BRIDGES
Mathematical Connections
in Art, Music, and Science

Mirror Curves

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Abstract

Mirror curves are present in ethnical art, as Tamil threshold designs or Tchokwe sand drawings. Historically, they are to be found in the art of most peoples surrounding the Mediterranean, the Black and Caspian Seas, in the art of Egyptians, Greeks, Romans, Byzantines, Moors, Persians, Turks, Arabs, Syrians, Hebrews and African tribes. Highlights are Celtic interlacing knotworks, Islamic layered patterns and Moorish floor and wall decorations. In this paper mirror curves are considered from the point of view of geometry, tiling theory, graph theory and knot theory. After the enumeration of mirror curves in a rectangular square grid, and a discussion of mirror curves in polyominoes and uniform tessellations, the construction of mirror curves is generalized to any surface.

1. Introduction and preliminaries

Let a connected edge-to-edge tiling of some part of a plane by polygons be given. Connecting the midpoints of adjacent edges we obtain a 4-*regular graph*: the graph where in every vertex they are four edges, called *steps*. A *path* in that graph is a connected series of steps, where each step appears only once. Every closed path in that graph is called a *component*. The set of all components of such graph is called a *mirror curve*. In every vertex we have three possibilities to continue our path: to choose left, middle, right edge. If the middle edge is chosen such vertex will be called a *crossing*. By introducing in every crossing the relation over-under, every mirror curve can be converted into a *knotwork design.*

The term "mirror curve" could be simply justified if we take a *rectangular square grid* RG[a,b] with the sides a and b, where that sides are mirrors, and the additional internal two-sided mirrors are placed between the square cells, coinciding with an edge, or perpendicular to it in its midpoint. In this case, a ray of light, emitted from one edge-midpoint, making with that edge a 45° angle, after the series of reflections will close a component. Beginning from different edge-midpoints, till exhausting the complete step graph, we obtain a mirror curve. It is easy to conclude that the preceding description could be extended to any connected part of a regular triangular, square or hexagonal tessellation, this means to any *polyamond*, *polyomino* or *polyhexe*, respectively.

After the historical remarks about Tamil and Tchokwe mirror curves and knotwork designs created by Leonardo and Dürer (Sections 2,3,4), the rules for the construction of one-component mirror curves are given in Section 5. Such algorithm rules are used for the combinatorial enumeration of mirror curves obtained from RG[a,b] with a minimal number of internal mirrors (Section 6). In the Sections 7, 8, 9 are considered black-white designs derived from mirror curves (so called Lunda designs), the use of mirror curves for a polyomino shape 0-1 notation, and Lunda polyominoes. In the Section 12 mirror curves are considered from the point of view of knot theory. In the last section, mirror curves are generalized to any surface.

2. Tamil Treshold Designs

"During the harvest month of *Margali* (mid-December to mid-January), the Tamil women in South India used to draw designs in front of the thresholds of their houses every morning. *Margali* is the month in which all kinds of epidemics were supposed to occur. Their designs serve the purpose of appeasing the god Siva who presides over *Margali*. In order to prepare their drawings, the women sweep a small patch of about a yard square and sprinkle it with water or smear it with cow-dung. On the clean, damp surface they set out a rectangular reference frame of equidistant dots. Then the curve(s) forming the design is (are) made by holding rice-flour between the fingers and, by a slight movement of them, letting it fall out in a closed, smooth line, as the hand is moved in the desired directions. The curves are drawn in such a way that they surround the dots without touching them."

P. Gerdes: Reconstruction and extension of lost symmetries: examples from the Tamil of South India [3].

The (culturally) ideal design is composed of a *single continuous line*. Names given to designs formed of a single "never-ending" line are normally *pavitram*, meaning "ring" and *Brahma-mudi* or "Brahma's knot". The object of the *pavitram* is to scare away giants, evil spirits, or devils.

Is it not strange that the design composed of two or several superimposed closed paths, are nevertheless called *pavitram*? Maybe the designs formed of a few never-ending lines are just degraded versions of originally single closed path figure?

Is it possible to construct a design rather similar to them, but made out of only one line? Slightly changing them, we may transform some imperfect, multi-linear designs into the ideal ones.

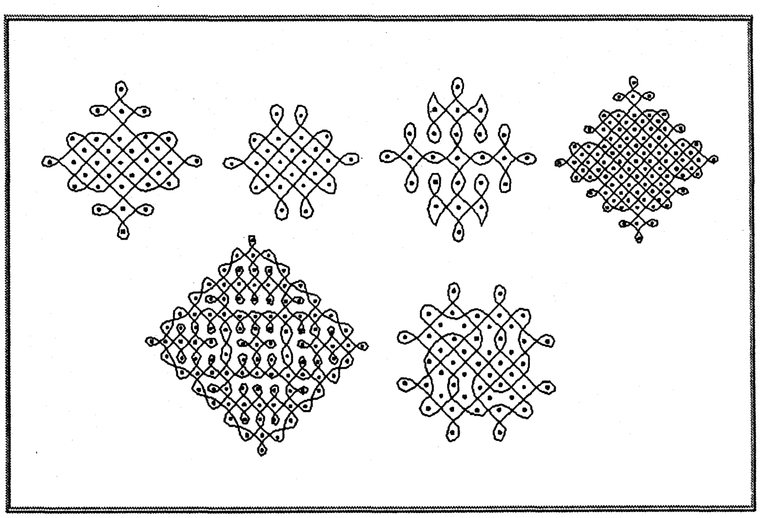

Figure 1: *Tamil threshold designs.*

3. Tchokwe Sand Drawings

"The Tchokwe people of northeast Angola are well known for their beautiful decorative art. When they meet, they illustrate their conversations by drawings on the ground. Most of these drawings belong to a long tradition. They refer to proverbs, fables, games, riddles, etc. and play an important role in the transmission of knowledge from one generation to the other."

