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Mathematical Connections
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A Polyhedral Byway

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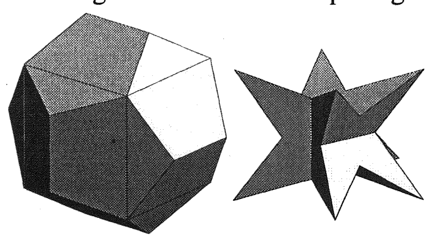
Abstract

An investigation to find polyhedra having faces that include only regular polygons along with a particular concave equilateral pentagon reveals several visually interesting forms. Less well-known aspects of the geometry of some polyhedra are used to generate examples, and some unexpected relationships explored.

The Inverted Dodecahedron

A regular dodecahedron can be seen as a cube that has roofs constructed on each of its faces in such a way that roof faces join together in pairs to form regular pentagons (Figure 1). Turn the roofs over, so that they point into the cube, and the pairs of roof faces now overlap. The part of the cube that is not occupied by the roofs is a polyhedron, and its faces are the non-overlapping parts of the roof faces, a concave equilateral pentagon (Ounsted[1]), referred to throughout as the concave pentagon.

Figure 1: An inverted dodecahedron made from a regular dodecahedron.



Are there more polyhedra that have faces that are concave pentagons? Without further restrictions this question is trivial: any polyhedron with pentagonal faces can be distorted so that the pentagons become concave (although further changes may be needed). A further requirement that all other faces should be regular reduces the possibilities, and we can also exclude those with intersecting faces. This still allows an enormous range of polyhedra, and no attempt will be made to classify them all. Instead a few with particularly interesting properties will be explored in some detail.

Regular Polyhedra with Pentagonal Faces

An useful starting point is to consider the regular dodecahedron, and three of the Archimedean polyhedra that have pentagons among their faces (Figure 2). Notice that in each case the pentagons lie in equivalent planes (they could be extended to make a regular dodecahedron), but they appear in different orientations. Neighbouring pentagons are either edge-to-edge or vertex–to-vertex, and they either touch, or they are one edge-length apart. The polyhedron is fully determined by this pentagonal framework.

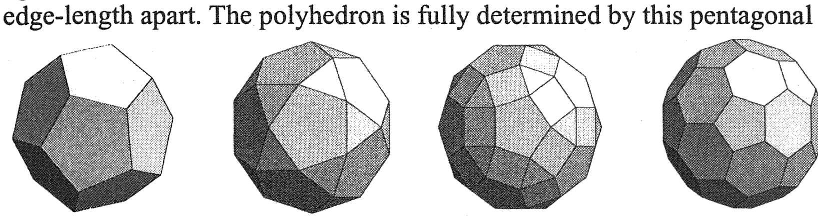


Figure 2. Well known polyhedra with pentagons among their faces.

Suppose that one of these polyhedra is cut, and part of it removed, leaving some concave pentagons (illustrated for the dodecahedron in Figure 3). New pentagons will fit into the v-shape (as in Figure 13), and the cut can be chosen so that they will lie in the correct positions to generate another of the illustrated polyhedra (except in the last example, the truncated icosahedron, when they are too far apart).

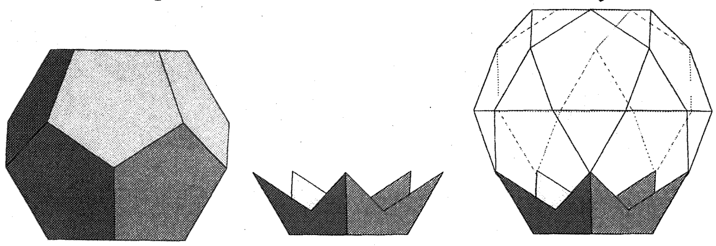


Figure 3. A piece can be cut from the regular dodecahedron that fits the icosidodecahedron.

If the pentagons are edge-to-edge then the cut must be chosen with 5-fold symmetry, if they are vertex-to-vertex then it must have 3-symmetry. Figure 4 shows polyhedra produced by fitting pieces together in this way.

