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Mathematical Connections
in Art, Music, and Science

The Linear Diophantine Equation in Music Theory

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In Western music, modulation usually occurs between *closely-related keys*, keys adjacent to each other on the circle of fifths. According to the *New Harvard Dictionary of Music*[1] "Because of the way in which sharps and flats are added to key signatures along the circle (of fifths), the number of pitches in common between the starting key and each successive key outward in either direction decreases by one." In recent works in music theory, microtonal versions of the circle of fifths have become known as *unidirectional P-cycles*. A *P-cycle* is a cycle of three or more sets from the same set-class such that there is a map between adjacent sets that leaves all but one of its notes fixed. The note that moves is changed by only a half-step. A *P*-cycle is said to be *unidirectional* if all the sets in the set-class are included in a single cycle. Clough and Myerson[2] were the first to construct an algorithm that generates the complete family of sets that support unidirectional P-cycles, and Clampitt[3] has shown that such sets are maximally even sets, as defined by Clough and Douthett[4], whose chromatic cardinality, *c*, and the diatonic cardinality, *d*, are co-prime. Such sets are also *generated sets* in the sense that it is possible to find a generating interval (in fact there are two) that will generate each set in the unidirectional P-cycle. This may be formalized by a linear Diophantine equation: $cI_g - dg = \pm 1$ where g and I_g are the chromatic and diatonic lengths of the generators. In this presentation, we will demonstrate how the linear Diophantine equation may be used to construct microtonal musical scales (i.e. Balzano's 20-fold system[5] and the Bohlen-Pierce 13-note scale[6]) having the modulation properties associated with the familiar circle of fifths.

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[1] "Modulation," in *The New Harvard Dictionary of Music,* ed. D. M. Randell (Belknap, Cambridge, MA, 1986). p. 503.
[2] J. Clough and G. Myerson, "Musical Scales and the Generalized Circle of Fifths," Am. Math. Monthly **93**(9), 695 - 701, (1986). J. Clough and G. Myerson, "Variety and Multiplicity in Diatonic Systems," J. Music Theory **29**, 249-270 (1985).
[3] D. Clampitt, "Pair-Wise, Well-Formed Scales, Structured and Transformational Properties," Ph. D. dissertation (SUNY at Buffalo, Buffalo, NY, April 1997).
[4] J. Clough and J. Douthett, "Maximally Even Sets," J. Music Theory **35**, 93 – 173 (1991).
[5] G. Balzano, "The Group-Theoretic Description of 20–Fold Pitch Systems," Comput. Music J. **4**. 66 – 84 (1980).
[6] M. V. Mathews, J. R. Pierce, A. Reeves, and L. A. Roberts, "Theoretical and Experimental Explorations of the Bohlen-Pierce Scale," J. Acoust. Soc. Am. **84**, 1214 – 1222 (1984).