Bridges
Mathematical Connections
in Art, Music, and Science

Hearing With Our Eyes: The Geometry of Tonal Space

Julian L. Hook
School of Music
Penn State University
University Park, PA 16802-1901
E-mail: jlh48@psu.edu

There is no intrinsic reason why musical structures should be represented graphically. Music is, after all, an auditory phenomenon. It consists of vibrations transmitted through the air as sound waves, received by our ears and processed by our brains as acoustic data. Visual aspects of music—the arrangement of musicians on a stage or of notes on a page, the gyrations of the conductor, the shine of the piano—are of secondary importance in our understanding and appreciation of a musical work.

Most human beings, however, are visually oriented. We rely on our eyes more than our ears or any of our other sense organs in finding our way around, identifying other people and objects, and learning new information. Because of this dependence on the visual world, most people find abstract concepts easier to grasp if they can somehow be *visualized*: if some sort of graphical, geometric representation can be devised showing, if only metaphorically, the important elements of the conceptual framework and their relationships with each other. Even if these elements exist in sound and time rather than in light and space, such a representation may help us to get our bearings and to interpret what we hear.

Pitches, chords, and key areas are examples of musical elements that lend themselves well to graphical depiction. Many music theorists through the centuries have drawn diagrams showing the appearance of "tonal space" from various perspectives. This paper offers a conceptual and historical overview of several of these geometric representations.

One-Dimensional Representations

The most obvious representation of musical pitch is a one-dimensional continuum, rather like an abstract sort of keyboard (Figure 1). Diagrams of this sort were drawn by the ancient Greeks (Figure 2), and more elaborately by Boethius in the sixth century [4, p. 104]. Despite its simplicity, this arrangement admits many possible variations, depending on the degree of continuity or discreteness desired, the particular pitches (and therefore the intervals) identified in the diagram, and the interpretation of the "distance" between pitches. The Euclidean representation in Figure 2, for example, locates pitches so that their distances from one endpoint of the line are proportional to the lengths of the vibrating strings needed to produce those pitches, so that the higher register (which appears at the *bottom* of the diagram) appears greatly compressed. In Figure 3, on the other hand, pitches are plotted at distances from the "zero point" in proportion to their *frequencies*, resulting in considerable *stretching* of the upper registers and therefore a fundamentally different concept of the "distance" between pitches. Perhaps it could be said that Figure 1 represents a pianist's view of pitch space, Figure 2 a violinist's, and Figure 3 an acoustician's.

The pitch space shown in Figure 1 is, in theory, infinitely long in both directions, and it is also infinitely fine: any number of pitches may be distinguished between C^4 and $C\sharp^4$. This diagram makes no attempt, however, to indicate any meaningful relationships between widely separated pitches, even at musically significant intervals such as the octave. An elegant device for depicting a relationship between octave-related pitches, advanced by Roger Shepard, involves bending the linear pitch space of Figure 1 into a helix, so that pitches separated by octaves align vertically (Figure 4). Depending on just how strong we believe the similarity of octave-related pitches to be, we can adjust the "stretch" of the helix—the vertical distance between successive loops. At maximum stretch, octave similarity is ignored and the helix is nothing more than the straight line of Figure 1. At the other extreme, octave-related pitches are considered fully equivalent, all the loops are compressed into a plane, pitches are replaced by pitch *classes*, and pitch

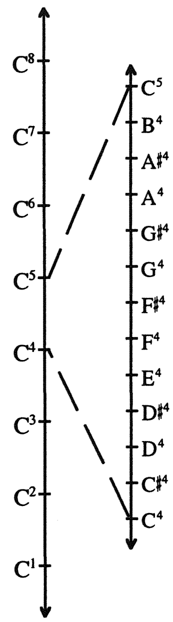

Figure 1: A one-dimensional representation of *pitch space,* with an enlargement showing finer detail in the octave from C^4 (middle C) to C^5.

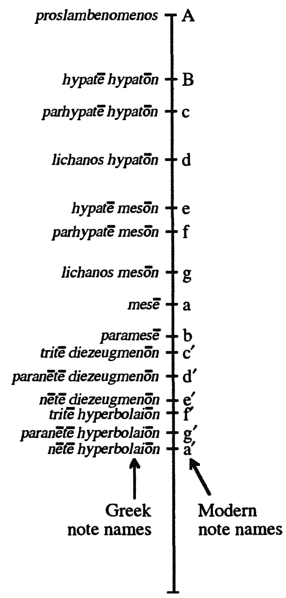

Figure 2: Pitch space as represented in the *Sectio canonis* attributed to Euclid (c. 300 B.C.E.) [2, p. 208].

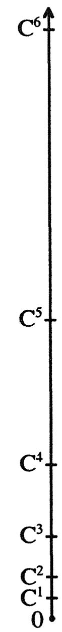

Figure 3: A frequency-based representation of pitch space.

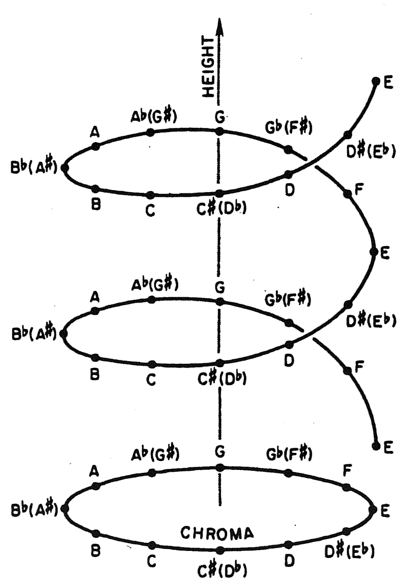

Figure 4: Roger Shepard's *pitch helix* [11, p. 114]. For historical precedents see [18].

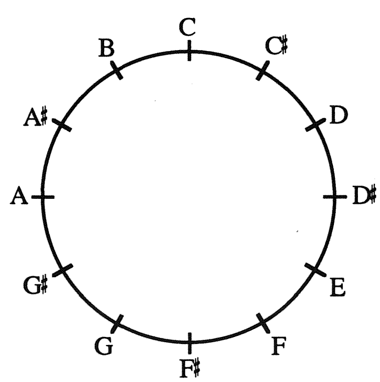

Figure 5: *Pitch-class space,* or the *chroma circle*: a collapsed version of Figure 4.

