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Hyperbolic Islamic Patterns — A Beginning

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Abstract

For more than a millennium, Islamic artists and craftsmen have used geometric patterns to decorate buildings, cloth, pottery, and other artifacts. Many of these patterns were "wallpaper" patterns — they were planar patterns that repeated in two different directions. Recently related patterns have also been drawn on the Platonic solids, which can conceptually be projected outward onto their circumscribing spheres, thus utilizing a second of the three "classical geometries". We extend this process by exhibiting repeating Islamic patterns in hyperbolic geometry, the third classical geometry.

Introduction

Islamic artists have long had a fascination for geometric patterns such as the one below in Figure 1 from the Alhambra palace. The purpose of this paper is to show that it is possible to create Islamic patterns in the hyperbolic plane, such as the one shown in Figure 2 which is related to the pattern of Figure 1.

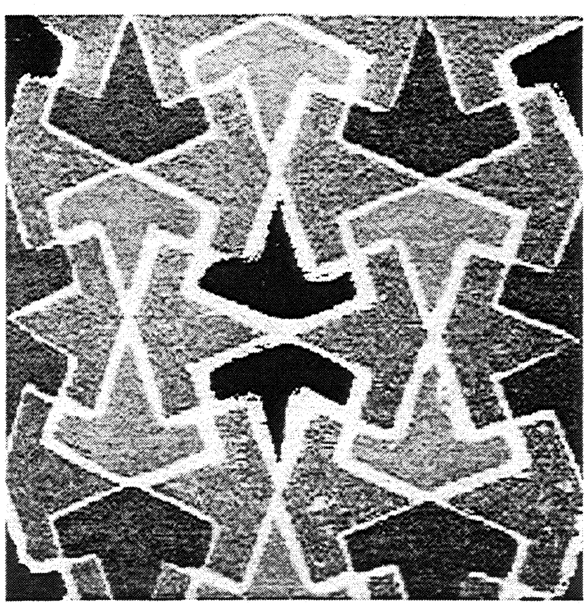

Figure 1: An Islamic pattern from the Alhambra.

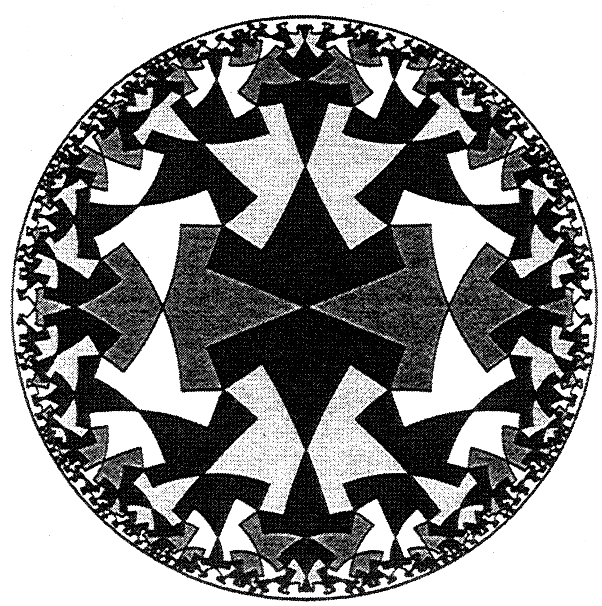

Figure 2: An Islamic hyperbolic pattern based on the Euclidean pattern of Figure 1.

The techniques for creating the original Islamic patterns were passed down from master to apprentice artisans, and have subsequently been lost. However, for more than 100 years, when it first became possible to print color reproductions, people have tried to analyze those patterns starting with Bourgoin [3]. The analysis of *wallpaper patterns*, patterns of the Euclidean plane that repeat in two different directions, became more precise when their 17 possible symmetry groups were classified. Abas and Salman have carried out the classification of hundreds of Islamic patterns with respect to these symmetry groups [2].