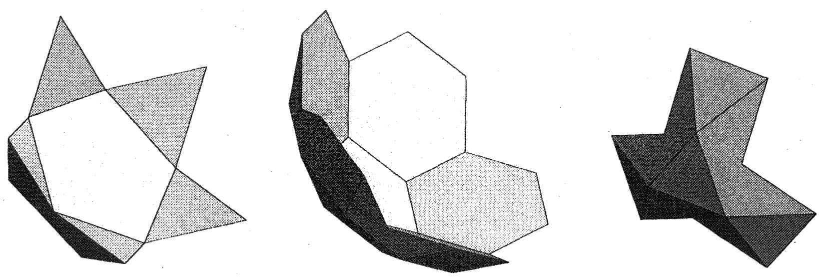


Figure 4. Cup shaped polyhedra made from pieces of those in Figure 2.

Alternatively the other pieces of the polyhedra could be used so that, for example, part of a dodecahedron can be fitted into an icosidodecahedron that has been cut to produce concave pentagons (Figure 5).

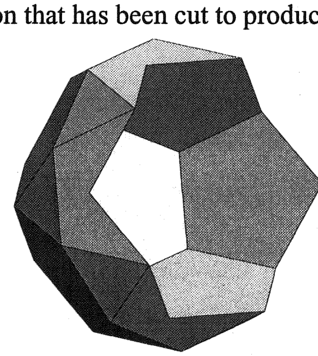


Figure 5. An alternative cup shaped polyhedron from an icosidodecahedron and a dodecahedron.

Joining Polyhedra

There is a different way of cutting two of these polyhedra that allows another type of construction. When the pentagons are touching (dodecahedron and icosidodecahedron) their diagonals define new, larger, pentagons. Concave pentagons are made by cutting along diagonals, which can be chosen to be adjacent edges of the larger pentagons, so that the cut is a regular skew polygon, and two such pieces will fit together (Figure 6). This process can be continued indefinitely, in different directions in the case of the dodecahedron.

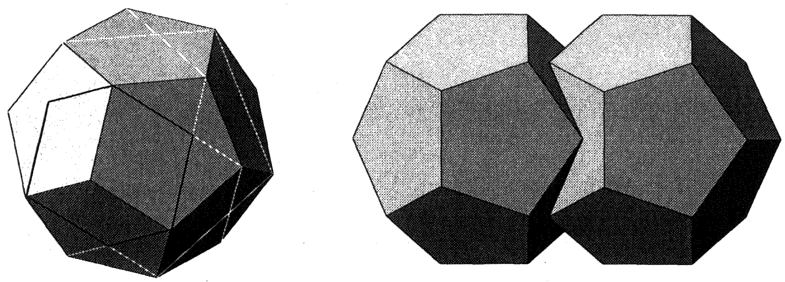
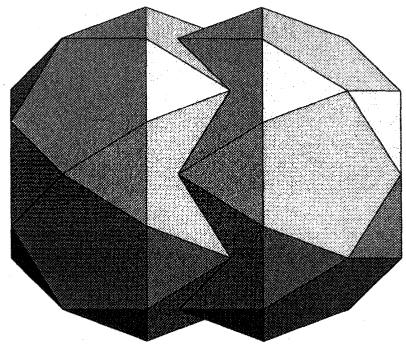


Figure 6. A regular skew cut on a dodecahedron to produce pieces that fit together.

The equivalent construction applied to an icosidodecahedron produces pieces like the one in Figure 5, which will also fit together (Figure 7). Obviously the half-dodecahedra on the inside of Figure 5 also fit together.

Figure 7. A similar construction can be applied to the icosidodecahedron.



Diminished Icosahedra

By removing a pentagonal pyramid from a regular icosahedron, a polyhedron having one pentagon among its faces, known as a diminished icosahedron or gyroelongated pentagonal pyramid, is produced (in general names of convex polyhedra follow those used in Johnson[2]). Two pyramids can be removed in two different ways to produce either a metabidiminished icosahedron or a pentagonal antiprism, and three can be removed forming a tridiminished icosahedron (Figure 8).