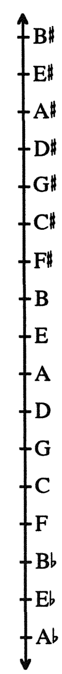

Figure 6: Pitch space arranged by fifths.

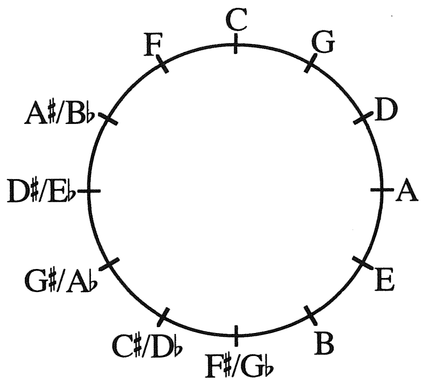

Figure 7: The *circle of fifths*, obtained from Figure 6 by identifying every twelfth pitch through enharmonic equivalence.

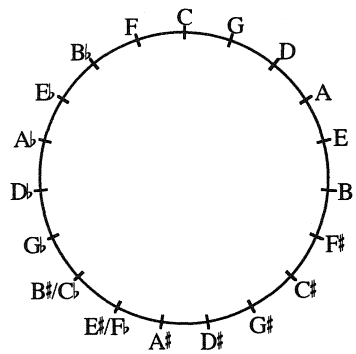

Figure 8: The circle of fifths in 19-note equal tuning, obtained from Figure 6 by identifying every 19th pitch.

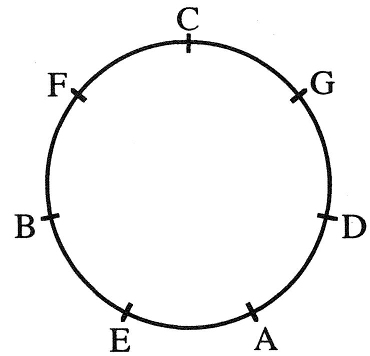

Figure 9: The *diatonic circle of fifths*, obtained from Figure 6 by identifying every seventh pitch.

space collapses into the *chroma circle* shown at the bottom of Figure 4 and redrawn in Figure 5 in orthogonal projection. Of course, Figure 5 can be obtained directly from Figure 1 by simply identifying the like-named pitches (every twelfth note), or by "wrapping the line into a circle."[1]

There are, of course, other ways to arrange pitches along a line besides the chromatic ordering. One of the most familiar of these is the ordering by fifths, shown in Figure 6. The line of Figure 6, like that of Figure 1, can be recast in circular form by identifying every twelfth pitch; the result is the familiar *circle of fifths* shown in Figure 7. It is worth noting, however, that the process of identification by which the circle of fifths is formed is fundamentally different from that by which the chroma circle of Figure 5 was obtained. Specifically, the two circles arise through two different equivalence relations. The chroma circle depends on *octave equivalence*—the notion that all C's, in all registers, are the same. The circle of fifths, on the other hand, depends on *enharmonic equivalence*—the notion that B♯ and C are actually the same pitch. Enharmonic equivalence, in turn, is predicated upon a particular system of tuning, twelve-note equal temperament. The circle of fifths would take on a very different appearance in other tuning systems. In Pythagorean tuning, with its acoustically pure fifths, the fifths do not form a circle at all; B♯ is *not* the same as C, and no succession of fifths will ever lead back to the original pitch class. In 19-note equal tuning—a system that has had many advocates—the concept of enharmonic equivalence works rather differently, and the circle of fifths takes on the form shown in Figure 8.[2]

Another way to obtain a circle of fifths is to assume not enharmonic equivalence but *diatonic equivalence*: disregard chromatic distinctions, ignore all ♯ and ♭ signs in Figure 6, and identify the pitches whose labels then coincide. The result is the *diatonic circle of fifths* shown in Figure 9. From a chromatic point of view, one of the intervals in Figure 9 (B-F) is not a perfect fifth but a diminished fifth; sequences in tonal music, particularly music of the Baroque era, often follow a path along this diatonic circle. It is important to recognize that Figure 9 was obtained as a transformation of Figure 6, but it could *not* have been obtained as a transformation of Figure 7: there is no consistent way to map the chromatic circle of fifths onto the diatonic one, since enharmonically equivalent pitches (such as B♯ and C) must be mapped to two different diatonic places (B and C).

In an attempt to incorporate information from the circle of fifths into his helical graph, Shepard

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[1] Despite the representations in three dimensions (Figure 4) and two dimensions (Figure 5), both of these diagrams are still correctly referred to as "one-dimensional" representations of pitch space. Locally, they still resemble the line of Figure 1; points that lie off the helix in Figure 4, or off the circle in Figure 5, do not represent pitches. Mathematically, Figures 4 and 5 could be described as embeddings of one-dimensional manifolds within spaces of higher dimension.

[2] In 19-note tuning [3], [17] the diatonic scale is very similar to that in 12-note tuning, but each major second comprises *three* chromatic steps (so between C and D there lie two distinct pitches, C♯ and D♭) and each minor second *two* (between E and F there lies a single pitch, E♯ = F♭).

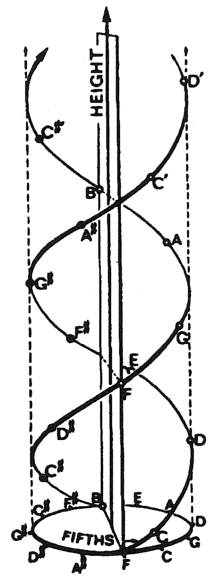

Figure 10: Shepard's double helix of musical pitch [23, p. 362].

devised the double helix shown in Figure 10. Whereas the projection of Figure 4 onto a horizontal plane was the chroma circle, the projection of Figure 10 is the circle of fifths. For any pitch on either strand of the double helix (for instance, G near the right side of the diagram), the closest pitches approximately above and below it on the other strand are those related to it by fifth (C and D). Each strand of the helix is now a whole-tone scale; the chromatic component appears to have been lost in this representation, but can be recovered as the projection of the two helixes onto the central vertical axis.