"...Just like the Tamils of South India, the Tchokwe people invented a similar mnemonic device to facilitate the memorization of their standardized drawings. After cleaning and smoothing the ground, they first set out with their fingertips an orthogonal net of equidistant points. The number of rows and columns depends on the motif to be represented. By applying their method, the Tchokwe drawing experts reduce the memorization of a whole design to that of mostly two numbers and a geometric algorithm. Most of they drawings display bilateral and/or rotational (90° or 180°) symmetries. The symmetry of their pictograms facilitates the execution of a drawing. This is important, as the drawings have to be executed smoothly and continuously. Any hesitation or stopping on the part of the drawer is interpreted by the audience as an imperfection and lack of knowledge, and assented with an ironic smile."

P. Gerdes: On ethnomathematical research and symmetry [1].

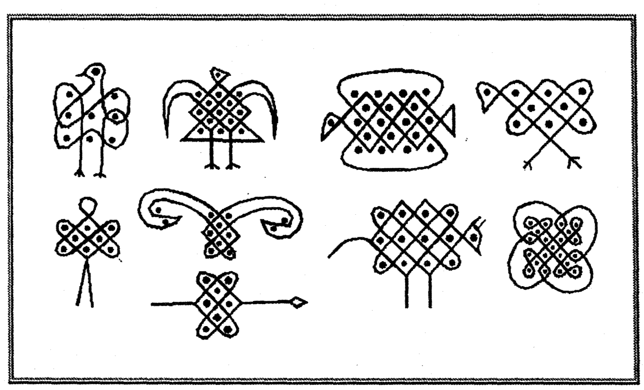

Figure 2: *Tchokwe sand drawings.*

4. Leonardo and Dürer

"Leonardo spent much time in making a regular design of a series of knots so that the cord may be traced from one end to the other, the whole filling a round space..."

G. Bain: Celtic Art - the Methods of Construction [2].

The construction of knot designs, closely connected with mirror curves, occupied the attention of two of the greatest painters-mathematicians: Leonardo and Dürer [2]. In some of their constructions, they very efficiently used the following geometrical property: for a rectangular square grid RG[a,b] with the sides a and b, if a, b are relatively prime the result is always a single closed curve covering uniformly the square tiling of a rectangle.

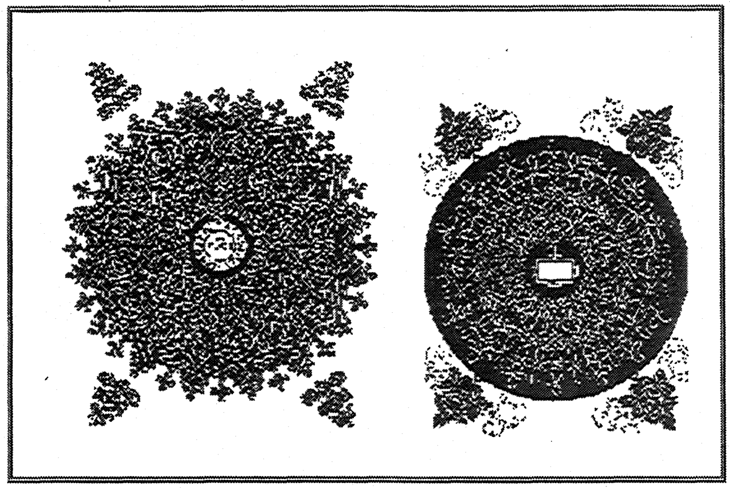

Figure 3: *Circular knot designs by Leonardo and Dürer.*

Let us notice one more beautiful geometrical property: mirror curves can be obtained by using only few different prototiles. For the construction of all the curves, with internal mirrors incident to the cell-edges, three prototiles are sufficient in the case of a regular triangular tiling, five in the case of square, and 11 in the case of hexagonal regular tiling [4].

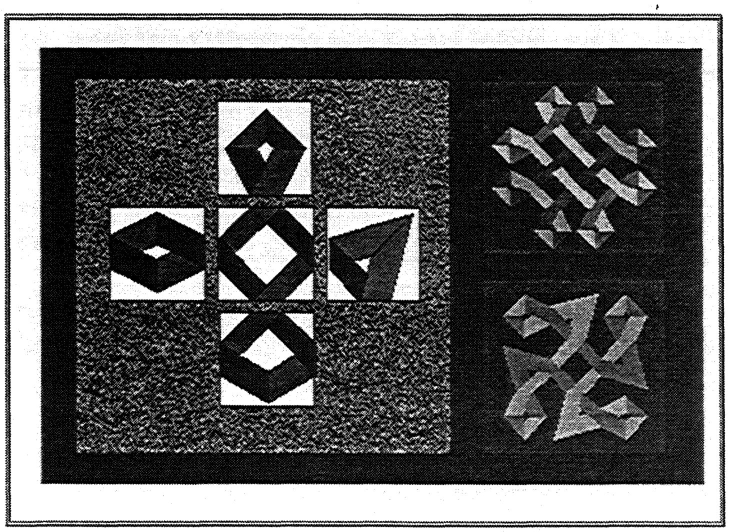

Figure 4: *Five knot prototiles for square grid.*

Using their combinations occurring in the 11 uniform Archimedean tilings [5], or the prototiles producing the impression of space structures and colored prototiles, we may obtain very artistic interlacing patterns belonging to the so called *modular design*: the use of few initial elements ("*modules*" - prototiles) for creating an infinite collection of designs [6]. The mirror curves obtained from Archimedean tilings remind us of the optical phenomenon: change of the direction of a light ray transferring from one physical environment to the other.

