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Flatland, Curved Space: How M. C. Escher Illustrated the History of Geometry

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Abstract

M. C. Escher's work *Circle Limit III* is a graphic representation of one of Henri Poincaré's relative consistency proofs for hyperbolic geometry. Poincaré, an opponent of the modern conception of mathematical truth, used this proof as the center point of an argument defending the competing view of mathematical truth put forward by Immanuel Kant against the charge that it is completely undermined by the existence of non-Euclidean geometries. This defense led Poincaré to limit the scope of geometry to spaces of constant curvature. This limitation was challenged by Hans Reichenbach who generalizes Poincaré's argument. This generalization, pictured in Escher's work *Balcony*, however, uses Poincaré's argument to support the modern view of mathematical truth that it was initially designed to attack.

1. Introduction

Friedrich Nietzsche argued that even the most radical revolution will become invisible when it triumphs completely. So it is with the modern view of mathematical truth. It is far too easy to see the entire history of mathematics through the lens of this conception of truth by taking the vital results that make up the history of mathematics to form a continuum leading smoothly to the present. But this placid vision does not contain the true spirit of history which resides in the philosophical struggles that lie just beneath the surface of each advance.

The philosophical struggles that lie beneath the establishment of the modern view of mathematical truth itself are in large part a result of the development and establishment of non-Euclidean geometry. Central to this story is Henri Poincaré with his relative consistency proofs for hyperbolic geometry. Poincaré took these relative consistency proofs to be more than mere formal results; he endowed them with philosophical content that he thought undermined the modern conception of mathematical truth which was just beginning to take hold in his time. Instead, Poincaré thought his relative consistency proofs saved a reformed version of the model of mathematical truth found in the works of Immanuel Kant. The philosophical result of these considerations is Poincaré's geometric conventionalism.

This conventionalism led Poincaré to limit the scope of geometry to spaces of constant curvature, excluding the work of Bernhard Riemann on generalized manifolds. Opposed to this limitation, we find Hans Reichenbach, an early 20th century philosopher of mathematics, who extended Poincaré's conventionalism to broaden its scope to include spaces of arbitrary curvature. Reichenbach's extension of

Poincaré's position brings it perfectly in line with the modern conception of mathematical truth making Poincaré a forefather of the very view he was attempting to defeat.

Both Poincaré and Reichenbach explicate their arguments via flatland, or more properly two-dimensional curvedland, examples. In visualizing these worlds whose geometries differ from our own, we are very fortunate to have the work of Dutch artist M. C. Escher. Escher's works *Circle Limit III* and *Balcony* are exact graphical representations of the arguments that Poincaré gives for his version of geometric conventionalism and Reichenbach posits for his extension of that view. Escher has illustrated this episode in the history of geometry.

2. Relative Consistency and Hyperbolic Fish

Due to the compartmentalization of the mathematical sciences and the humanities, students (and professors!) are often unaware of how eminent an influence Euclid's *Elements* was upon the entire history of western thought. Because of its breadth and rigor, René Descartes, the discoverer of analytic geometry and father of modern philosophy, thought that Euclid's work should stand as the model of all reasoning. In *Discourse on Method*, he wrote:

> "Those long chains of utterly simple and easy reasonings that geometers commonly use to arrive at their most difficult demonstrations had given me occasion to imagine that all the things that can fall within human knowledge follow from one another in the same way [2, p.10]."

Considering the "geometric method" as the hallmark of well-reasoned intellectual activity is seen throughout the development of western thought. We find writers modeling their treatises directly upon Euclid in areas as disparate as Baruch Spinoza's metaphysical discourse *Ethics* and Isaac Newton's master work of physics *Mathematical Principles of Natural Philosophy.*

Because of this privileged place reserved for Euclid, one cannot overstate the shock that occurred to the foundation of western thought when Nikolai Lobachevski and János Bolyai independently produced a geometric system whose basic axioms were not those of Euclid. Positing an alternative geometry questioned self-evidence as legitimate grounds for rational belief. This struck at the heart of our understanding of truth itself in virtually every account from Plato to Kant.