M. C. Escher was always fascinated with wallpaper patterns and created his first prints of them in the early 1920's. But his interest was sparked into a consuming passion by his visit in 1936 to the Alhambra, a Moorish palace in Granada, Spain, which was richly decorated with a large number of Islamic patterns. From that point on Escher filled notebooks with drawings of wallpaper patterns, many inspired by those Islamic patterns. For example, Schattschneider (page 18 of [9]) shows a sketch of his interlocking "weightlifters" superimposed on the pattern of Figure 1; that weightlifter pattern later became Escher's Notebook Drawing Number 3.

The goal of this paper is to take a first step toward combining Islamic patterns and hyperbolic geometry. Recently artists have created patterns on cubes, tetrahedra, dodecahedra, and icosahedra (see Plates 8, 12, 15, and 16 of [2]). Of course patterns on regular polyhedra can be thought of as spherical patterns by projecting them outward onto the polyhedrons' circumscribing spheres. Thus Islamic patterns will have been created in each of the three *classical geometries*: Euclidean, spherical, and hyperbolic geometry.

We will begin with a brief discussion of Islamic patterns, followed by a review of hyperbolic geometry, regular tessellations, and symmetry groups. Then we show a series of hyperbolic Islamic patterns that are related to existing Euclidean Islamic patterns. Finally, we indicate some directions for future work.

Islamic Patterns

Islamic artisans have been decorating texts, buildings, and other artifacts with geometric patterns for more than a thousand years. Artists working in a religious setting could hint at the infinitude of God by drawing potentially infinite repeating patterns (pages 1 and 2 of [2]). There are various kinds of 2-dimensional Islamic repeating patterns, including spirals, star patterns, key patterns, and "Y" patterns. Other kinds of Islamic patterns include arabesques (flower and intertwining vine patterns), and Kufic patterns, which are words written in stylized Arabic script. Of course many patterns fall into more than one category.

European interest in Islamic art was initiated by Owen Jones' color reproductions of various kinds of Islamic art in his book *The Grammar of Ornament* [6], first published about 150 years ago. Since then, a number of people have classified many of the wallpaper patterns [2, 5], and others have made guesses as to how the patterns were originally created [5, 8]. Recently Abas and Salman presented methods for the computer generation of such patterns [1], and Kaplan has designed a program to draw Islamic star patterns [7]. We will continue these endeavors by suggesting a more general framework for classifying the patterns, and then using that classification as a basis for creating new Islamic patterns in the hyperbolic plane.

Many repeating Islamic patterns seem to have been built upon the framework of a regular tessellation of the Euclidean plane. Such regular tessellations generalize to the hyperbolic plane. In the next sections, we will discuss those tessellations and their symmetry groups, which generalize some of the 17 symmetry groups of wallpaper patterns.

Hyperbolic Geometry and Regular Tessellations

Hyperbolic geometry is the least familiar of the classical geometries. This is probably because the entire hyperbolic plane cannot be embedded in 3-dimensional Euclidean space in a distance preserving way — unlike the sphere and the Euclidean plane. However, there are useful *models* of hyperbolic geometry in the Euclidean plane, which must perforce distort distance.

We will use the *Poincaré circle model* for the same reasons that made it attractive to Escher: (1) it is conformal (i.e. the hyperbolic measure of an angle is equal to its Euclidean measure) — consequently a transformed object has roughly the same shape as the original, and (2) it lies entirely within a *bounding circle* in the Euclidean plane — allowing an entire hyperbolic pattern to be displayed. In this model, the hyperbolic points are the interior points of the bounding circle and the hyperbolic lines are interior circular arcs perpendicular to the bounding circle, including diameters. Figure 3 shows the hyperbolic lines of reflection symmetry of Figure 2 For example, Figure 3 shows the hyperbolic lines of reflection symmetry superimposed on the pattern of Figure 2.

Two-dimensional hyperbolic geometry satisfies all the axioms of 2-dimensional Euclidean geometry except the Euclidean parallel axiom, which is replaced by its negation. Figure 4 shows an example of this hyperbolic parallel property among the reflection lines in Figure 3: there is a line, ℓ, (the vertical diameter), a point, P, not on it, and more than one line through P that does not intersect ℓ.