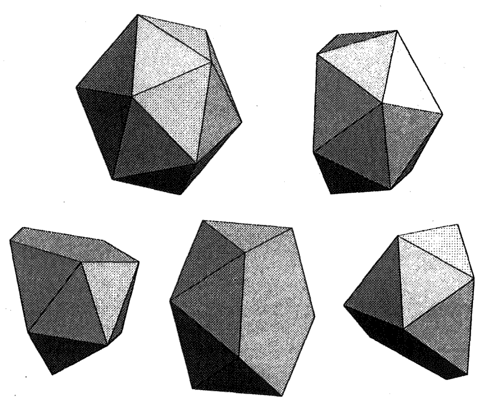


Figure 8: The regular icosahedron, diminished in various ways.

Pentagonal faces that are produced in this way can be converted into concave pentagons by cutting the polyhedra in appropriate ways. Some of the results are described below.

The Antiprism

Moving a pair of triangular faces of the pentagonal antiprism so as to create a slit can be done so that the top and bottom faces become concave pentagons (Figure 9). The resulting polyhedron does not fulfil the requirements, since it has two rhombic faces that are not fully regular, but is sufficiently striking visually to be worth including. Its surface area is the same as the original antiprism's, since reference to the icosahedron reveals the rhombic faces to be derived from pentagons, exactly matching the piece cut out to produce the concave form. Rather unusually it has only rotational symmetry of order 2.

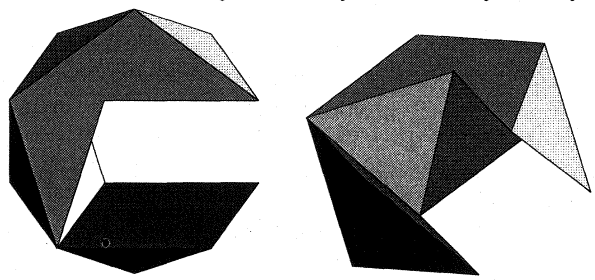
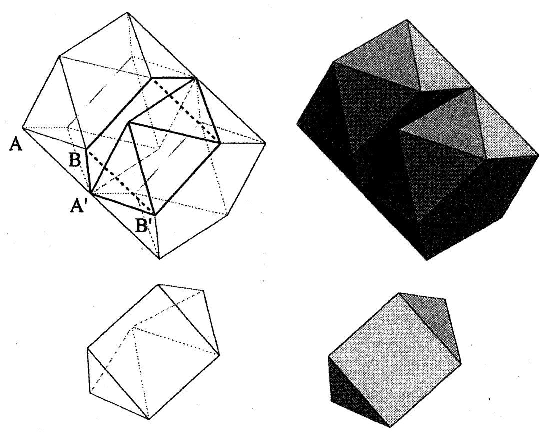


Figure 9: Two views of a pentagonal antiprism with a slot removed.

The Wedge[3]

A polyhedron that does fulfil all the requirements can be produced from a pentagonal antiprism by making a copy and translating it so that pentagonal faces overlap (Figure 10). This immediately generates a concave pentagon that forms part of a polyhedron with three of the triangles from the antiprism. In the figure point A has been translated to A', and B to B'. Clearly BB' = AA', so that triangle A'B'B is equilateral, and mirror symmetry ensures that the all other faces of the resulting "wedge" polyhedron are regular.

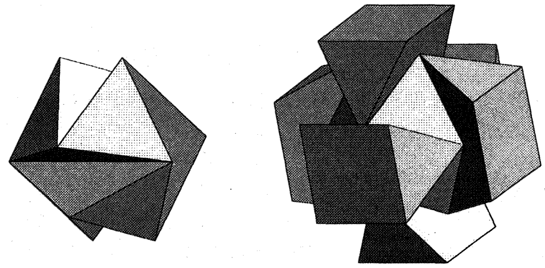
Figure 10: The construction of the wedge polyhedron from two antiprisms stacked together.