While the infamous, counter-intuitive results we find in Lobachevski's "Geometrical Researches on the Theory of Parallels" were not contradictory, a finite set of non-contradictory theorems does not guarantee that the set of axioms is consistent. A contradiction may occur in an as yet unproved theorem. Discovery of an inconsistency in Lobachevski's system would have been generally well received as it would show the primacy of Euclidean geometry and thus allow classical understandings of the means of obtaining rational beliefs to remain intact.

But Euclidean geometry not only would not, but could not surface as logically superior. This was demonstrated by the relative consistency proofs by Eugeno Beltrami, Felix Klein, and Henri Poincaré. These proofs showed that as long as Euclidean geometry remained free of contradictions, so would the non-Euclidean geometries. Since nobody wanted to surrender plane geometry, they now also could not jettison the alternatives.

A relative consistency proof is based upon the fact that contradictions arise not from the content of the theory, but from the form of the axioms making up the basis of the theory. Consistency is a formal and not semantic property. If one can translate the meanings of the basic terms of a theory S_1 into the language of a theory S_2 so that the reinterpreted axioms of S_1 become theorems of S_2, then the only way that S_1 could contain a contradiction is if S_2 has inconsistent theorems, i.e., if S_2 is also inconsistent. If we can translate the axioms of S_1 into theorems of S_2 and are willing to believe that S_2 is consistent, then we must also consider S_1 to be consistent.

Poincaré provided us with one of the most famous relative consistency proofs by finding a

Euclidean model of the hyperbolic axioms. In other words, Poincaré succeeded in translating every basic Lobachevskian geometric term—e.g., space, line, and distance—into the Euclidean language. He begins by specifying a circle in Euclidean space, termed the limit circle.

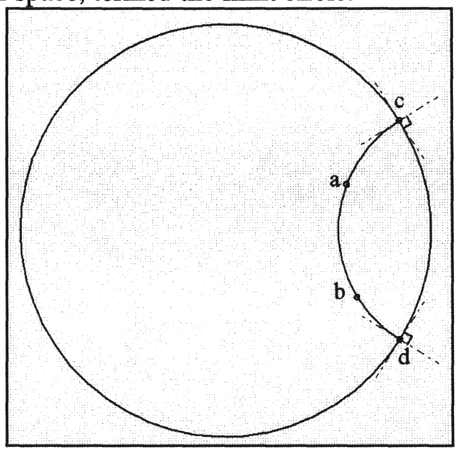

Figure 1: *A line in Poincare's relative consistency proof*

The term "space" in our reinterpreted Lobachevskian system, "space$_L$," is taken to represent the area inside, but not including the limit circle as in Figure 1. Lines$_L$ are open diameters of the limit circle or open arcs whose tangents are perpendicular to the tangents of the limit circle at the points of intersection. To determine the distance$_L$ between two points a and b in the interior of the limit circle, one finds the line$_L$ connecting them and the points, c and d, at which the line$_L$ intersects the limit circle. The distance$_L$ between a and b is $(\frac{1}{2})\log\{[(c-a)/(c-b)]/[(d-a)/(d-b)]\}$.

This reinterpretation is pictured in Escher's *Circle Limit III*, Figure 2. The area inside of the circle is space$_L$ under Poincaré's translation. The fish are arranged in Euclidean circular arcs, but also in reinterpreted Lobachevskian equidistant curves$_L$. They get smaller as they approach the limit circle in our Euclidean notion of length, but they are all the same length$_L$. In this print, we easily see some of the famous Lobachevskian results. For example, consider any given equidistant curve. We can find two other equidistant curves that intersect with each other, yet neither intersects the original equidistant curve.