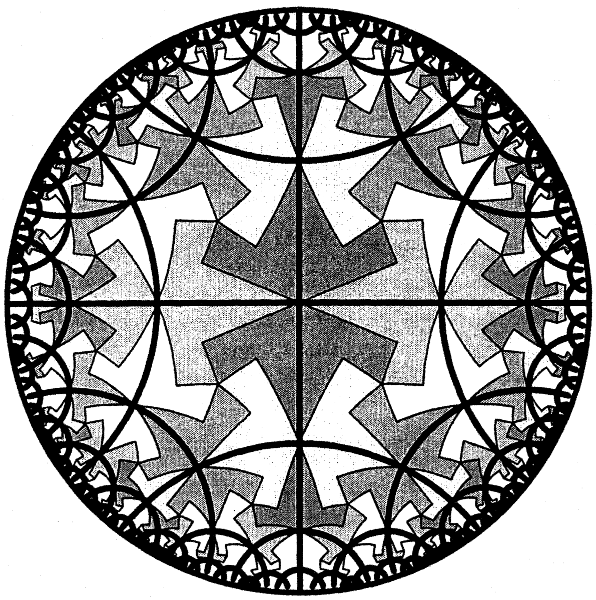

Figure 3: The tessellation $\{5,4\}$ superimposed on the pattern of Figure 2.

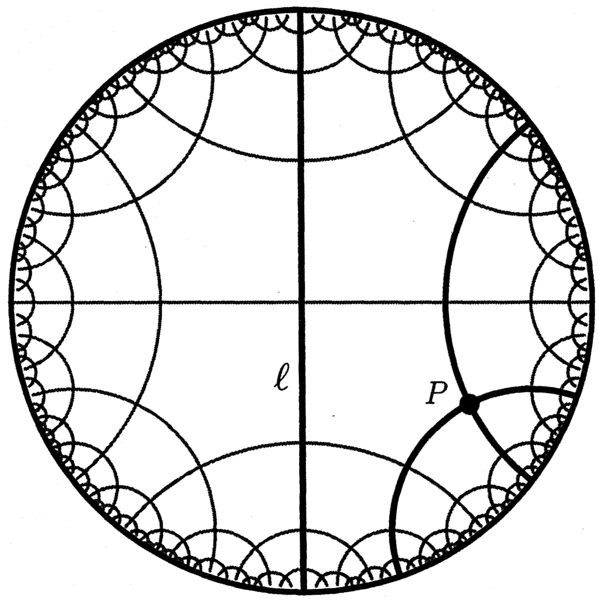

Figure 4: An example of the hyperbolic parallel property: a line ℓ, a point P not on ℓ, and two lines through P not meeting ℓ.

Equal hyperbolic distances in the Poincaré model are represented by ever smaller Euclidean distances toward the edge of the bounding circle (which is an infinite hyperbolic distance from its center). All the curvilinear pentagons (actually regular hyperbolic pentagons) in Figure 3 are the same hyperbolic size, even thought they are represented by different Euclidean sizes.

The curved pentagons that meet four at a vertex in Figure 3 form the regular tessellation $\{5,4\}$ More generally, in any of the classical geometries the Schläfli symbol $\{p,q\}$ denotes. the *regular tessellation* by regular p-sided polygons, or *p-gons*, meeting q at a vertex. We must have $(p-2)(q-2) > 4$ to obtain a hyperbolic tessellation; if $(p-2)(q-2) = 4$ or $(p-2)(q-2) < 4$, one obtains tessellations of the Euclidean plane and the sphere, respectively. One of the vertices of the $\{5,4\}$ is centered in the bounding circle in Figure 3 — but note that the center of the bounding circle is not a special point in the Poincaré model, it just appears so to our Euclidean eyes.