A More Symmetric Polyhedron

The convex hull of the wedge can be seen to be a pentahedron, the faces of which are a square, two equilateral triangles and two pentagonal trapezia (generated from a pentagon by cutting along one of its diagonals). These fit into the concave sections of a type of icosahedron first described by Jessen[4] to form a polyhedron with the same symmetry as the inverted dodecahedron (Figure 1). Its faces comprise 12 concave pentagons, 20 triangles and 6 squares (Figure 11).

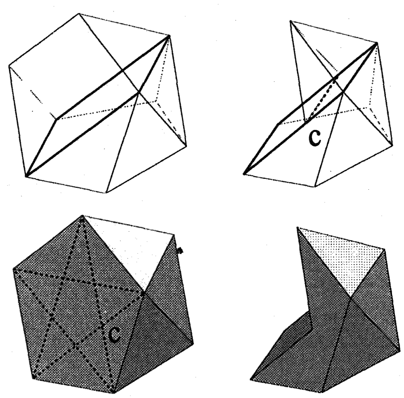
Figure 11: A Jessen type icosahedron converted to a polyhedron with concave pentagonal faces.



The Chair

By inspection of the tridiminished (or metabidiminished) icosahedron it is apparent that the rectangle formed by opposite edges of the regular icosahedron has golden proportions, since the long edge is a diagonal of a regular pentagon (Figure 12). Cutting out a section defined by the point of intersection of two diagonals (C in the figure) makes two of the pentagons concave, and generates two new faces, a square and an equilateral triangle. The new vertex, C, where a triangle, a square and a regular pentagon meet, is a concave form of vertex B in the wedge.

Figure 12: The construction of the chair polyhedron from a tridiminished icosahedron.

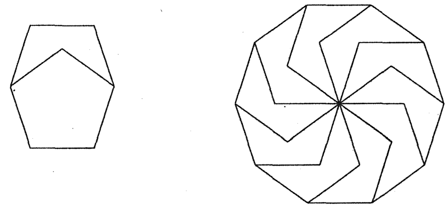


Dissections

It is obvious from its construction that a wedge will fit onto an icosahedron, or a polyhedron derived from one, such as a tridiminished icosahedron. Also a chair will fit onto a wedge. This provides limitless possibilities for constructing new shapes, only some of which will be considered here.

Noting how things fit together in 2-D will help in understanding some of the 3-D examples. The concave pentagon can fit onto a pentagon to produce an equilateral hexagon with opposite sides parallel and two lines of mirror symmetry, and 10 concave pentagons can fit together to form a decagon (Figure 13).

Figure 13: How the concave pentagon fits in two dimensions.



Fitting a wedge to a tridiminished icosahedron produces a peculiar polyhedron with the hexagon in Figure 13 as one of its faces. Two of these polyhedra will fit back to back (because of the symmetry of the hexagon) forming a convex polyhedron with regular faces known as a bilunabirotunda (Figure 14). Some of the properties of this remarkable polyhedron are described by Stewart[5].

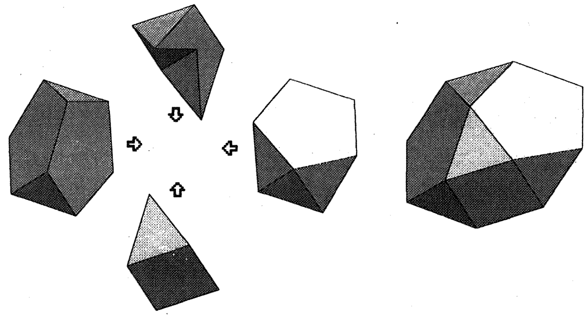


Figure 14: Two tridiminished icosahedra and two wedges combine to form a bilunabirotunda.

Wedges and chairs can fit together in different ways, but if they are arranged so that the concave pentagons fit together to produce a decagon, as in Figure 13, then the resulting form (it is not a proper polyhedron under most definitions) is clearly related to a pentagonal rotunda (half an icosidodecahedron), with a pit in the middle (Figure 15).