5. Mirror Curves

The imitation of the three-dimensional arts of plaiting, weaving and basketry was the origin of interlaced and knotwork interlaced designs. There are few races that have not used it as a decoration of stone, wood and metal. Interlacing rosettes, friezes and ornaments are to be found in the art of most peoples surrounding the Mediterranean, the Black and Caspian Seas, Egyptians, Greeks, Romans, Byzantines, Moors, Persians, Turks, Arabs, Syrians, Hebrews and African tribes. Their highlights are Celtic interlacing knotworks, Islamic layered patterns and Moorish floor and wall decorations [2].

Their common geometrical construction principle, discovered by P. Gerdes, is the use of (two-sided) mirrors incident to the edges of a square, triangular or hexagonal regular plane tiling, or perpendicular to its edges in their midpoints [1, 7, 8, 9]. In the ideal case, after the series of consecutive reflections, the ray of light reaches its beginning point, defining a single closed curve. In other cases, the result consists of several such curves. For example, to the Celtic designs from G. Bain's book "Celtic Art" [2], correspond the following mirror-schemes:

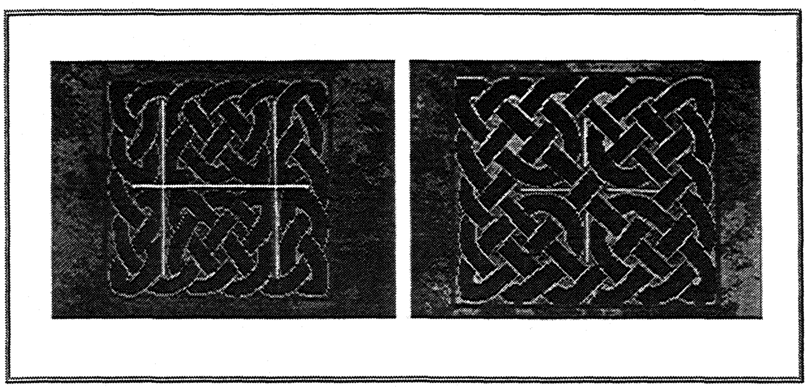

Figure 5: *Celtic knots with mirror-schemes.*

Trying to discover their common mathematical background, they appear two questions: how to construct such a perfect curve - a single line placed uniformly in a regular tiling, this means, how to arrange the set of mirrors generating it, and how to classify the curves obtained. In principle, any polyomino (polyiamond or polyhexe [10]) with mirrors on its border, and two-sided mirrors between cells or perpendicular to the internal cell-edges in their midpoints, could be used for the creation of the corresponding curves.

For their construction in some polyomino (polyiamond or polyhexe), we propose the following method. First, we construct all the different curves in it without using internal mirrors, starting from different cell-edge midpoints and ending in them, till the polyomino is exhausted, i.e., uniformly covered by k curves. After that, we may use "curve surgery" in order to obtain a single curve, according to the following rules:

1. any mirror introduced in a crossing point of two distinct curves connects them into one curve;

2. depending on the position of a mirror, a mirror introduced into a self-crossing point of an (oriented) curve either does not change the number of curves, or breaks the curve into two closed curves.

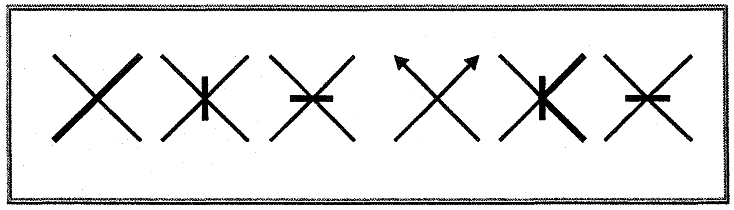

Figure 6: *Rules for introduction of a mirror.*

In every polyomino we may introduce k-1, k, k+1,..., 2A-P/2 internal two-sided mirrors, where A is the area and P is the perimeter of the polyomino. Introducing the minimal number of mirrors k-1, we first obtain a single curve, and in the next steps we try to preserve that result.

In the case of a rectangular square grid RG[a,b] with the sides a, b, the initial number of curves, obtained without using internal mirrors is k = gcd(a,b) (gcd- greatest common divisor), so in order to obtain a single curve, the possible number of internal two-sided mirrors is k-1, k, ..., 2ab-a-b. According to the rules for introduction of internal mirrors, we propose the following algorithm for the production of mono-linear designs: in every step, each of the first internal k-1 mirrors must be introduced in crossing points belonging to different curves. After that, when the curves are connected and transformed into a single line, we may introduce other mirrors, taking care about the number of curves, according to the rules mentioned. For example:

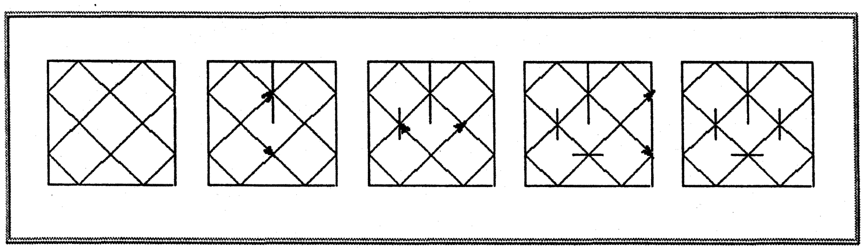

Figure 7: *Derivation of mono-linear designs in RG[2,2].*

The symmetry of such curves is used for the classification of the Celtic frieze designs by P. Cromwell [11], and for the reconstruction of Tamil designs by P. Gerdes [3, 12]. From the ornamental heritage, at first glance appears that symmetry is the mathematical basis for their construction and possible classification [1,3,11]. But, the existence of such asymmetrical curves suggests another approach.

