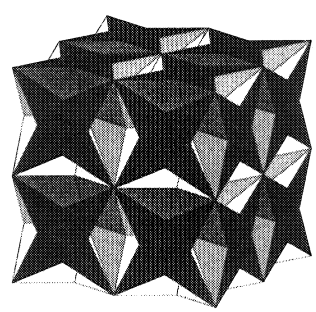


Figure 16.

When they are at the mid-points of the octagonal prisms space is filled by stellated rhombic dodecahedra (Fig. 5), and as the vertices begin to move away from the mid-points the faces become kite-shaped, and the space becomes connected into congruent halves (Fig. 6). The faces rotate as the vertices move, one set of dihedral angles approaches 180° (Fig. 7), and the faces become coplanar in sets of six (Fig. 8), forming the regular hexagons of ($6^4$). As the vertices move further the edges that form the square holes (dark in the diagram) open out into convex octagons (Fig. 9), and when they coincide with the mid-points of the faces of ($6\ 4^3$) the octagons are regular (Fig. 10). As the vertices move further the other set of dihedral angles approaches 180° (Fig. 11) and the faces become coplanar in sets of four (Fig. 12), forming the squares of ($4^6$). The square holes become four-pointed stars, but in a different orientation to those in Figs. 6 and 7 (Fig. 13) as the vertices that were coincident in Fig. 5 get further apart and the other set of moving vertices come closer together. Eventually (Fig. 16) vertices meet at the mid-points of the rhombitruncated cuboctahedra, generating another space-filling by finite polyhedra, here cut in half.

There are more possibilities with (4 6 4 6) since two types of rotation are possible: about diagonals of hexagons, or about diagonals of squares. Alternatively the vertices of the dual can be seen as moving on lines joining mid-points of adjacent truncated octahedra either through squares, or through the hexagons. In either case space remains divided into congruent halves, but if both movements are combined the resulting pieces are unequal. The figures have been arranged in a grid: moving down the page (in numerical order) the vertices are moving on lines through the squares, moving across the page (e.g. Figs. 17, 22, 27) the vertices are moving on lines through the hexagons. Unlike ($6\ 4^3$), which is based on a space-filling with two types of polyhedra, the mid-points of the truncated octahedra in (4 6 4 6) are equivalent, and lines that join them are symmetric about their mid-points. Whether a vertex moves towards one truncated octahedron or the other makes no difference to the resulting dual, and only half the ranges need be considered.

Fig. 17 shows part of the same structure as Fig. 3 viewed from a different angle. As vertices move from the mid-points of the squares those in the hexagons begin to flatten out (Fig. 18) until the faces become coplanar and the hexagons of ($6^6$) are produced (Fig. 19). Moving further causes the triangular holes to close up into three pointed stars (Fig. 20), and when the vertices reach the centres of the truncated octahedra (Fig. 21) a space-filling of finite polyhedra is produced.

Starting again from Fig. 17 and moving the vertices from the mid-points of the hexagons begins to flattens out the 4-vertices (Fig. 22) until the faces become coplanar at the same time as the vertices coincide at the centres of the truncated octahedra, producing a higher, cubic symmetry (Fig. 27).

Starting from Fig. 19 instead of Fig. 17, and moving the vertices from the mid-points of the hexagons produces an infinite polyhedron (Fig. 24) that does not divide space equally. Eventually (Fig. 29) a space filling with cubic symmetry is produced again, but there are two types of polyhedron. The convex one is Catalan's triakis hexahedron, the dual of the truncated octahedron.

The same process starting from Fig. 21 produces a space filling of rhombic dodecahedra (Fig. 31). This can also be considered as the end of the sequence moving down the page from Fig. 27. The faces move until they become coplanar in pairs. The concave polyhedron in the middle of the cluster has been completely squeezed out.