Assuming for simplicity that $p \geq 3$ and $q \geq 3$, there are five solutions to the "spherical" inequality $(p-2)(q-2) < 4$: $\{3,3\}$, $\{3,4\}$, $\{3,5\}$, $\{4,3\}$, and $\{5,3\}$. These tessellations may be obtained by "blowing up" the Platonic solids: the regular tetrahedron, the octahedron, the icosahedron, the cube, and the dodecahedron, respectively, onto their circumscribing spheres. In the Euclidean case, there are three solutions to the equality $(p-2)(q-2)=4$: $\{3,6\}$, $\{4,4\}$, and $\{6,3\}$, the tessellations of the plane by equilateral triangles, squares, and regular hexagons. Bourgoin used $\{6,3\}$ and essentially $\{4,4\}$ in some of his classifications of Islamic patterns [3]. Wilson shows how some hexagonal Islamic patterns can also be thought of as patterns based on $\{3,6\}$ (see Plate 38 of [10]). Of course there are infinitely many solutions to the hyperbolic inequality $(p-2)(q-2)>4$, and hence infinitely many regular hyperbolic tessellations.

This completes our treatment of hyperbolic geometry and regular tessellations. Next, we complete our theoretical considerations with a discussion of repeating patterns and their symmetry groups.

Repeating Patterns and Symmetry Groups

A *repeating pattern* in any of the classical geometries is a pattern made up of congruent copies of a basic subpattern or *motif.* One copy of the motif in Figures 1 and 2 is the right half of the top black polygonal figure in the center of the pattern; the left half may be obtained from it by reflection in the vertical symmetry axis. In the discussion below, we assume that a repeating pattern fills up its respective plane. Also, it is useful that hyperbolic patterns repeat in order to show their true hyperbolic nature.

The regular tessellation, $\{p,q\}$, is an important kind of repeating pattern since it forms a framework for many Euclidean Islamic patterns and for the hyperbolic Islamic patterns presented in this paper. The radii and perpendicular bisectors of the edges of a p-gon divide it into $2p$ right triangles whose other angles are π/p and π/q. Any one of these right triangles can serve as a motif for the tessellation.

A *symmetry operation* or simply a *symmetry* of a repeating pattern is an isometry (distance-preserving transformation) that transforms the pattern onto itself. For example reflection in the axis of symmetry of any of the polygons of Figures 1 or 2 is actually a symmetry of the whole pattern (if color is ignored in Figure 1). A reflection across a hyperbolic line in the Poincaré circle model is an inversion in the circular arc representing that line (or an ordinary Euclidean reflection across a diameter). Reflections across the radii and perpendicular bisectors of the edges of each p-gon are symmetries of $\{p,q\}$. Reflections are basic kinds of isometries in that the other isometries can be decomposed into a finite succession of reflections. For example, in each of the classical geometries, the composition of reflections across two intersecting lines produces a rotation about the intersection point by twice the angle of intersection. There is a 4-fold rotation symmetry, i.e. a rotational symmetry by $\pi/4$ about the trailing tips of the "arms" of the polygons of Figure 1. Similarly, there is a 5-fold rotation symmetry about the trailing tips of the polygon arms in Figure 2. The points of 5-fold rotational symmetry are at the centers of the overlying pentagons in Figure 3.

There are $\pi/4$ and $\pi/5$ rotation symmetries about the trailing tips of the "arms" of the polygons of Figures 1 and 2, respectively. The points of $\pi/5$ rotational symmetry are at the centers of the overlying pentagons in Figure 3.

The *symmetry group* of a pattern is the set of all symmetries of the pattern. The symmetry group of the tessellation $\{p,q\}$ is denoted $[p,q]$ (using Coxeter's notation [4]) and can be generated by reflections across the sides of the right triangle with angles of π/p and π/q; that is, all symmetries of $\{p,q\}$ may be obtained by successively applying a finite number of those three reflections. Note that such a right triangle can also serve as a motif for its "dual" tessellation $\{q,p\}$, and so the symmetry groups $[p,q]$ and $[q,p]$ are isomorphic, i.e. "the same" mathematically. This is denoted: $[p,q]\cong[q,p]$. In the Euclidean case, $[4,4]$ is the wallpaper group $p4m$, and $[3,6]$ or $[6,3]$ is the group $p6m$. In the Islamic patterns that Abas and Salman classified, the groups $p6m$ and $p4m$ appeared most frequently (see Figure 5.1 of [2]). Abas and Salman show a Kufic pattern on a dodecahedron with symmetry group $[3,5]$ (Plate 8 of [2]); they also show a star pattern on a cube with symmetry group $[3,4]$ (see Plate 16 of [2]). Of course, these can be considered to be spherical patterns.