First criterion that we may use is the geometrical one: two curves are equal *iff* there is a similarity transforming one into the other. This means, that one curve can be obtained from the other by a combined action of a proportionality and isometry. Instead of considering the curves, we may consider the equal mirror arrangements defined in the same way. Having the algorithm for the construction of such perfect curves and the criterion for their equality, we may try to enumerate them: to find the number of all the different curves (i.e., mirror arrangements) which can be derived from a rectangle with the sides a, b, for a given number of mirrors m (m = k-1, k, ..., 2ab-a-b).

The other point of view to the classification of such perfect curves is that of knot theory [13]. Every such curve can be simply transformed into an interlacing knotwork design, this means, into the projection of some alternating knot. We return to this connection in Section 11.

6. Enumeration

The problem is: find the number of all different mono-linear curves (i.e. the corresponding mirror arrangements) which could be derived from a rectangular grid RG[a,b] with the sides a, b, covered by k curves, for a given number of mirrors m ($m = k$-1, k, ..., $2ab$-a-b). Unfortunately, we are very far from the general solution of this problem. Reason for this is: every introduction of an internal mirror changes the whole structure, so it behaves like some kind of "Game of Life" or cellular automata, where a local change results in the global change.

Till now we have only few combinatorial results [14], obtained for some particular cases by the author, and generalized by G. Baron. Let a rectangular grid RG[a,b], k = gcd(a,b), be given, and let the minimal number k-1 of two-sided internal mirrors be introduced incident to the cell-edges. If t = (ab-lcm(a,b)):(k(k-1)) = $4xy$ (lcm - lowest common measure), $x = a$:(2k), $y = b$:(2k), we have the following results, where for different k are given the conditions for a, b, and the number of curves:

a) with k-1 only edge-incident mirrors, and a non equal to b,

a_1) for k odd: $(4k)^{k-2}t^{k-1} + 2(4k)^{(k-3)/2}t^{(k-1)/2}$;

a_2) for k even: $(4k)^{k-2}t^{k-1} + (4k)^{(k-2)/2}zt^{(k-2)/2}$,

with $z = x$ for $a = 0$ mod 2k, $b = k$ mod 2k, and $z = x+y$, for $a = b = k$ mod 2k;

b) with k-1 edge-incident or edge-perpendicular mirrors, and a non equal to b,

b_1) for k odd: $2(8k)^{k-2}t^{k-1} + 4(8k)^{(k-3)/2}t^{(k-1)/2}$,

b_2) for k even: $2(8k)^{k-2}t^{k-1} + 2(8k)^{(k-2)/2} zt^{(k-2)/2}$,

with $z = x$ for $a = 0$ mod 2k, $b = k$ mod 2k, and $z = x+y$ for $a = b = k$ mod 2k.

For $a = b$ we have to put $t = 1$, $z = 1$, divide the numbers by 2, and get

a) for k-1 only edge-incident mirrors

a_1) for k odd: $2^{2k-5} k^{k-2} + 2k\text{-}3^{k(k-3)/2}$,

a_2) for k even: $8k^{2k-5} k^{k-2} + 2k\text{-}3^{k(k-2)/2}$,

b) with k-1 edge-incident or edge-perpendicular mirrors,

b_1) for k odd: $(8k)^{k-2} + 2(8k)^{(k-3)/2}$,

b_2) for k even: $(8k)^{k-2} + (8k)^{(k-2)/2}$.

Even for some smaller rectangles (e.g., a = 6, b = 3), and minimal number of mirrors (k-1 = 2), the number of the different curves obtained is very large. G. Baron also derived formulas for the case $a = b$ with the larger groups of symmetries and, finally, constructed for $k = a = b$ equal 2 or 3 and the maximum number of mirrors all different mirror-schemes. There is only one for k =2 and 28 for k=3.

For example, there are 52 different arrangements of two edge-incident mirrors in a rectangle 6×3 producing perfect curves. Among them, only 8 are symmetrical- 4 mirror-symmetrical and 4 point-symmetrical.