One could, of course, arrange pitches in order by any musically meaningful interval—major thirds, for example. Whether or not such a diagram closes into a circle will depend, once again, on our understanding of enharmonic equivalence. Diatonic versions of these circles may also be obtained. Some of the circles will not contain every pitch class: a "circle of major thirds" in standard twelve-note equal tuning is nothing but an augmented triad (although the diatonic "circle of thirds" includes every diatonic pitch class). The usefulness of such a diagram in isolation may be questioned; it can, however, form a valuable component of a more complex construction in higher dimensions, to which we shall next turn our attention.

Two-Dimensional Representations

As the number of musical relationships to be depicted increases, one-dimensional representations of pitch space quickly become inadequate. Pitch space organized by fifths naturally includes pitches related by other musically significant intervals such as thirds, but the arrangement fails to elucidate those relationships: there is nothing in Figure 6 or 7, for example, to indicate any connection between the pitches C and E other than the fact that they are separated by four fifths. To convey our musical understanding that a major third is something of importance in its own right, rather than just a multiple fifth, it becomes necessary to diagram pitches on a *surface*, in which one can move from pitch to pitch in two different dimensions.

Figure 11 presents one such representation. Along a horizontal axis in this diagram, pitches are organized by minor thirds; along a vertical axis, by major thirds. Other intervals, obtained by adding or

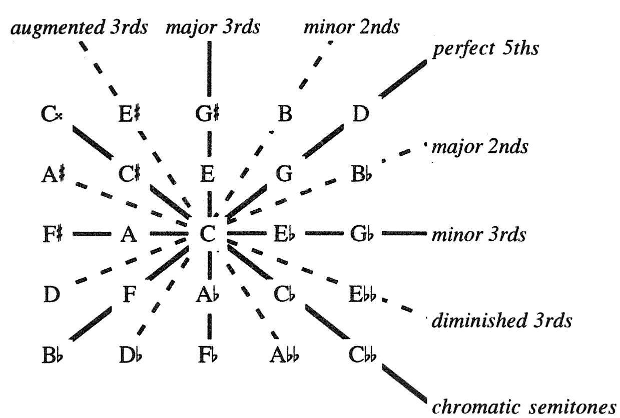

Figure 11: A two-dimensional representation of pitch space.

C	E♭	F♯	A	C	E♭	F♯	A	C
A♭	B	D	F	A♭	B	D	F	A♭
E	G	B♭	C♯	E	G	B♭	C♯	E
C	E♭	F♯	A	C	E♭	F♯	A	C
A♭	B	D	F	A♭	B	D	F	A♭
E	G	B♭	C♯	E	G	B♭	C♯	E
C	E♭	F♯	A	C	E♭	F♯	A	C

Figure 12: A simplified version of Figure 11 assuming enharmonic equivalence. The dashed rectangular blocks are all identical.

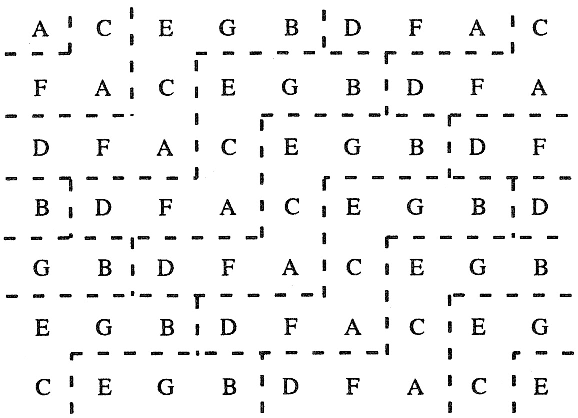

Figure 13: A two-dimensional representation of diatonic pitch space.

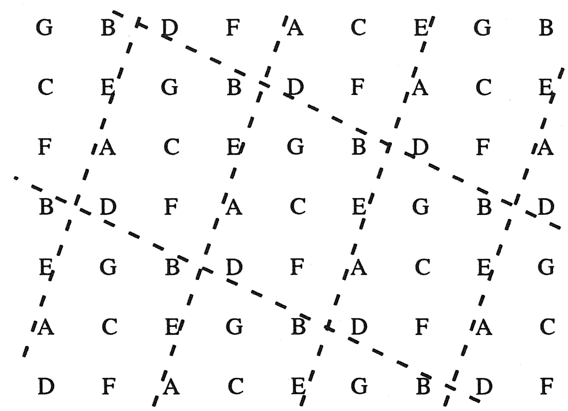

Figure 14: Another two-dimensional representation of diatonic pitch space.

subtracting major and minor thirds, then appear in the diagonals of the diagram: semitones along the northwest-to-southeast diagonal, and the circle of fifths along the southwest-to-northeast. Other repeating interval patterns, including the whole-tone scale, show up in "knight's-move" diagonals, indicated by dashed lines in the diagram. (All the lines shown intersect at the central C, but there is actually nothing privileged about that C; any other line in the diagram parallel to one of those depicted will consist of the same intervals.) Of course, many different combinations of two intervals could have been chosen for the two primary axes, with corresponding variation in the appearance of other intervals. The particular representation chosen in Figure 11 is attractive because all the most musically significant intervals take simple geometric forms here; in fact, several music theorists have constructed diagrams of pitch space closely akin to Figure 11.[3]

No conception of enharmonic equivalence is presupposed in Figure 11, and it is impossible to expand the diagram by one row or column in any direction without introducing triple sharps or flats. If we identify enharmonically equivalent pitches, however, an apparently larger tract of pitch space can be viewed easily, as shown in Figure 12. This figure is only "apparently" larger than Figure 11 because there are, of course, only twelve pitch classes represented here. Each rectangular block indicated by dashed lines encompasses all twelve; since all the blocks are identical, any one would theoretically suffice for the whole diagram, with the understanding that opposite edges of the rectangle are sewn together. Topologically, that is, Figure 12 is a torus; this torus stands in the same relation to the plane of Figure 11 as does the circle of Figure 7 to the line of Figure 6, but in a higher-dimensional space. Many distinctive musical features may be noted in Figure 12. Major triads (such as C-E-G) form triangles of one characteristic orientation, minor triads (A-C-E) of another. Each vertical column outlines an augmented triad, each horizontal row a diminished-seventh chord. A pair of adjacent rows forms an octatonic collection; a pair of adjacent columns forms a hexatonic collection [5].