It has been known for about 100 years that there are exactly 17 possible symmetry groups of patterns of the Euclidean plane that repeat in at least two different directions — these are often referred to as the *wallpaper groups*. Many sources have descriptions of these groups, including Abas and Salman [2]. This completes our discussion of repeating patterns and introduces symmetry groups. In the following sections, we will examine several patterns and their symmetry groups.

Patterns with Symmetry Group $[p, q]$

The arabesque pattern in Figure 5 below has symmetry group $p6m$ (or $[6,3]$). I made it up by repeating one of the hexagonal arabesques that appeared on the frontispiece of a Koran produced in Iran in 1313, and reproduced as Plate 81 of [10]. Only a slight distortion is required to deform a hexagon arabesque of Figure 5 to obtain a hyperbolic hexagon arabesque that will fit in a hexagon of the $\{6,4\}$ tessellation. The resulting pattern, which has symmetry group $[6,4]$ is shown in Figure 6. Patterns with symmetry group $[p,q]$ can be recognized by their large number of reflection lines.

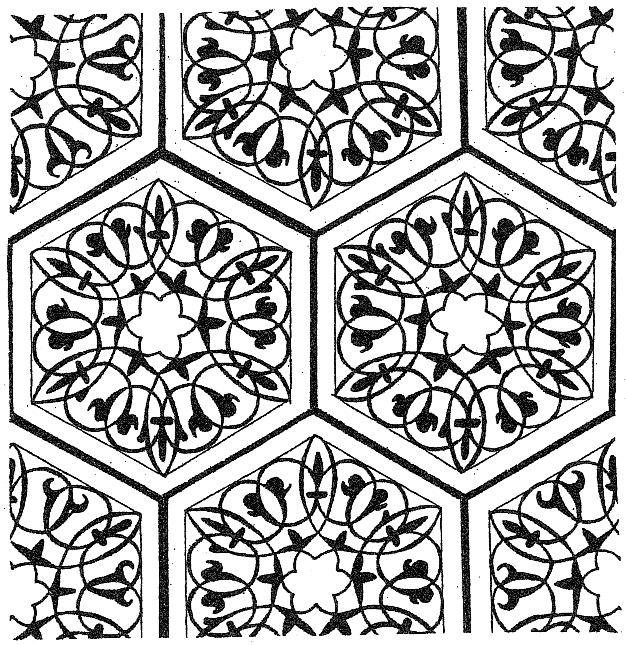

Figure 5: An Islamic pattern with symmetry group $p6m$ (= $[6,3]$).

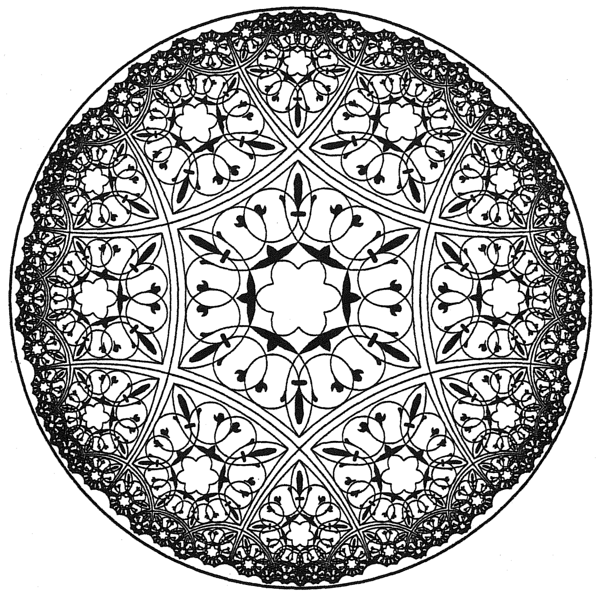

Figure 6: An Islamic hyperbolic pattern with symmetry group $[6,4]$ that is based on the Euclidean pattern of Figure 5.