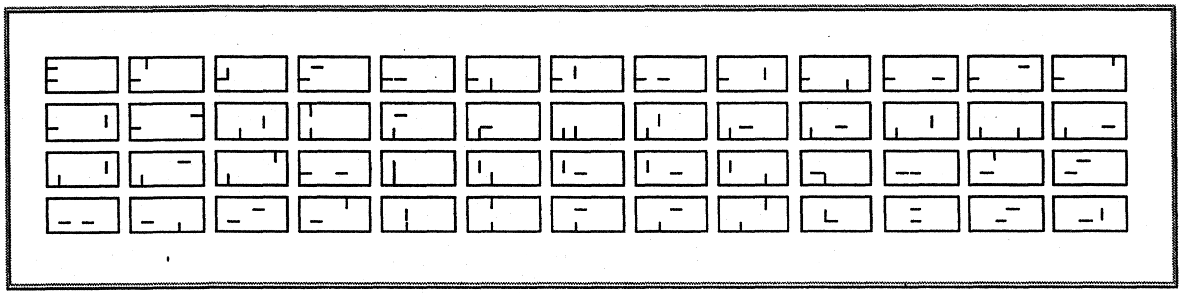

Figure 8: *52 arrangements of two edge-incident mirrors in a rectangle 6×3.*

7. Lunda Designs

If we enumerate the small squares through which the singular mirror curve passes by 1, 2, 3, ... until the closed curve is complete, and then reduce all the numbers modulo 2 (replacing every number by its remainder, when dividing it by 2), the result will be a 0-1 (or "black"-"white") mosaic: a *Lunda design* [7, 8, 9]. Lunda designs possess the local equilibrium property: the sum of the integers in every two border unit squares with the joint reference point is the same, and the sum of the integers in the four unit squares between two arbitrary neighboring grid points is always twice the preceding sum. From this, results the global equilibrium property: the sum in each row is equal, and the same holds for the columns. This local, and global equilibrium property resulting from it holds as well if we enumerate the curve and reduce all the numbers modulo 4.

In particular, enumerating a regular curve (with the mirrors incident to the grid edges) and reducing all the numbers modulo 4, we obtain four-colored Lunda designs, where every reference point is orderly surrounded by numbers 0,1,2,3 and the disposition of the sequences around the points is alternately clockwise and anti-clockwise.

The correspondence between mono-linear mirror-curves (i.e., the corresponding arrangements of mirrors) and Lunda designs is many-to-one, so the same Lunda design could originate from several classes, consisting of different mirror arrangements. Open question: try to define such classes of mirror arrangements.

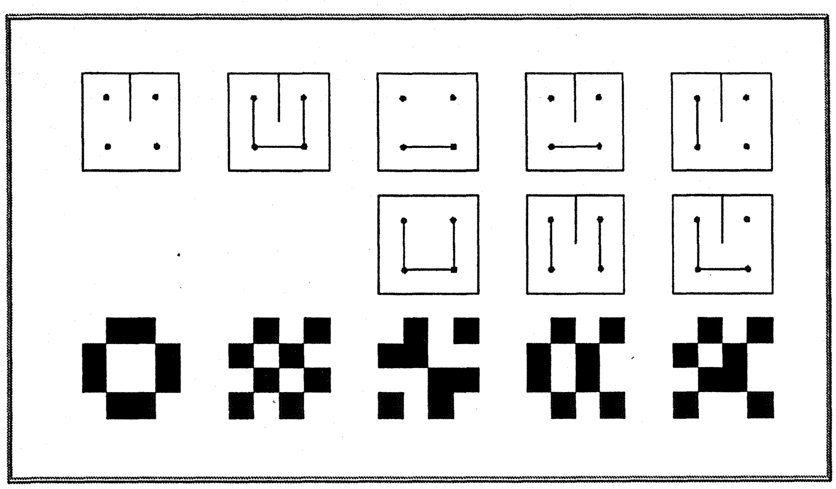

Figure 9: *Mirror arrangements and their corresponding Lunda designs.*

8. Polyominoes

A plane topological disk (i.e., a plane region without "holes"), consisting of n edge-to-edge adjacent squares is called a *polyomino* [10]. Instead of squares we could use n equilateral triangles or n regular hexagons, and to obtain, respectively, polyiamonds or polyhexes. We will restrict our discussion to polyominoes (with polyiamonds and polyhexes the situation is absolutely the same). For every polyomino we could distinguish its shape and orientation ("left" and "right" form). For polyominoes not having a reflective symmetry, we may distinguish (or not) their "left" and "right" form. According to this we have two possible equality criteria: a) regarding only the shape (without distinguishing "left" or "right" form); b) regarding both, the shape and orientation. Till now, there is no formula for calculating the number of different polyominoes, but only some results (for smaller values of n), obtained by empirical (computer) derivation.

In every border square cell of a polyomino we could introduce two-sided mirrors perpendicular to the internal edges in their midpoints. After a series of reflections, the ray of light will "describe" the shape: a closed Dragon curve. If we denote a reflection in a border mirror by 0, and a reflection in an internal mirror by 1, we have 0-1 words (or symbols) for polyominoes, where these words are cyclically equivalent (this means, could be read starting from any sign 0 or 1 and ending in it). For n=1 we will have only one polyomino 0000, for n=2 the polyomino 00010001, for n=3 two polyominoes: 000101000101 and 000100100011, for n=4 five of them: 0001010100010101, 0001010001100011, 001001001001, 0001001100010011, and 0001001010001011, *etc.*

From their binary symbols we could directly make conclusions about the symmetry: every reversible word denotes polyominoes with a sense-reversing symmetry (they don't have "left" and "right" form); irreversible symbols correspond to the polyominos appearing in the "left" and "right" form (e.g., 0001001100010011 or 0001001010001011).

That symbols (or binary numbers) we could translate into hexadecimal numbers and to every polyomino assign exactly one such number. For example, this could be the minimum of all such cyclic-equivalent symbols (e.g. to the polyomino 00010001 correspond cyclically equivalent symbols 00100010, 01000100, 10001000 and the minimum of them is 00010001 = 11 in the hexadecimal system. Hence, we have the notation for polyominoes where to every polyomino corresponds exactly one such number, and vice versa. (Open question: find the general algebraic form of the number determining a polyomino? Namely, some numbers will determine "opened polyominoes", "hollow polyominoes" or "overlapping polyominoes", that are not included in our definition, and other will determine "real" polyominoes).