The higher-dimensional analog of Figure 9, diatonic space, is shown in Figure 13. As in the one-dimensional case, it should be noted that Figure 13 is derived from Figure 11, *not* from Figure 12. The particular representation shown in Figure 13, corresponding to the arrangement of Figure 11, is less than ideal because of the inconvenient zigzag shape of the repeating modules encompassing all seven diatonic pitch classes. (Actually there are several possible shapes for these "tiles," of which only one is shown.) Another version, mathematically equivalent but visually simpler, is shown in Figure 14; here the tiles can be conveniently represented as parallelograms.

Hugo Riemann in 1914–15 [21] presented the *Tonnetz* shown in Figure 15. (Several similar

[3] Also encountered in the literature are a representation using major thirds on one axis and perfect fifths on the other [15, p. 21], and one based on the chromatic and whole-tone scales [23, p. 375].

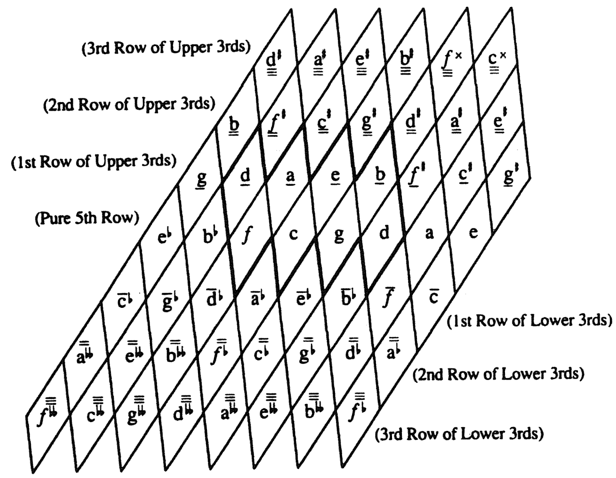

Figure 15: Hugo Riemann's *Tonnetz* [21, p. 102].

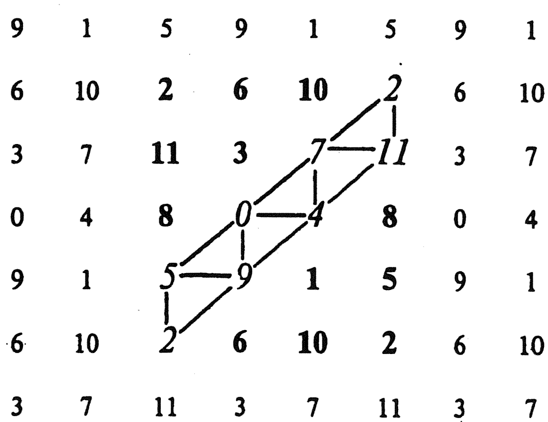

Figure 16: Balzano's group-theoretic model of pitch space [1, p. 73].

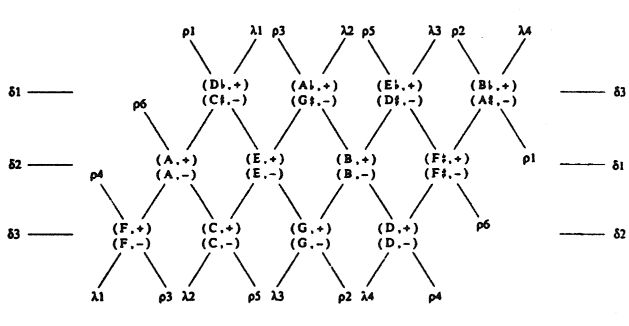

Figure 17: Hyer's representation [9, p. 133].

diagrams had appeared previously [8], [20].) Riemann used this table to map out harmonies: every major triad appears as an upward-pointing triangle and every minor triad as a downward-pointing triangle. Comparison of Figure 15 with Figure 11 shows that they are essentially identical; only the orientation is slightly different. (The vertical major thirds of Figure 11 are angled from southwest to northeast in Figure 15, and other relationships are altered accordingly.) Riemann was uncomfortable with the idea of enharmonic equivalence, which explains the presence of numerous double sharps and flats in his diagram (and also explains why we must compare Riemann's table to Figure 11 rather than to Figure 12). In fact, he was so careful about tuning differences that he distinguished the pitch of a pure major third (for example, E in the triad C-E-G) from the corresponding pitch obtained by fifths (the E at the right side of the middle row of the diagram). These pitches theoretically differ by a ratio of 81/80, the *syntonic comma*; Riemann indicated comma differences by over- and underscores. In an extreme example, the pitch at the lower left-hand corner of the diagram is labeled F♭♭ with a triple overscore; this pitch is three syntonic commas (almost two-thirds of an equal-tempered semitone) higher than the F♭♭ that would appear 25 places to the left of C in the middle row (and of course it differs also from the various E♭'s and D♯'s in the diagram, to all of which it is enharmonically equivalent by most musicians' way of thinking).

Riemann's table was motivated by acoustical and harmonic considerations; he was surely unaware of its rather profound mathematical properties. In recent years, however, an explosion of activity in transformational theories of harmony has revived interest in the *Tonnetz* and related figures [6]. Gerald Balzano's diagram (Figure 16) is constructed algebraically, from a study of the group-theoretic structure of the twelve-note equal tuning system. Balzano depicts pitch classes with the numbers 0 through 11 and lays out major thirds horizontally and minor thirds vertically; his diagram can thus be regarded as a sideways version of Figure 12. The bold rectangle at the center is the repeating module from which the whole diagram is generated, corresponding to the dashed outline in Figure 12. Triangles indicate major and minor triads; the pitches linked by the lattice of triangles are the pitches of the C major scale, corresponding precisely to the zigzag segment highlighted by Riemann in Figure 15.[4]

Yet another diagram similar to Figure 12 is Brian Hyer's in Figure 17. Major and minor thirds appear along the two diagonal axes, giving the configuration an orientation similar to Riemann's. Hyer shows only one copy of the repeating module here; the connections of opposite sides are indicated by labels on the axes. Thus as one moves southwest from G along line λ3, one disappears off the diagram, only to re-emerge at the other end of line λ3 and arrive at E♭.