Patterns with Symmetry Group $[p, q]^+$

The subgroup of $[p,q]$ that consists of orientation-preserving transformations is denoted $[p,q]^+$ by Coxeter [4], and can be generated by rotations of $2\pi/p$ and $2\pi/q$ about the p-gon centers and vertices of the tessellation $\{p,q\}$. Since there are no reflections in $[p,q]^+$ (because reflections reverse orientation), spiral patterns often have symmetry groups of the form $[p,q]^+$. Again, because of the duality between p and q, the symmetry groups $[p,q]^+$ and $[q,p]^+$ are isomorphic. The symmetry groups $p4$ ($\cong [4,4]^+$) and $p6$ ($\cong [6,3]^+ \cong [3,6]^+$) are the only two Euclidean groups of this type. Figure 7 shows a spiral pattern from the Alhambra (in [6]) with symmetry group $p6$. Figure 8 shows a hyperbolic version of this pattern having symmetry group $[7,3]^+$. Patterns with symmetry group $p4$ or $p6$ appear less frequently that patterns with symmetry group $p4m$ or $p6m$ among the patterns classified by Salman and Abas (see Figure 5.1 of [2]), but more frequently than some patterns with less symmetry. Abas and Salman show a pattern on an icosahedron that appears to have symmetry group $[3,5]^+$ (Plate 12 of [2]);

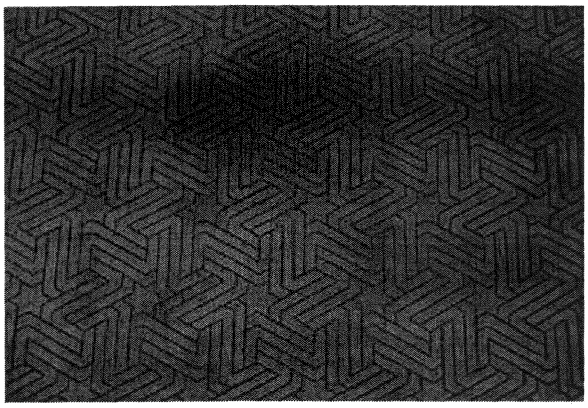

Figure 7: A pattern from the Alhambra with symmetry group $p6$ (= $[6,3]^+$).

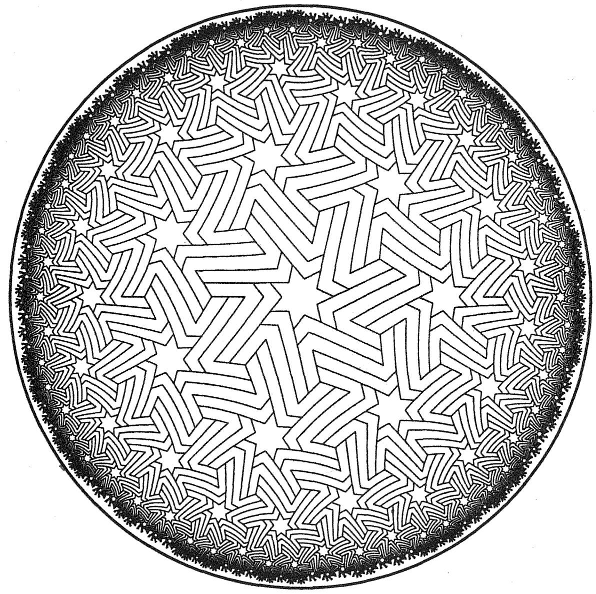

Figure 8: A hyperbolic pattern based on the pattern of Figure 7, with symmetry group $[7,3]^+$.