Every (n+1)-omino we derive from some n-omino by adding to it a single square. Certainly, the addition operation is a positional one, that is, the result depends from the position where we add the new square. Here we have the following addition rules: i) a0+0000=a10001 (1-edge contact); ii) a0110+0000=a1001 (2-edge contact); iii) a0110110+0000=a1010 (3-edge contact), where a never ends with 1.

This "algebra" could be successfully used for the computer enumeration of polyominoes. In each step we need to derive (n+1)-minoes from n-minoes by adding a square, to test the equality of polyominoes obtained and to make a complete list of the different (n+1)-minoes obtained. The main problem in such a derivation will make “undesired” edge contacts (e.g., in parallel edges, producing “hollow” polyominoes).

9. Lunda Polyominoes and Lunda Animals

Polyominoes (either black or white) appearing in Lunda designs will be called *Lunda polyominoes* [7]. The possible shape of Lunda polyominoes is conditioned by the local equilibrium condition for Lunda designs. Therefore, some polyominoes are inadmissible (e.g., 001001001001). On the other hand, in Lunda polyominoes are included also "hollow" polyominoes. By introducing the concept of *Lunda-animals*, P. Gerdes in his book "Lunda geometry: Designs, Polyominoes, Patterns, Symmetries" [7] obtained the first approximation for the total number of different Lunda n-ominoes.

Lunda-animal is a Lunda m-omino with a unit square at one of its ends, representing a head. A Lunda-animal walks in such a way that after each step the head occupies a new unit square, and each other cell occupies the preceding one. In other words, two subsequent positions of a Lunda-animal have a Lunda-tetromino in common. How many different positions $p_5(n)$ of a Lunda 5-omino are possible after n steps? P. Gerdes proved that: $p_m(n) = f(n+3)$ for $m = 1,2,3,...,8$, where $f(n)$ is the famous Fibonacci sequence 0,1,1,2,3,5,8,13,21,34... given by the recurrence formula: $f(n+1) = f(n) + f(n-1)$. It is interesting that for every Lunda m-omino for $m<9$ the result is the same, so $p_m(n) = f(n+3)$ for $1 \leq m \leq 8$. From $m = 9$ onwards, $p_m(n) < f(n+3)$. Open problem: try to find the general formula for $p_m(n)$.

10. Graphs and Knots

Every Eulerian graph [14] is a projection of some knot or link and *vice versa*. Such a projection is called *regular* if the graph is 4-regular, i.e., if the valence of every vertex is 4. Otherwise, the projection is irregular. By slightly changing it, it is always possible to turn some irregular projection of a knot or link into a regular one. Two knot or link projections are isomorphic (or simply, equal or same) if they are isomorphic as graphs on a sphere. Trying to find all non isomorphic projections of alternating knots and links with n crossings, we need to find all non isomorphic 4-regular planar graphs with n vertices and *vice versa*. Among them, we could distinguish graphs with or without digons. If we denote digons by colored edges, we could imagine the trefoil knot as a triangle with all colored edges, the knot 4_1 as a tetrahedron with two colored nonadjacent edges, Borromean rings as an octahedron [15], *etc*. After that, you could replace every digon by a chain of digons, and obtain different families of knots and links. Such "geometrical" approach to knots and links is presented in my papers "Geometry of links" [16] and "Ordering Knots" [17]. For example, the family of knots and links generated by the knot 4_1 is illustrated at Fig. 10.

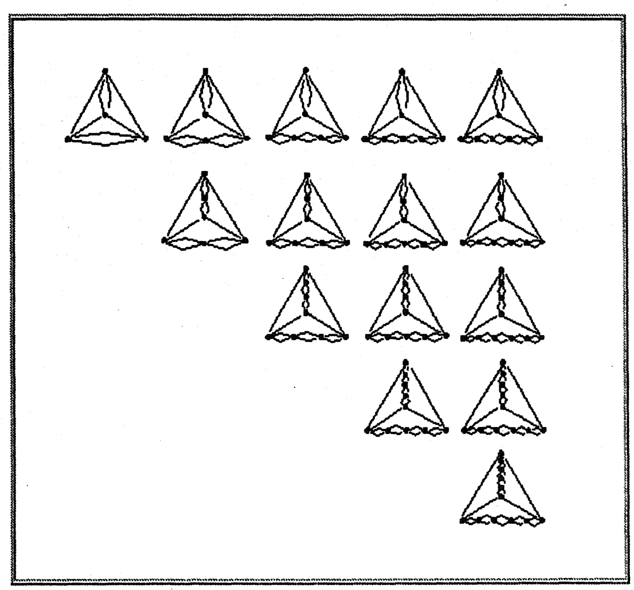

Figure 10: *The family of knots and links generated by the knot 4_1.*

11. Knots and Mirror Curves

The other point of view to the classification of mirror curves is that of knot theory [13, 16, 17]. As mentioned before, every such curve can be simply transformed into an interlacing knotwork design, this means, into a projection of some alternating knot. In the history of ornamental art, such curves occur most frequently as knotworks, then as plane curves. Even the name *Brahma-mudi* (Brahma's knot) denoting such Tamil curves refers us to the knots. In order to classify them, we will first transform every such knot projection into a reduced (proper) knot projection - knot projection without loops, by deleting cells with loops.