Representations of Chords and Key Areas

Hyer's diagram depicts one aspect of musical structure not explicitly indicated in any of our earlier figures. To this point, our diagrams have been presented as models of *pitch* only. It is desirable, however, to be able to describe relations among *chords* and *key areas*, as well as among single pitches, by means of such geometric constructions. Some new considerations arise in conjunction with this more complex objective. Hyer solves this problem in a fairly rudimentary way: each node in Figure 17 is labeled not with a single pitch (C) but with the names of two chords ((C, +) and (C, –), indicating C major and C minor triads).

Some of the earlier diagrams can easily be employed to model chords or key areas successfully. This is particularly true of the diatonic space of Figure 14. We can regard each of the seven diatonic pitch names as standing for one of the seven diatonic triads in the key of C major; the diagram then takes on the appearance of Figure 18. This representation closely accords with what Fred Lerdahl calls *chordal space*, whose "core" he diagrams as in Figure 19. Lerdahl's 3 × 3 array serves in place of one parallelogram module, although it is slightly redundant in that two chords (iii and vi) appear twice.

In attempting to model relationships among more chords than just those that are diatonic in one key, a fundamental difficulty arises in the need to include both major and minor triads. In some cases, as in Figure 17, this

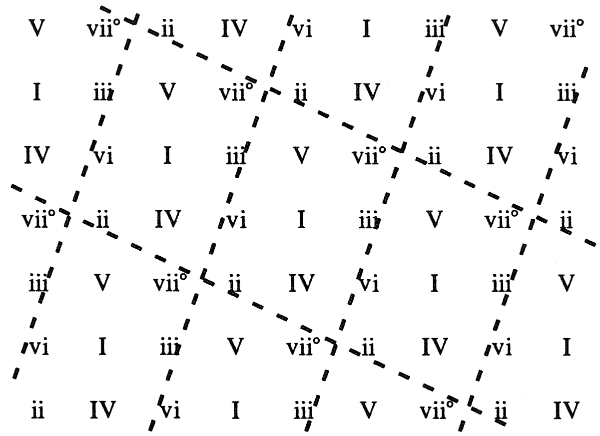

Figure 18: Diatonic chordal space.

iii	V	vii°
vi	I	iii
ii	IV	vi

Figure 19: The "core" of chordal space as represented by Lerdahl [14, p. 57].

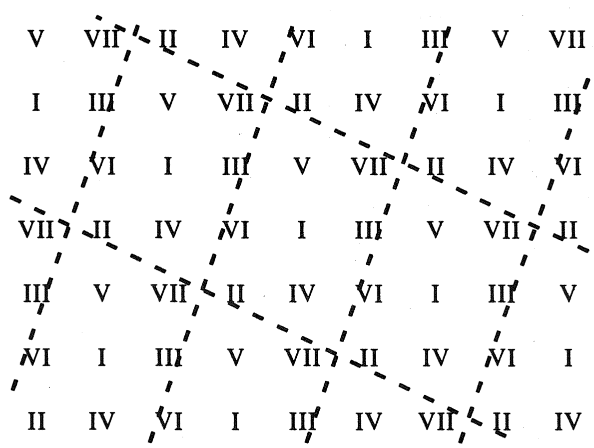

Figure 20: A mode-neutral representation of chordal space.

[4] Readers who observe that Balzano's diatonic scale, like Riemann's, appears to contain only six triads are invited to discover for themselves where the seventh one has gone.

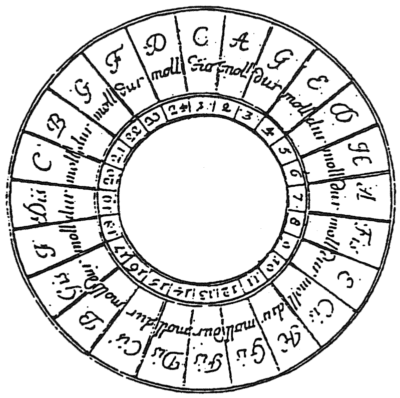

Figure 21: The circle of fifths as represented by Johann David Heinichen [7, p. 837].

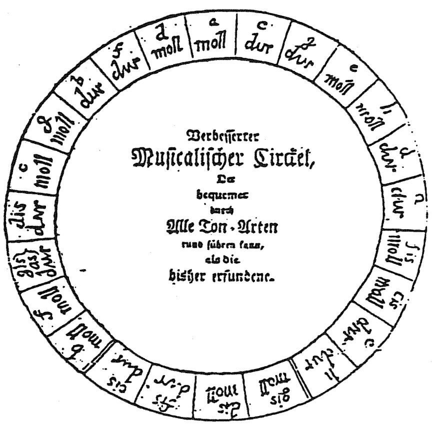

Figure 22: The circle of fifths as depicted by Johannes Mattheson [19, p. 131].

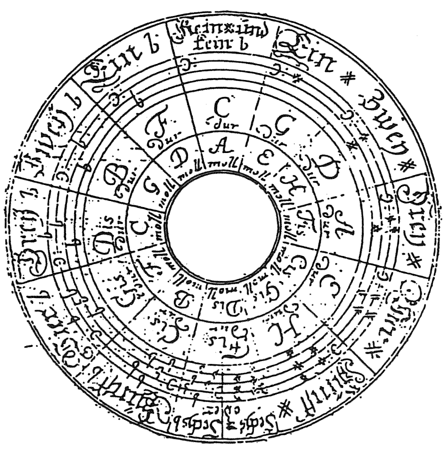

Figure 23: The circle of fifths as depicted by David Kellner [10, p. 60].

problem can be circumvented by identifying the two. It is occasionally useful to recast Figure 18, diatonic chordal space, in the mode-neutral fashion shown in Figure 20. Whereas in Figure 18 each symbol represents one of the seven diatonic triads in a major key, in Figure 20 there may be several possible interpretations of any one symbol: in C major, for example, "III" could represent E major, E minor, E♭ major, or E♭ minor.[5]