Patterns with Symmetry Group $[p^+, q]$

There is another subgroup of $[p, q]$ that contains rotational symmetries about the centers of the p-gons and reflections across the sides of the p-gons in the tessellation $\{p, q\}$. In this case, q must be even for the reflections to be consistent; $q/2$ reflection (mirror) lines intersect at each vertex of $\{p, q\}$. This subgroup is denoted $[p^+, q]$ by Coxeter, where the superscript + is used to signify an orientation-preserving symmetry [4]. The pattern of Figure 2 has symmetry group $[5^+, 4]$ if color is ignored. In $[p^+, q]$, p and q play different roles, so $[p^+, q]$ is a different group than $[q^+, p]$ (unless $p = q$). The Euclidean instances of $[p^+, q]$ are: $p4g$ (= $[4^+, 4]$), and $p31m$ (= $[3^+, 6]$). Figure 1 shows a pattern from the Alhambra with symmetry group $p4g$ if color is ignored. Note that $[6^+, 3]$ is not a valid group since 3 is not even. Islamic patterns with symmetry groups $p4g$ and $p31m$ appear with slightly less frequency than those with symmetry groups $p6$ and $p4$ among those classified by Salman and Abas (Figure 5.1 of [2]).

The pattern shown in Figure 9 is from the Alhambra and was one of the ones copied by Escher during his 1936 visit to that palace. It is an example of a "Y" pattern — so named because their motifs have the symmetry of a somewhat spread out Y whose arms make angles of $2\pi/3$ with each other. These are popular patterns in Islamic decoration. The Y patterns have symmetry group $p31m$ with each of the arms of the Y lying on reflection lines.

The pattern we show in Figure 10 is what we call a hyperbolic "X" pattern, which is like a Y pattern except that the motifs have four arms instead of three. The pattern of Figure 10 is the X patterns corresponding to the Y of Figure 9. The hyperbolic X patterns have symmetry group $[p^+, 8]$, where $p \geq 3$ (in contrast to the Y patterns which have symmetry group $[p^+, 6]$).

We note that we designed a hyperbolic Y pattern (with symmetry group $[4^+, 6]$) corresponding to Figure 9, but Figure 10 turned out to be aesthetically more pleasing. Also, we note the Figure 10 has the same symmetry group as Escher's *Circle Limit II* pattern of crosses. As an aside, about 15 years ago Coxeter reversed the process by providing the Euclidean Y pattern (with symmetry group $p31m$) corresponding to the X pattern of *Circle Limit II*.

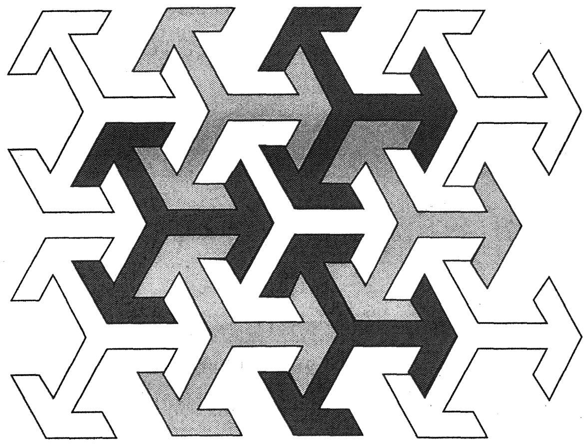

Figure 9: A "Y" pattern from the Alhambra with symmetry group $p31m$ (= $[3^+, 6]$).

Figure 10: A hyperbolic "X" pattern based on the Euclidean "Y" pattern of Figure 9, having symmetry group $[3^+, 8]$.

Patterns with Symmetry Groups $[p, q, r]$ and (p, q, r)

The symmetry group $[p, q, r]$ can be generated by reflections across the sides of a triangle with angles π/p, π/q, and π/r. We have already seen a special case of this: when $r = 2$, $[p, q, r] \cong [p, q]$. Also, when more than one of p, q, and r are equal to 2, we obtain symmetry groups of spherical patterns. So for the rest of this section, we will assume that p, q, and r are all greater than or equal to three. If p, q, and r are all equal to three, $[3, 3, 3] \cong p3m1$. At least two of p, q, and r must be equal in order to obtain a pattern based on a regular tessellation: $[p, p, q]$ is based on the tessellation $\{2q, p\}$, where each $2q$-gon is subdivided into $2q$ triangles of angles π/p, π/p, and π/q.