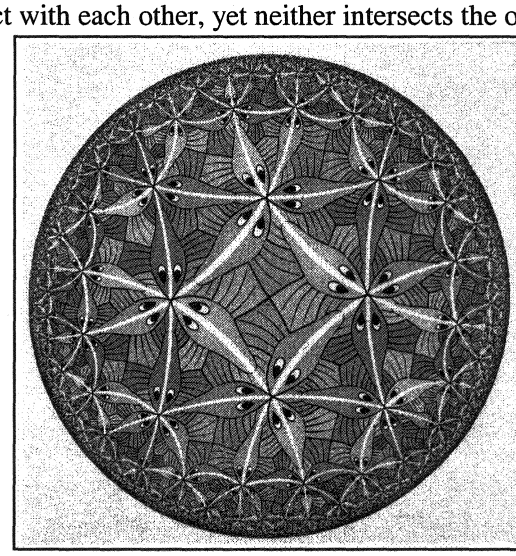

Figure 2: *M.C. Escher's "Circle Limit III"*
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This representation is not accidental. Escher first saw a graphical representation of Poincaré's disk in the great geometer H. S. M. Coxeter's article "Crystal Symmetry and its Generalizations" and wrote to his acquaintance Coxeter that this image,

> "...gave me quite a shock. Since a long time I am interested in patterns with 'motives' getting smaller and smaller till they reach the limit of infinite smallness. The question is relatively simple if the limit is a point in the center of a pattern. Also a line-limit is not new to me, but I was never able to make a pattern in which each 'blot' is getting smaller gradually from a centre towards the outside circle-limit, as shows your figure 7. I tried to figure out how this figure was geometrically constructed, but I succeeded in only finding the centres and radii of the largest inner-circles. If you could give me a simple explanation how to construct the following circles, whose centres approach gradually from the outside till they reach the limit, I should be immensely pleased and very thankful to you! [1, p.19]."

Indeed, it is not only Escher, but all of us who ought to be thankful for Coxeter's reply. The result was a series of block lithographs entitled *Circle Limit I-IV*. The first in this series, Escher himself considered "awkward." [4, p.126] But the third, he considered a triumph. To his son, he wrote of *Circle Limit III*,

> "I've been killing myself, first to finish that litho and then, for four days with clenched teeth, to make another nine good prints of that highly painstaking circle-boundary-in-color. Each print requires twenty impressions: five blocks, each block printing four times. All of this with the odd feeling that this piece of work means a 'milestone' in my development and that, besides myself, there will never be anyone else who'll realize that [5]."

Little did Escher know of the philosophical ramifications that lie within his labors.

3. Conventional Wisdom and the Plane Truth

The modern view of mathematical truth colors the standard interpretation of Poincaré's discussion. Because the modern view is so deeply entrenched and because Poincaré was so great a mathematician, Poincaré's philosophy of mathematics and the modern approach are often wrongly united.

The modern view of mathematical truth turns on no notion deeper than deductive closure. A mathematical proposition is true just in case it is entailed by whatever set of axioms the mathematician has chosen to start from. We freely select the rules and the starting point and whatever follows is taken as true. Mathematical truth as deductive dependence, is a relative notion. Consider a certain proposition in set theory that requires the axiom of choice for its derivation. Is the axiom of choice true? There is no answer here. If you want to include the axiom of choice, include it. If not, then don't. First decide, then we'll talk about truth.

It is not difficult to see why Poincaré's conventionalism, in which the choice of geometric system is free to be selected at whim, is thought to spring from this view. But, in fact, Poincaré's geometric conventionalism was an explicit attempt to avoid it.

The modern view stems from the axiomatic project of David Hilbert, but has its roots in the rationalism of the 16th and 17th century. We see this classical view most clearly stated in the writings of David Hume [8] who considered mathematical truths to be "relations of ideas," i.e., statements whose denials are contradictions. As a result, mathematical propositions are really nothing more than interesting restatements of the principle of the excluded middle. While on the one hand the strength of this foundation is undeniable, it also means that mathematical truths are devoid of true content, i.e., mathematical truths are not really true *of* anything.

The modern view extends this to give as a grounding to mathematics not only the logical proposition "A or not A," but also whatever axiom set the mathematician chooses to start from. But since these are freely selected, again mathematical truth is not really true *of* anything.