Frequently, however, it is necessary to distinguish major from minor—a distinction that has been graphically problematic ever since Johann David Heinichen constructed the earliest known circle of fifths in 1728. Figure 21 shows Heinichen's arrangement. Each major key is paired with its relative minor, with the odd result that major seconds appear between the alternate pairs of keys. Johannes Mattheson, a few years later, tried the alternate configuration shown in Figure 22. The major seconds are gone in Mattheson's scheme, but are replaced by even stranger goings-on: C major and G major are adjacent, but two minor keys separate C major from F major. As Peter Westergaard has pointed out [25], the problem with both of these schemes lies in the attempt to depict two different types of relationship (circle-of-fifths and relative major/minor) in one dimension (around the perimeter of the circle). David Kellner, in 1737, was apparently the first to solve the problem neatly by nesting a circle of minor keys inside a circle of major keys so that relative keys align (Figure 23). Kellner's circle accurately models the modern conception of "closely related keys": the keys directly adjacent to C major around the outer circle or directly or diagonally adjacent to it on the inner circle are exactly the five other keys having no more than one sharp or flat in their key signature.[6]

How, then, do we incorporate minor keys into a two-dimensional representation of pitch space such as Figure 12? A solution analogous to Kellner's concentric circles would be to "thicken" Figure 12, creating two parallel planes, one for major keys and one for minor, as shown in Figure 24. Each major key in the near plane is aligned with its relative minor in the more distant one. The rectangular tiles of Figure 12 have now become 2×3×4 rectangular blocks, each containing all 24 keys. Since Figure 12 is topologically a torus, Figure 24 has the structure of two nested tori, like a tire and an inner tube.

Figure 24 has a certain theoretical appeal, but as a depiction of the musical relations among key areas, it leaves much to be desired. For one thing, the three-dimensional configuration makes some relationships a bit difficult to visualize. A more serious shortcoming is that the dia-

[5] Figure 20 is similar to Lerdahl's *scale-degree space* [14, p. 100]. Lerdahl uses scale-degree numbers $\hat{1}$–$\hat{7}$ rather than Roman numerals I–VII.

[6] Figures 21–23 are all reproduced in [25].

gram suggests far too many "close" key relationships: if horizontal, vertical, front-to-back, and diagonal steps are allowed, fully eighteen of the 24 keys lie within one step of C major; this includes such seemingly remote keys as B major and B♭ minor, but excludes some apparently much nearer ones like D major and G minor.

An ultimately more satisfactory map of tonal space is shown in Figure 25. Here the vertical axis is the circle of fifths and the horizontal axis alternates between relative keys and parallel keys.[7] The dashed rectangle includes all 24 key areas. One might complain that the dual function of the horizontal axis is objectionable for the same reason as the dual functions of Heinichen's and Mattheson's circles of fifths, but in fact the offense is a much lesser one in this case. Heinichen interrupted what should have been a continuous series of keys—the circle of fifths, generated by the relation labeled *D* in Figure 25—by inserting the relative keys. Neither the relative relation nor the parallel relation by itself generates such a continuous series; each relation is self-inverting, as indicated by the double-headed arrows labeled *R* and *P* in Figure 25. Only by combining the two can a continuous series of keys be generated. (The notations *R*, *P*, *D*, and *L* are in widespread use in "neo-Riemannian" transformational theory [6], [9]. *D* can be read "is the dominant of"; *L* is Riemann's *Leittonwechsel*.) Anyone who is still disturbed by the dual-purpose axis is invited to imagine the diagram folded into a staircase in three dimensions, as shown in Figure 26.

It is reasonable to ask whether a diagram such as Figure 25 (or 26) can be said to model musical understanding of key relationships in any quantifiable way. One indication that it does lies in the observation that a simple and musically meaningful measure of distances between keys can be read directly from Figure 25. The distance between two keys is taken to be the number of steps needed to get from one to the other in the diagram, allowing horizontal, vertical, and diagonal moves. The distances of all 24 keys from C major and

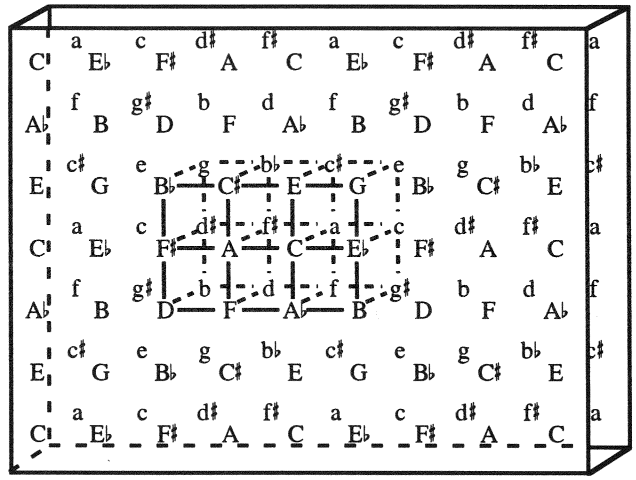

Figure 24: A two-layer representation of tonal space. Major keys lie in the front layer, minor keys in the back.

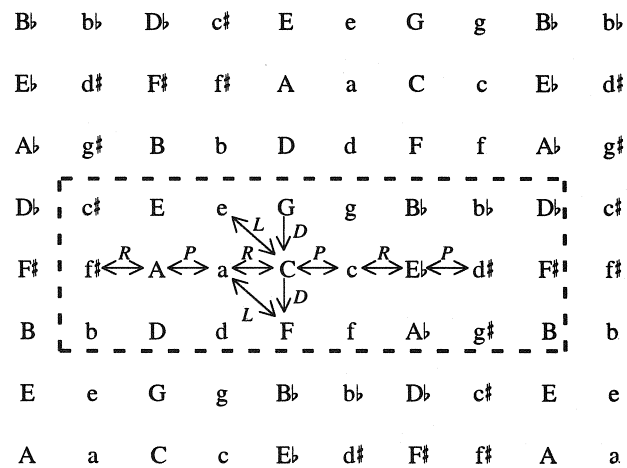

Figure 25: A two-dimensional representation of tonal space.