This way, we obtain reduced knot projections with the minimal number of crossings. Two such projections or knot diagrams are equal if they are isotopic in the projection plane as graphs, where the isotopy is required to respect over-crossing respectively under-crossing. In order to classify our curves, treated as knot projections, we will define an invariant of knot (or link) projections [4]. Let a reduced oriented knot diagram D with generators $g_1, ..., g_n$ be given. If the generators g_i, g_j, g_k are related as in the left figure, then $a_{ii} = t$, $a_{ij} = 1$, $a_{ik} = -1$; if they are related as in the right figure, then $a_{ii} = -t$, $a_{ij} = 1$, $a_{ik} = -1$; in all the other cases $a_{ij} = 0$. The determinant $d(t) = |a_{ij}|$ is the polynomial invariant of D.

The writhe of D, denoted by $w(D)$, is the sum of signs of all the crossing points in D, where the sign is +1 if the crossing point is "left", and -1 if it is "right". It is the visible property of every knot projection: $|w(D)|$ is the type of the knot projection.

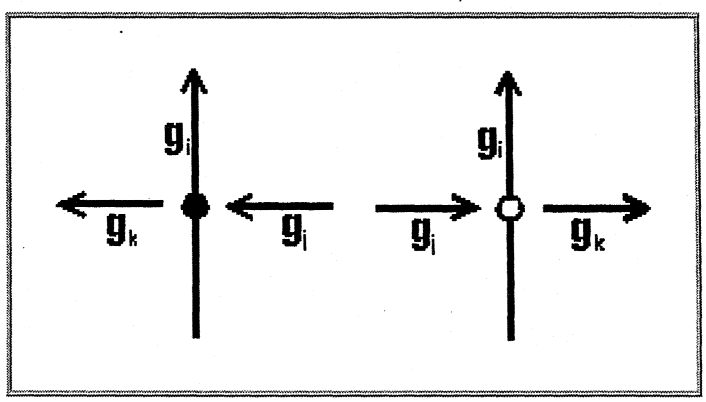

Figure 11: *"Left" and "right" crossing.*

There are some important properties of the integer polynomial invariant $d(t)=c_n t^n+... +c_1 t$:

- for every alternating knot projection, the degree of $d(t)$ is n and $|c_n|=1$;
- for every knot projection $|c_1|$ is equal to the type of the knot projection (i.e., $|c_1|=|w(D)|$);
- $d(t)$ and $d(-t)$ correspond to the obverse (enantiomorphic, mirror symmetrical) knot diagrams;
- for n=0 mod 2, the change of the orientation of alternating knot projection results in the change of $d(t)$ to $d(-t)$;

- ♦ for n=1 mod 2 a change of orientation of an alternating knot projection results in the change of $d(t)$ to $-d(-t)$.

According to the last three properties, in the set of all polynomials $d(t)$ we may distinguish even functions ($d(t) = d(-t)$), containing only even degrees of t, corresponding to amphichiral knot projections, and odd functions ($d(t) = -d(-t)$), containing only odd degrees of t, which are invariant to a change of orientation of the knot projection. For example, to the knot 3_1 corresponds odd projection polynomial t^3+3t, and to the amphicheiral knot 4_1 the even polynomial t^4-2t^2.

Let us also notice that this polynomial projection invariant makes a distinction not only between non-isomorphic knot projections of prime knots (e.g., two projections of the knot 7_5, to which correspond, respectively, the polynomials $t^7+3t^5-4t^3-7t$ and $t^7+2t^5+t^4-4t^3-7t$, but also between non isomorphic knot projections of composite knots (e.g., three non isomorphic projections of $4_1\#3_1$, with their projection polynomials $t^7+t^5-2t^3+5t^2-3t$, $t^7+t^4-3t^3-2t^2+3t$, $t^7+t^5+2t^4-2t^3+t^2-3t$, respectively).

Rectangular square grid RG[2,2] is the minimal RG from which we could derive some nontrivial alternating knot (different from unknot) - the knot 3_1. From RG[3,2] we could obtain the knots 7_4, 6_2, $3_1+\#3_1+$, 5_1, 5_2, 4_1 and 3_1, where different mirror-arrangements could give the same projection.

It is possible to derive every knot projection from some RG? Which knot projections could be obtained from a particular RG? Which mirror-arrangements in some RG result in the same knot projection? Find the minimal RG for a given knot! Could you obtain several non-isomorphic projections of some knot from the same RG? These and many other problems connected with mirror curves represent an open field for research.

12. Mirror Curves on Different Surfaces

The construction of mirror curves described before is not dependent on the metrical properties or on the geometry of the surface, so the same principle of construction can be applied to any tiling (e.g., on a sphere [7, 9] or in the hyperbolic plane [18]). Let any edge-to-edge tiling of a part of any surface be given. After connecting midpoints of adjacent edges, we obtain a 4-regular graph. Using the rules for the introduction of mirrors, it can be converted in a single mirror curve. Open question: for any tiling, find a general formula for the number k of curves, before mirrors are introduced.

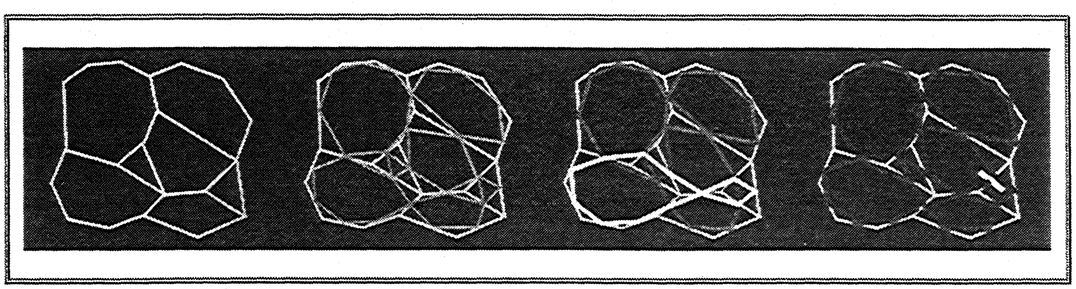

Figure 12: *Construction of a mirror curve for a given tiling.*

In the same way, from those mirror curves we may obtain the corresponding Lunda designs, *etc*. All non-isomorphic Lunda designs on a regular octahedron are given at Fig. 13. Open question: enumerate non-isomorphic Lunda designs obtained from regular polyhedrons.