Poincaré [13] refused to accept this assertion of the vacuousness of mathematical propositions. He argued that the work of the mathematician is creative and creativity would have no place in deriving mere empty propositions. Furthermore, mathematical results are based upon mathematical induction, a process whose ampliative nature is radically contrary to the deductivist notions underlying the axiomatic approach. Mathematical propositions must be seen to have real, deep, and interesting content.

This also was the view of Immanuel Kant [9]. Kant held that mathematical truths belonged to a strange group of propositions he termed *synthetic a priori*. Like empty truths of logic that do reduce to the principle of the excluded middle and unlike observational statements, mathematical truths are knowable without experience. This is what the *a priori* means. But similar to observational statements and unlike logical truths, they say more than just "A or not A." This is why such statements are synthetic instead of analytic. They synthesize additional beliefs that go beyond the meanings of the terms in the propositions and therefore require more than mere linguistic analysis to determine their truth.

The determination of the grounding for such synthetic a priori statements was Kant's central project in many of his works, most notably his *Critique of Pure Reason*. The basis for belief in synthetic a priori statements that Kant proposes is psychological. Synthetic a priori statements are not truths of the external world of experience, but rather are the principles by which we create the external world of experience from the raw manifold of our jumbled perceptions. In the same way that an operating system must be loaded onto a computer in order to run programs, so too the synthetic a priori statements must not only be in the mind prior to experience, but such notions are the internal instructions by which the mind forms experiences out of the raw input it gets from the senses. And just as one purchases a computer with the operating system already loaded, so too do we get as a package deal a mind and the synthetic a priori propositions when created human. Further, just as one would be hard pressed to purchase a machine without a Microsoft operating system, so too are we restricted in the form of the synthetic a priori. All humans have the same basic internal instructions for constructing experience. Kant knew nothing of Linux.

Poincaré agreed with Kant on this basic outline. Mathematical knowledge, according to Poincaré, possesses content and springs from an intrinsic mathematical intuition that is naturally a part of the human mind. There are mathematical truths that do not dissolve down to the principle of the excluded middle, but that we as humans were unable to deny. The truths of arithmetic, for example, Poincaré asserted are simply a part of how the human mind works. Arithmetic, Poincaré argues, is based upon iterative processes and "[m]athematical induction—i.e., proof by recurrence—is,..., necessarily imposed on us, because it is only the affirmation of a property of the mind itself [13, p.13]."

But Kant's picture runs into trouble with the advent of non-Euclidean geometry. Plane geometry, Kant asserted, was the way in which we construct the space of experience and is the only possible space which may be so constructed by the human mind. Of course, at Kant's time this seemed indubitable. The idea of a geometry other than Euclid's was considered absurd. But in Poincaré's era, there it stood in all of its relatively consistent glory.

It is in response to this problem in Kant that Poincaré adds life to his relative consistency proof [13]. He asks us to imagine a world enclosed in a sphere. In this world, the temperature is not uniform but decreases as we move from the center towards the surface of the sphere which is itself at absolute zero. Further, all objects in the world have the same coefficient of thermal expansion and transported objects instantaneously reach thermal equilibrium. What would such a world look like? Just like Escher's *Circle Limit III*.

If we were to people the world with beings, say fish, with minds like our own, what sort of geometry would be natural to them? Poincaré comes to the conclusion that they would possess radically different intuitive geometric notions than our own despite having similar minds. "[B]eings like ourselves, educated in such a world, will not have the same geometry as ours [13, p.68]." The fish would think that lines_L are real lines. They would think that distances_L are real distances. We would, of course, correct their mistaken notions and tell them that they, in fact, live in a finite space and shrink as they move in what they mistakenly believe to be a line. They would certainly take umbrage at our claim to geometric superiority, especially when our hypotheses are so bizarre. They would steadfastly maintain that they remain precisely the same size when they move in a line and that their space is infinite in all directions as they can move arbitrarily far in any direction.