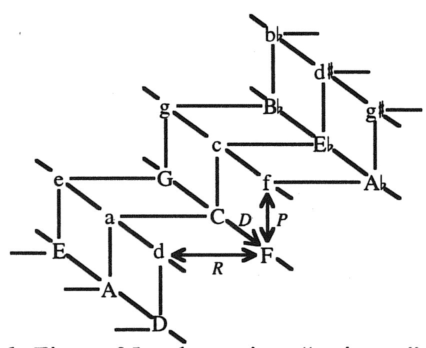

Figure 26: Figure 25 redrawn in a "staircase" configuration.

[7] The first theorist to diagram tonal space in this way was apparently Gottfried Weber [24], whose *Tabelle aller Tonarten-Verwandtschaften* is an exact mirror image of Figure 25. The scheme of Figure 25 also lies at the core of Arnold Schoenberg's well-known "chart of the regions" [22, p. 20]. Interestingly, Schoenberg's descriptions of the nearness of key relationships [22, pp. 68–69] bear little relation to the geometry of his diagram.

from C minor are tabulated in Figure 27. The table for C major divides the other 23 keys into three categories, at distances of 1, 2, and 3 from C major. The keys at distance 1 include all the traditional diatonically related keys as well as the parallel minor and the minor dominant and subdominant. At the other extreme, the five keys at distance 3 from C major are the tritone-related major key (F♯), its relative and parallel minor keys (d♯ and f♯), the minor Neapolitan (c♯), and the "hexatonic pole" (g♯) [5].[8]

These five distantly related keys all have another intriguing property: they are equally distant from C major *in more than one direction* in the diagram. This observation can be related to the perception of ambiguity that often accompanies the use of distantly related keys in close proximity, such as the notorious juxtaposition of G minor and D♭ major in the March to the Scaffold from Berlioz's *Symphonie fantastique*. From G minor near the center of Figure 25, one can move to D♭ major in three steps in any of three different directions (east, south-southeast, or northwest); the listener has no clues to signal which of the relationships is the one intended.

A remarkable study in music perception by Carol Krumhansl and E. J. Kessler has provided further support for the validity of Figure 25. Krumhansl and Kessler obtained a measure of perceived inter-key distances, applied multidimensional scaling techniques, and derived the perceptual map of keys shown in Figure 28 (another torus). Here the circle of fifths unfolds in the opposite direction from that in Figure 25, but otherwise the similarities of the two arrangements are striking. Moreover, the keys positioned closest to C major in Figure 28 are precisely those at distance 1 according to Figure 27—a notable result, especially in light of the complex psychoacoustic considerations that come into play in any attempt to measure listeners' perceptions in a musical context.[9]

Theorists are accustomed to saying that developmental passages in tonal music "explore" various key areas; some have attempted to depict this musical space exploration graphically.[10] A diagram such as Figure 25 provides a consistent and systematic methodology for mapping journeys through tonal space. Lerdahl has analyzed several pieces of music in this way. Lerdahl points out, however, that Figure 25 fails to distinguish between *chords* and *regions* (key areas); in a path through Figure 25, one cannot tell if a given harmony is a tonic, a dominant, or something else. To address this problem, Lerdahl refines the diagram into the *chordal/regional space* shown in Figure 29. The large-scale organization here, indicated by the boldface key areas, follows that of Figure 25. But each key area carries its own cluster of "satellites," a miniature representation of chordal space in that key, corresponding to Figure 19. Lerdahl traces the trajectories of Chopin Preludes and other musical works through the spaces of Figures 25 and 29, and points out that numerous tonal relationships are clarified by the deeper structure of the latter arrangement [14, pp. 94–98].

Diagrams such as those studied here exhibit a pleasing balance between the abstract and the concrete. On the one hand, as mentioned at the outset, they are mere geometric figures intended somehow to convey information about something that is not intrinsically geometric at all. On the other hand, the elements represented here—pitches, chords, and keys—are familiar to every musician, and the relationships among them are presented in ways that seem to model the listening experience with some accuracy. In some ways, apparently, it is only with our eyes that we can make sense of what we hear.

[8] Lerdahl has devised a more complex notion of "regional distance" [14, p. 69]. The simpler version given here, however, seems intuitively at least as accurate as Lerdahl's. In his system, for example, the Neapolitan key D♭ is at a distance 23 from C major, while e♭, f♯, and b♭ are all at the nearer distance of 21.

[9] For discussions of some of these considerations see [11], [18], and [23, pp. 369–370].

[10] An early example, often reproduced, is a diagram constructed by Alfred Lorenz in 1926 portraying the large-scale tonal structure of Wagner's opera *Tristan und Isolde* [16, p. 178].

Distances from C major				Distances from C minor			
Major key	**Distance**	**Minor key**	**Distance**	**Major key**	**Distance**	**Minor key**	**Distance**
C	0	c	1	C	1	c	0
D♭	2	c♯	3	D♭	2	c♯	2
D	2	d	1	D	2	d	2
E♭	2	d♯	3	E♭	1	d♯	2
E	2	e	1	E	3	e	2
F	1	f	1	F	1	f	1
F♯	3	f♯	3	F♯	3	f♯	3
G	1	g	1	G	1	g	1
A♭	2	g♯	3	A♭	1	g♯	2
A	2	a	1	A	3	a	2
B♭	2	b♭	2	B♭	1	b♭	2
B	2	b	2	B	3	b	2

Figure 27: Table of inter-key distances derived from Figure 25.