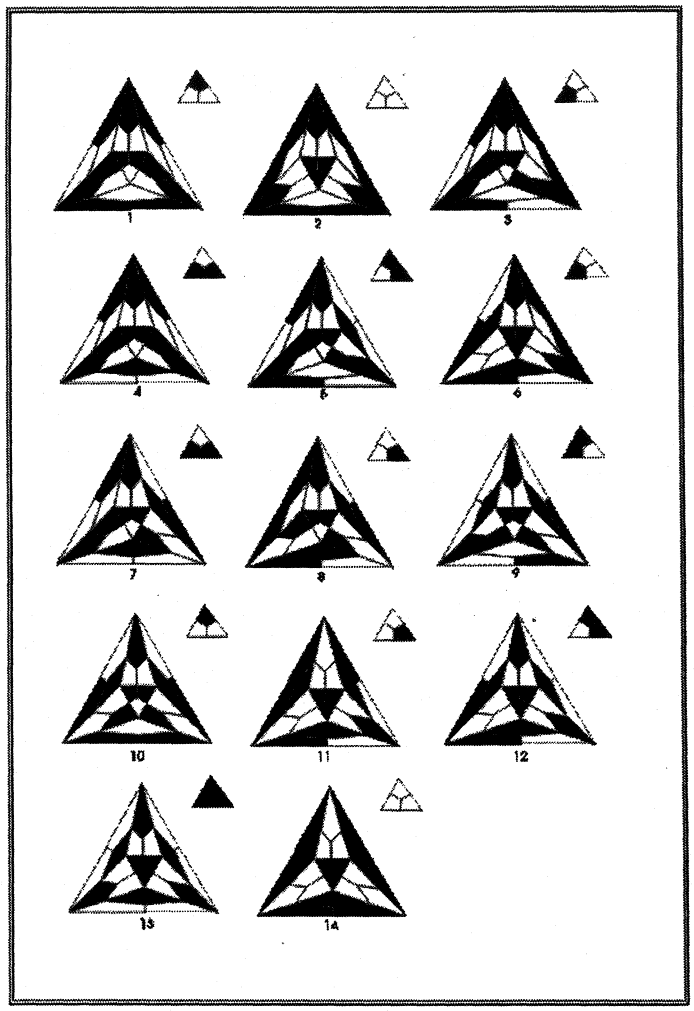

Figure 13: *Lunda designs on a regular octahedron.*

References

[1] P. Gerdes, On ethnomathematical research and symmetry, Symmetry: Culture and Science 1, 2 (1990), 154-170.

[2] G. Bain, *Celtic Art - the Methods of Construction*, Dover, New York, 1973.

[3] P. Gerdes, Reconstruction and extension of lost symmetries, Comput. Math. Appl. 17, 4-6 (1989), 791-813 (also in *Symmetry: Unifying Human Understanding*, II, Ed. I.Hargittai).

[4] S.V. Jablan, Mirror generated curves, Symmetry: Culture and Science 6, 2 (1995), 275-278.

[5] B. Grünbaum, G.C. Shephard: *Tilings and Patterns*, W.H.Freeman, New York, 1987.

[6] S.V. Jablan, Modularity in Art & Mirror Curves, http://members.tripod.com/modularity/

[7] P. Gerdes, Lunda Geometry - *Designs, Polyominoes, Patterns, Symmetries*, Universidade Pedagogica, Mocambique, 1996.

[8] P. Gerdes, On mirror curves and Lunda designs, Comput. & Graphics 21, 3 (1997), 371-378.

[9] P. Gerdes, *Geometry from Africa: Mathematical and Educational Explorations*, Mathematical Association of America, Washington DC, 2000.

[10] S. Golomb, *Polyominoes: Puzzles, Patterns, Problems and Packings*, Princeton University Press, New York, 1994.

[11] P.R. Cromwell, Celtic knotwork: mathematical art, The Math. Intelligencer 15, 1 (1993), 36-47.

[12] P. Gerdes, Extensions of a reconstructed Tamil ring-pattern, in *The Pattern Book: Fractals, Art and Nature*, Ed. C.Pickower. World Scientific, Singapore, 1995, 377-379.

[13] C.C. Adams, *The Knot Book*, Freeman, New York, 1994.

[14] F. Harary, F. E. Palmer, *Graphical Enumeration*, Academic Press, New York, London, 1973.

[15] S.V. Jablan, Are Borromean Links so Rare?, Forma, 14, 4 (1999), 269-277 (also in Visual Mathematics, http://members.tripod.com/vismath/).

[16] S.V. Jablan, Geometry of Links, Novi Sad J. Math. 29, 3 (1999), 121-139.

[17] S.V. Jablan, Ordering Knots, Visual Mathematics, 1 (1999), http://members.tripod.com/vismath/

[18] D. Dunham, Hyperbolic Celtic Knot Patterns, Bridges: Mathematical Connections in Art, Music, and Science, Conference Proceedings, 13-23, 2000.