The orientation-preserving subgroup of $[p, q, r]$ is denoted (p, q, r), and can be generated by any two of the rotations by $2\pi/p$, $2\pi/q$, and $2\pi/r$ about the vertices of the triangle mentioned above. Because all of these symmetries are orientation-preserving, patterns with symmetry group (p, q, r) are chiral, that is all of the motifs rotate in the same direction. If p, q, and r are all equal to three, $(3, 3, 3) \cong p3$. Figure 11 shows another pattern that Escher copied from the Alhambra, with symmetry group $p3$. Figure 12 shows a hyperbolic pattern based on that of Figure 11 and having symmetry group $(3, 3, 4)$, which happens to be the symmetry group of Escher's most famous hyperbolic pattern *Circle Limit III*.

Conclusions and Future Work

We have developed a theoretical framework that allows us to create hyperbolic Islamic patterns that are related to Euclidean patterns. We have shown how to do this for each of the 17 "wallpaper" groups with 3-, 4-, or 6-fold rotational symmetries. One direction of future work would be to create hyperbolic patterns that were related to the remaining wallpaper groups. For some of those wallpaper groups, it is not clear what the appropriate hyperbolic generalization is. Other directions of future work could include creating hyperbolic versions of other kinds of Islamic patterns, such as Kufic or star patterns. For example, it should be possible to generate hyperbolic star patterns by combining the methods above and those of Kaplan [7].

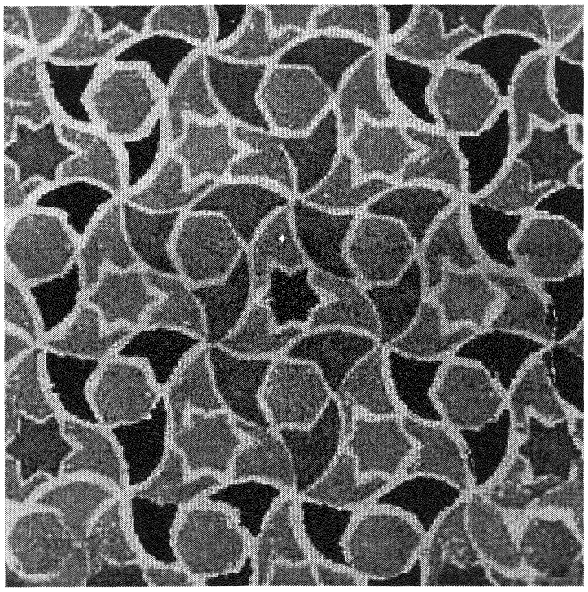

Figure 11: A pattern from the Alhambra with symmetry group $p3$ (= (3,3,3)).

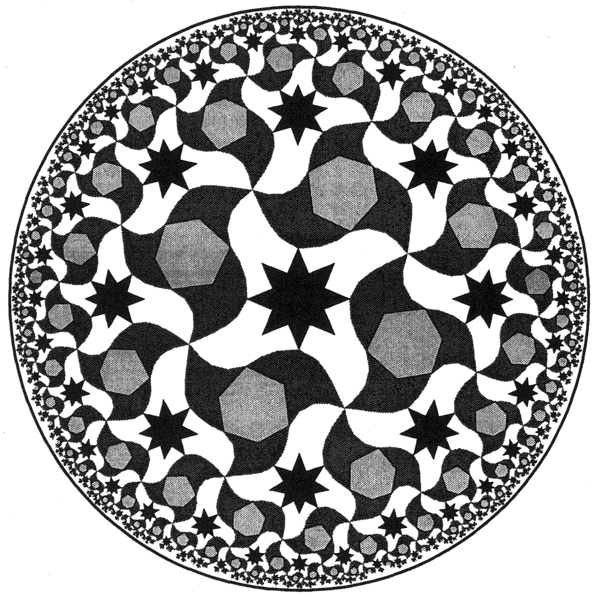

Figure 12: A hyperbolic pattern based on the pattern of Figure 11, with symmetry group (3,3,4).

Acknowledgment

I would like to thank the many students who have worked on the program used to generate the hyperbolic patterns.

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