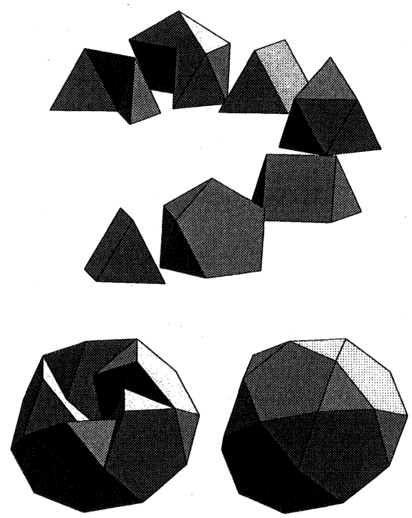


Figure 15: Chairs and wedges will fit together to form a shape with a convex hull that is a pentagonal rotunda.

Star Polyhedra

The central part of the pit consists of five concave pentagons joined to form a pentagonal pyramid, just as three concave pentagons form triangular pyramids in the inverted dodecahedron (Figure 1). These are features of the small stellated dodecahedron and the great stellated dodecahedron respectively, and the inverted dodecahedron can be seen as a great stellated dodecahedron with some of the pyramids removed (Figure 16).

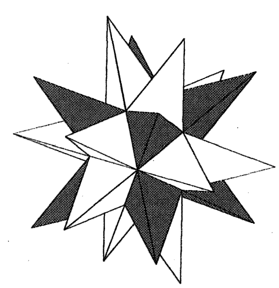
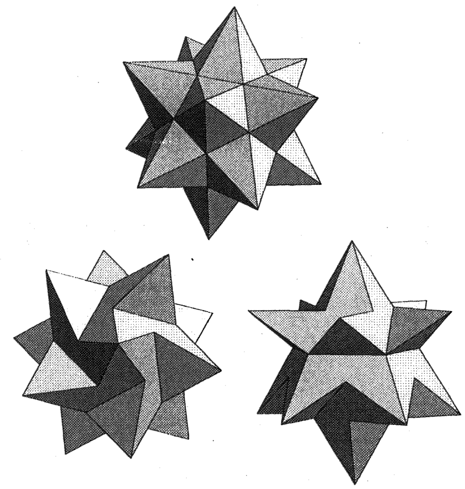


Figure 16: Removing twelve of the pyramids from a great stellated dodecahedron leaves an inverted dodecahedron.

It is clear from Figure 15 that if five concave pentagons are joined then triangles are needed to fill in the spaces created. In fact there are several polyhedra based on this construction. If the small stellated dodecahedron is thought of as a dodecahedron with pentagonal pyramids on its faces then triangular faces can be removed in pairs and replaced by an equilateral triangle. This also changes the base of the pyramid, which must be compensated, so that every equilateral triangle is matched with a concave pentagon. Since there are 32 vertices a simple application of Euler's formula shows there can be a maximum of 15 triangle/pentagon pairs. Figure 17 shows an example with five-fold rotational symmetry.

Figure 17: The small stellated dodecahedron, and two views of a polyhedron derived from it that has five-fold symmetry, and will fit into the pit in Figure 15. It can be split into two with a single cut in an obvious way.



Another Dissection

Instead of simply fitting together wedges and chairs an alternative is to start by fitting wedges onto an icosahedral core, then fitting chairs to the wedges, and so on. This is illustrated in the following sequence.