With this science fiction scenario, Poincaré is arguing that minds like ours are not intrinsically limited to Euclidean geometry. This seems like a radical break with Kant. Poincaré appears to be surrendering the doctrine of the synthetic a priori. This appearance is further amplified by the philosophical ramifications that Poincaré gleans from the fish's experience.

Poincaré argues that the disagreement between us and the fish needs not to be resolved, but dissolved as it is only illusory. The key is the "dictionary" that Poincaré used in creating the relative consistency proof in the first place. The Lobachevskian terms were translated into the Euclidean language. When we realize that we are using Euclid-speak and they are using Lobachevski-speak, we will then be able to translate our propositions back and forth before speaking to one another and then we would suddenly realize complete agreement upon everything. Geometry, therefore, becomes a conventional matter of language. As French is no more true than German, so one geometric system is no more true than any other.

Again, what could be less Kantian than this freedom of choice in determining mathematical truth? Indeed, it seems strongly correlated with the modern view. But this appearance is only on the surface. Poincaré's full explication returns the Kantian flavor, while expanding the view beyond an orthodox reading of Kant.

Poincaré begins by drawing a distinction between what he calls changes of position and changes of state. A change of position is exactly what it sounds like. An object has undergone a change of position when it is no longer at the same location. A change of state, on the other hand, is an alteration of the properties of an object other than location, say, its size or shape. The study of changes in position alone, i.e., to speak of behaviors of invariable solids, is the definition of geometry. The study of variable solids, i.e., the study of changes of state, is not mathematics at all, but rather the purview of science, in particular physics.

In this way, Poincaré sees hyperbolic and elliptical geometries not as systems competing with Euclid's, but as coming together with plane geometry to form a generalized geometry. In Euclid one may construct similar figures of any size. This is not true in the non-Euclidean systems. But while one cannot enlarge or shrink a figure without deformation, one is free to move it arbitrarily without deformation. Hence the hyperbolic and elliptical systems are different geometric languages, but they are *geometric*. This definition of geometry in terms of the change of position/change of state distinction gives us a generalized picture of geometry in which Euclid is a special case.

The next step in generalizing geometry naturally seems to be the move to spaces of arbitrary curvature, i.e, the generalized manifold geometry of Riemann. For Poincaré, however, this move crosses the line. No longer are we generalizing geometry because in spaces of non-constant curvature we cannot arbitrarily change the position of an object without a change of state. There suddenly is no longer even the possibility of invariable solids. Since geometry is defined as the study of invariable solids, there is no geometry here.

Why do we require invariable solids? Why can we not think of the world as being made of rubber bands? Simply because we as humans cannot conceive of not being able to move things without distorting them. The concept of ideal invariable solids is where Poincaré draws the new Kantian line in the sand. We may disagree with Escher's fish about when an object is or is not moved without deformation, but in disagreeing we are all starting with the presupposition that such movement without deformation is at least in principle possible in our space. The synthetic a priori is not abandoned as it first seemed, but its scope is altered to account for the relative consistency of the non-Euclidean geometries of spaces of constant curvature. As these systems provide alternative means of describing changes of positions, we are free to conventionally speak in terms of whichever of these geometric systems we choose, but we may only speak in terms of these systems. What we cannot do, according to Poincaré, is deny that movement without physical variation is possible. Just as in arithmetic, we see Poincaré in league with Kant, only he expands the geometric notions in Kant to enable accounting for non-Euclidean geometry.

4. Gödel, Escher, Reichenbach

The limitation of geometry to the study of spaces of constant curvature seemed quite artificial to many. Amongst the voices eager to overcome the barrier was Hans Reichenbach, a philosopher of mathematics of the first half of the 20th century who as an undergraduate attended the lectures of David Hilbert and after obtaining his Ph.D. was one of the small handful of students to attend Albert Einstein's first seminar on general relativity at the university at Berlin in 1919. In his most well-known work, *Philosophy of Space and Time* [14], Reichenbach argues that Poincaré was correct that geometric truths are conventional, but that such conventionality must be extended to include the generalized geometry of Riemann. Aware of Poincaré's flatworld argument, Reichenbach sets out his own. While there are similarities between the scenarios, what is most important are the aspects in which they are intentionally made to differ.