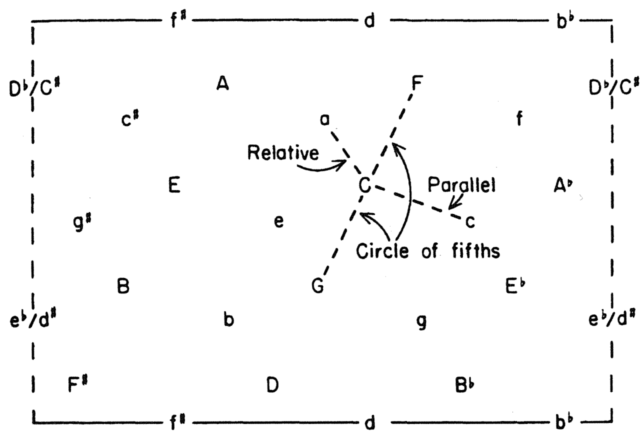

Figure 28: The map of musical keys according to perceptual studies by Krumhansl and Kessler [12, p. 55].

iii	V	vii°	III	V	VII	iii	V	vii°	III	V	VII	iii	V	vii°
vi	**B**	iii	VI	**b**	III	vi	**D**	iii	VI	**d**	III	vi	**F**	iii
ii	IV	vi	ii°	iv	VI	ii	IV	vi	ii°	iv	VI	ii	IV	vi
iii	V	vii°	III	V	VII	iii	V	vii°	III	V	VII	iii	V	vii°
vi	**E**	iii	VI	**e**	III	vi	**G**	iii	VI	**g**	III	vi	**B♭**	iii
ii	IV	vi	ii°	iv	VI	ii	IV	vi	ii°	iv	VI	ii	IV	vi
iii	V	vii°	III	V	VII	iii	V	vii°	III	V	VII	iii	V	vii°
vi	**A**	iii	VI	**a**	III	vi	**C**	iii	VI	**c**	III	vi	**E♭**	iii
ii	IV	vi	ii°	iv	VI	ii	IV	vi	ii°	iv	VI	ii	IV	vi
iii	V	vii°	III	V	VII	iii	V	vii°	III	V	VII	iii	V	vii°
vi	**D**	iii	VI	**d**	III	vi	**F**	iii	VI	**f**	III	vi	**A♭**	iii
ii	IV	vi	ii°	iv	VI	ii	IV	vi	ii°	iv	VI	ii	IV	vi
iii	V	vii°	III	V	VII	iii	V	vii°	III	V	VII	iii	V	vii°
vi	**G**	iii	VI	**g**	III	vi	**B♭**	iii	VI	**b♭**	III	vi	**D♭**	iii
ii	IV	vi	ii°	iv	VI	ii	IV	vi	ii°	iv	VI	ii	IV	vi

Figure 29: Lerdahl's *chordal/regional space* [14, p. 96].

References

[1] G. J. Balzano, "The Group-Theoretic Description of 12-Fold and Microtonal Pitch Systems," *Computer Music Journal* 4/4 (1980), 66–84.

[2] A. Barker, ed. *Greek Musical Writings*. Cambridge, U.K.: Cambridge University Press, 1984.

[3] E. Blackwood, *The Structure of Recognizable Diatonic Tunings*. Princeton: Princeton University Press, 1985.

[4] Boethius, *Fundamentals of Music*. Trans. C. Bower. New Haven: Yale University Press, 1989.

[5] R. Cohn, "Maximally Smooth Cycles, Hexatonic Systems, and the Analysis of Late-Romantic Triadic Progressions," *Music Analysis* 15 (1996), 9–40.

[6] R. Cohn, "Neo-Riemannian Operations, Parsimonious Trichords, and Their *Tonnetz* Representations," *Journal of Music Theory* 41 (1997), 1–66.

[7] J. D. Heinichen, *Der General-Baß in der Composition*. Dresden, 1728; reprinted Hildesheim, Georg Olms, 1969.

[8] O. Hostinsky, *Die Lehre von den musikalischen Klangen: Ein Beitrag zur aesthetischen Begrundung der Harmonielehre*. Prague: H. Dominicus, 1879.

[9] B. Hyer, "Reimag(in)ing Riemann," *Journal of Music Theory* 39 (1995), 101–138.

[10] D. Kellner, *Treulicher Unterricht im General-Baß*. Hamburg, 1737; reprinted Hildesheim, Georg Olms, 1979.

[11] C. Krumhansl, *Cognitive Foundations of Musical Pitch*. New York: Oxford University Press, 1990.

[12] C. Krumhansl, "Perceptual Structures for Tonal Music," *Music Perception* 1 (1983), 28–62.

[13] F. Lerdahl, "Tonal Pitch Space," *Music Perception* 5 (1988), 315–349.

[14] F. Lerdahl, *Tonal Pitch Space*. New York: Oxford University Press, 2001.

[15] D. Lewin, *Generalized Musical Intervals and Transformations*. New Haven: Yale University Press, 1987.

[16] A. Lorenz, *Das Geheimnis der Form bei Richard Wagner*, vol. 2: *Der musikalische Aufbau von Richard Wagners 'Tristan und Isolde.'* Berlin, 1926.

[17] M. J. Mandelbaum, "Multiple Divisions of the Octave and the Tonal Resources of 19-Tone Temperament," Ph.D. dissertation, Indiana University, 1961.

[18] E. W. Marvin, "*Tonpsychologie* and *Musikpsychologie*: Historical Perspectives on the Study of Music Perception," *Theoria* 2 (1987), 59–84.

[19] J. Mattheson, *Kleine General-Baß Schule*. Hamburg, 1735; reprinted Laaber: Laaber-Verlag, 1980.

[20] A. von Oettingen, *Harmoniesystem in dualer Entwicklung: Studien zur Theorie der Musik*. Dorpat and Leipzig: W. Gläser, 1866.

[21] H. Riemann, "Ideen zu einer 'Lehre von den Tonvorstellungen,'" *Jahrbuch der Bibliothek Peters* 21–22 (1914–15). Trans. R. W. Wason and E. W. Marvin, *Journal of Music Theory* 36 (1992), 69–117.

[22] A. Schoenberg, *Structural Functions of Harmony*. London: Williams and Norgate, 1954.

[23] R. N. Shepard, "Structural Representations of Musical Pitch," in D. Deutsch, ed., *The Psychology of Music*. New York: Academic Press, 1982, 343–390.

[24] G. Weber, *Versuch einer geordneten Theorie der Tonsetzkunst*. Mainz, 1817–21.

[25] P. Westergaard, "Geometries of Sounds in Time," *Music Theory Spectrum* 18 (1996), 1–21.