## The Limiting Cases

In all of the examples at the limits of the series sets of vertices coincide. Space-fillings of finite polyhedra result, some of which are well known, such as the stellated rhombic dodecahedra of Fig. 5, the cubes of Fig. 27, or the rhombic dodecahedra of Fig. 31. Others are fairly obvious variations of these, however two cases are not so obvious. The polyhedra in Fig. 16 are particularly difficult to visualize since they are cut in half, and they are illustrated more clearly in Fig. 32. The way they pack can be imagined by referring to Fig. 4, remembering that the arms of the stars are in the octahedral prisms, and the centres are at the centres of the rhombitruncated cuboctahedra, meeting in groups of six at right angles. Fig. 32 shows two of them. Now consider the other twelve surrounding octahedral prisms (holes in Fig. 4). They have their own sets of stars, arms of which fit in the gaps like the third one in Fig. 32.

Fig. 21 is interesting as an example of a space-filling polyhedron with tetrahedral symmetry and another view of it appears in Fig. 33.

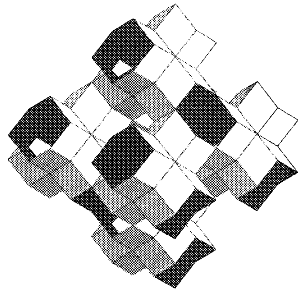


**Figure 17.**

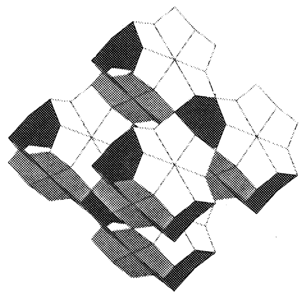


**Figure 18.**

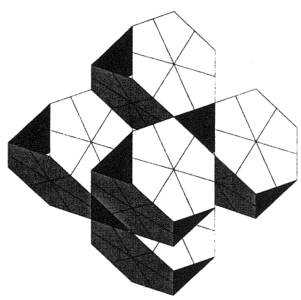


**Figure 19.**

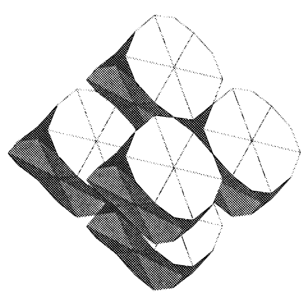


**Figure 20.**

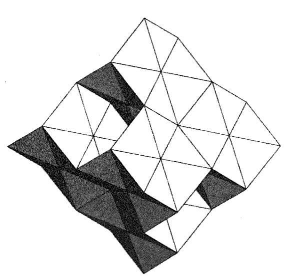


**Figure 21.**

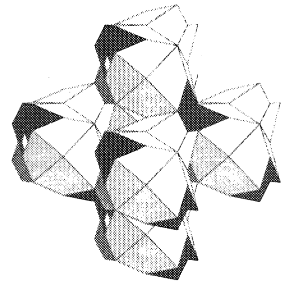


**Figure 22.**

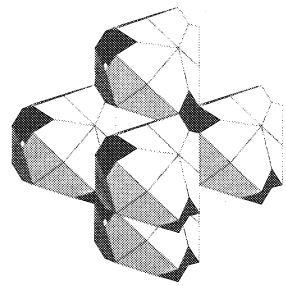


**Figure 23.**

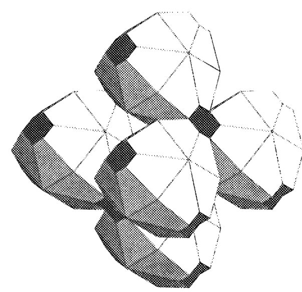