Reichenbach invites us to "imagine a big hemisphere made of glass which merges gradually into a huge glass plane [14, p.11]". See figure 3. Below this is a flat, opaque surface parallel to the plane section of the surface above. This lower plane is endowed with a strange physical property, the length of an object placed upon the plane becomes that of an object projected down from the surface above.
This physical property Reichenbach calls a "universal force." Beneath the plane section of the top surface, therefore, the behavior of measuring rods conforms perfectly to Euclid as the universal force does nothing to the rods. Under the hemispheric hump, however, the universal force shrinks the rods in such a fashion that measuring results are exactly those as would be observed in the curved region of the top surface. If we were to take a circle of wire of unit circumference and measure its radius, we would measure the expected Euclidean result of $1/(2\pi)$ in the outer reaches, but a distance greater than that in the central region.

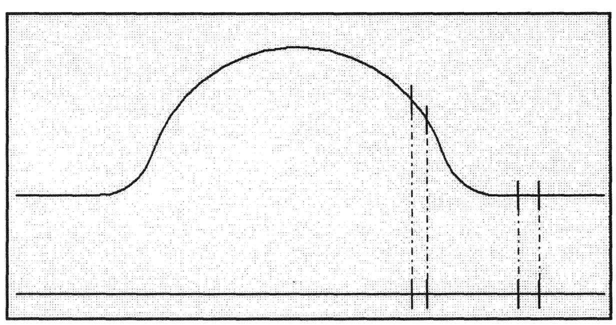

Figure 3: *Reichenbach's flatworlds*

Reichenbach's bare sketch of the situation is stylized in Escher's work *Balcony* shown in Figure 4. In this print, we believe ourselves to see, as Reichenbach describes, a bulging hemisphere rising smoothly from a plane surface altering the geometry of the building. But we must remember that this is a print on a flat piece of paper. Just as in Reichenbach's lower flat world, the geometry remains the same—it is the physical relations are changed. Where Reichenbach makes use of his strange "universal forces," Escher uses a more mundane impetus.

"The...print gives the illusion of a town, a block of houses with the sun shining on them. But...it's a fiction for my paper remains flat. In a spirit of deriding my vain efforts and trying to break up the paper's flatness, I pretend to give it a blow with my fist at the back, but once again it's no good: the paper remains flat and I have created only the illusion of an illusion. However, the consequence of my blow is that the balcony in the middle is about four times enlarged in comparison with the border objects [4, p.66]."

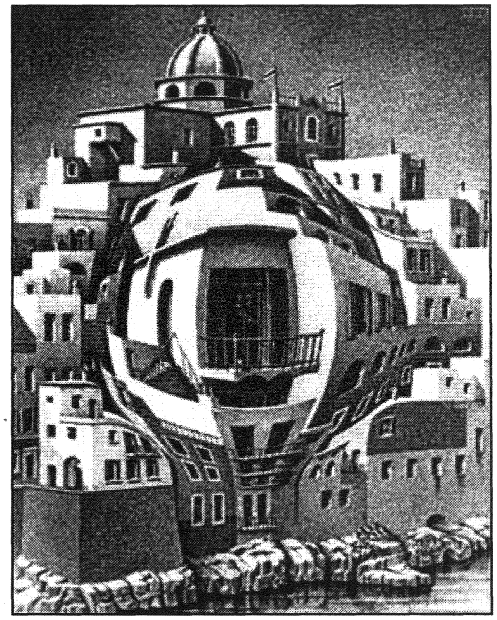

Figure 4: *M.C. Escher's "Balcony"*
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Just as in Reichenbach's scenario we have what is truly a flat surface giving the illusion of a change in geometry when it really is a physical change. The balcony is not truly bulging, but it is enlarged relative to its surroundings. What we thought was a result of a change in geometry is in fact a physical alteration.

Reichenbach follows Poincaré in peopling his world with beings with minds like ours. He asks whether the people on the flat plane would conclude that their world is flat with an odd force that alters the size of objects in the center region or whether their world has no such strange forces and has a hemispheric hump arising in the center like their neighbors above. Is there really a fact of the matter as to whether the change has a geometric or physical root? Can we expect to be believed when we tell the people standing on Escher's balcony who think that they have remained the same size, but now are outside of the plane of the building that contrary to how they think they see it, they are actually remaining in the plane and have become expanded to about four times their previous size?

Like Poincaré, Reichenbach argues that because we are able to translate back and forth between those who say that the alterations possess a geometric explanation and those who assert their genesis in a physical cause, the choice is merely linguistic and therefore conventional. We have an argument for conventionality that does not seem different at all from that of Poincaré. Intertranslatability entails conventionality.

But there is an important difference between the two scenarios. Unlike Poincaré's world, Reichenbach's is a space of non-constant curvature. It is important to see that this is not an accidental aspect. Reichenbach makes reference to Poincaré in this section of his book and so could have very well simply appropriated Poincaré's example. If he had limited himself to the area under the hemispheric hump, he would have had a Poincaréan example. But Reichenbach intentionally embeds this hump in a flat plane. He intentionally runs through Poincaré's argument in a space of variable curvature forcing its generalization.

His motivation is summed up in a claim made in the introduction to *Philosophy of Space and Time*. Reichenbach writes that "a philosophy of space and time is nowadays always a philosophy of relativity [14, xiv]." Not long after Poincaré published his argument, Einstein's general theory of relativity was born. Indeed, we know from Einstein himself [3] that he had read Poincaré's argument for geometric conventionalism and was explicitly influenced by it. The central equations of this theory, the Einstein field equations, connect the distribution of mass and energy in the universe with the metric tensor of spacetime. In giving a relativistic account of gravitation, Einstein takes geometry and puts it into play as a physical entity. Reichenbach's extension of Poincaré's argument for geometric conventionalism is a direct result of the fact that in general relativity, for any non-trivial mass-energy distribution, the geometry that best describes the space of experience is a space of variable curvature.

Our best scientific theory allows Reichenbach to save Poincaré's geometric conventionalism, but forces him to abandon the distinction between changes of position and changes of state as entirely untenable. We can no longer bracket mathematics off from physics as Poincaré desires. Mathematics is no longer an idealization which we correct with physical explanations, but rather has become an integral part of the physical explanation itself. Where Poincaré had to make use of a contrived temperature distribution for his example, Reichenbach's strange universal forces are nothing other than gravity itself as understood in the relativistic sense.

But recall that this distinction was put in place by Poincaré to safeguard the Kantian understanding of mathematical truth as containing content. Poincaré did what he did to try to account for non-Euclidean geometries without completely abandoning Kant's doctrine of the synthetic a priori. What Reichenbach's move does is to take this crutch away. The full blown geometric conventionalism that comes out of Reichenbach now leaves geometry on a purely relativistic footing. There is no deeper truth of mind or world underlying our choice of geometric system. We see this choice of geometry as the paramount example of the modern conception of mathematical truth.

We have now obtained the vantage point from which the Escher-like irony may be fully appreciated. Henri Poincaré formulated his argument for geometric conventionalism as an attempt to save Immanuel Kant's picture of mathematical truth from the modern conception. That argument inspired Albert Einstein's discovery of the general theory of relativity which adopted the generalized Riemannian viewpoint for describing spatio-temporal geometry. In searching for an epistemological foundation that could support the weight of this new theory, Hans Reichenbach pulled out Poincaré's argument for geometric conventionalism. But in so doing, Reichenbach enthroned the modern view, supplanting the Kantian view for good with the very argument that had initially been designed to save it. Like the red ants crawling in Escher's *Möbius Strip*, sometimes if you follow one side out far enough you end up on exactly the other side.

Acknowledgement

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