**Figure 24.**

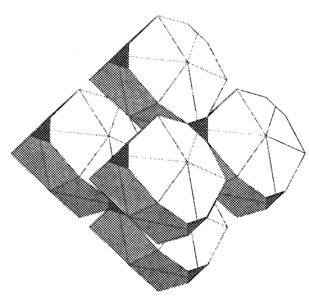


**Figure 25.**

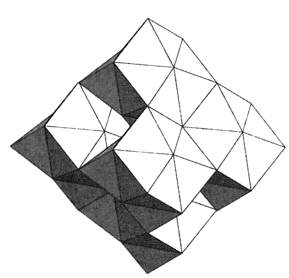


**Figure 26.**

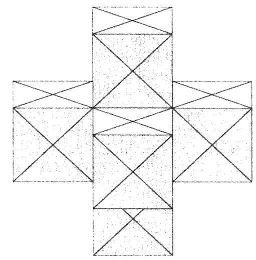


**Figure 27.**

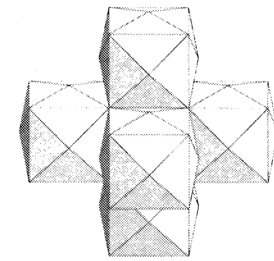


**Figure 28.**

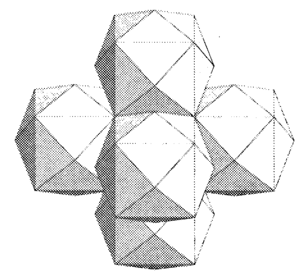


**Figure 29.**

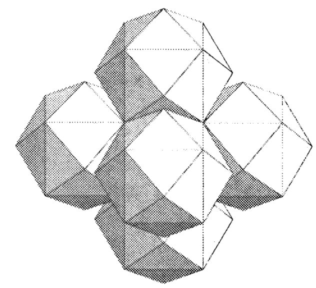


**Figure 30.**

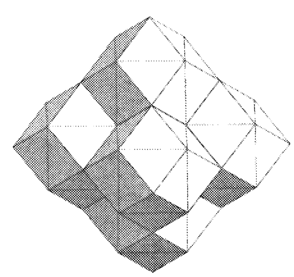


**Figure 31.**

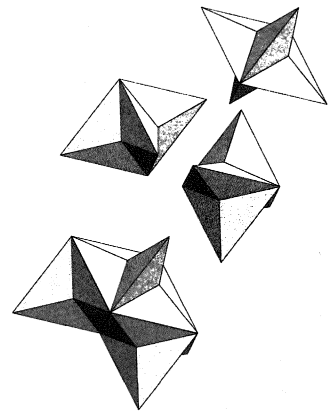


**Figure 32.**
Three of the polyhedra in Fig. 16, showing how they fit together. Their eight outside edges form the regular body centred cubic network. The faces are isoceles triangles having edge lengths in the ratio 1·5419:1 The face angles are approximately 101·5°, 39·2° and 39·2°.

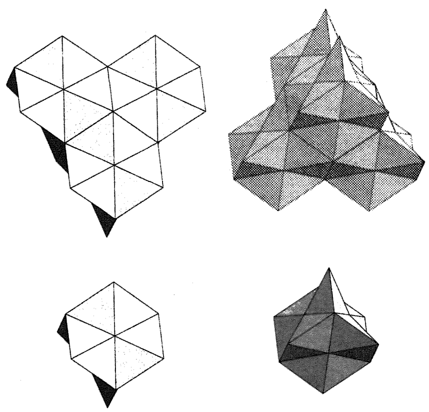


**Figure 33.**
Two views of the polyhedron in Fig. 21, showing how it fits together. The faces are isoceles triangles having edge lengths in the ratio 1·0445:1 The angles are approx. 63°, 58·5° and 58·5°.

## Software

The original data for the diagrams were calculated using a program written in BASIC and passed to !PolyDraw (for RISCOS machines) which generated the illustrations. Animated versions of the transformations have been produced using VRML.

## References


[1] Rouse Ball, W.W. and Coxeter H.S.M., Mathematical Recreations and Essays, Twelfth Edition, University of Toronto Press, 1974. p.152-3.

[2] e.g. Wachman A.M., Burt M. and Kleinman. Infinite Polyhedra. Technion, 1974.