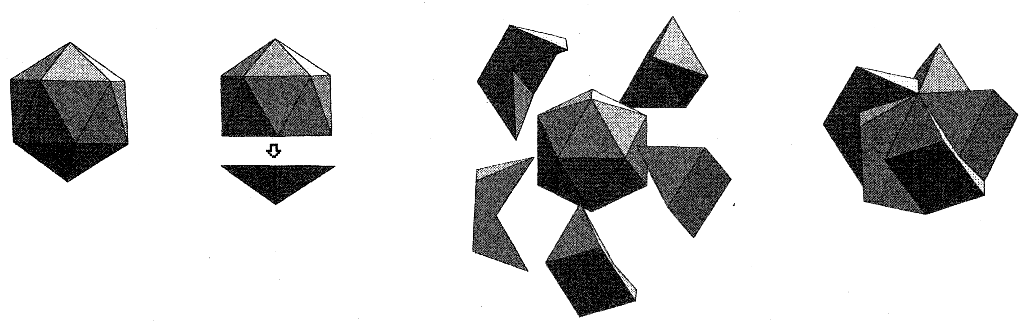


Figure 18: Fitting the first layer of wedges around a diminished icosahedron.

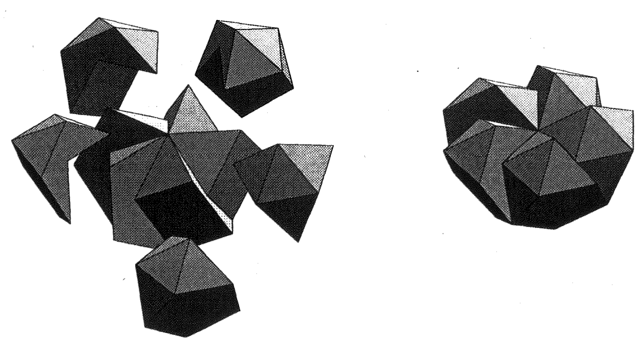


Figure 19: A second layer of chairs fits in the gaps...

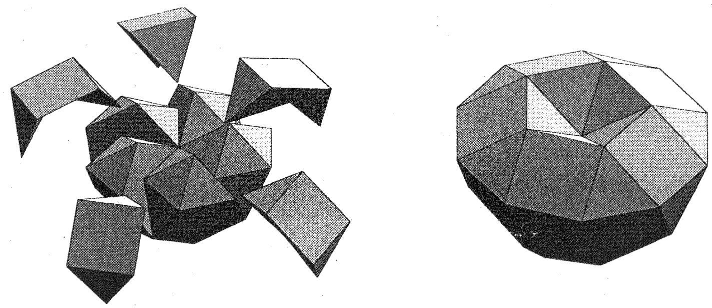


Figure 20: ...leaving room for one more layer of wedges.

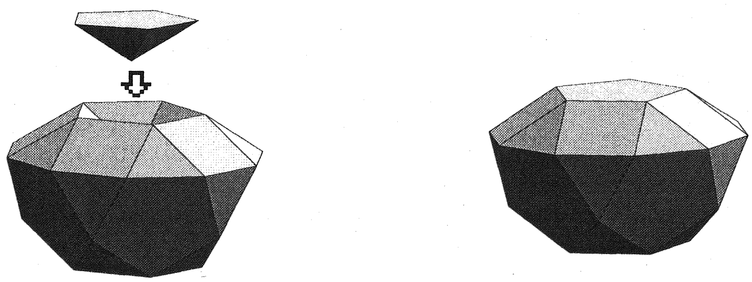


Figure 21: Replacing the pentagonal pyramid completes the pentagonal cupola rotunda.

Afterword

To some extent the course of an investigation is influenced by the nature of any aids that are used to assist thinking (including pencil and paper or computer software). In this case extensive use was made of Mathematic Activity Tiles (MATs), which are cheaply produced cardboard polygons (available from the Association of Mathematics Teachers[6]) that can be quickly joined edge-to-edge using latex cement, so the focus was on the faces of polyhedra. This is in contrast to the more common approach, which is typically concerned with issues of symmetry (for example finding all polyhedra with regular faces whose vertices are alike) and is much more like that in references Johnson[2] and Stewart[5]. One consequence is the discovery of polyhedra with low degrees of symmetry, which as a result have a visual interest that more symmetrical polyhedra often lack. Of course other construction apparatus, such as Zometool, could be used instead, and might well lead in different directions.

Acknowledgement

I would like to thank George Hart for his helpful ideas and suggestions, and the encouragement he has given me.

References

[1] Ounsted J. An Unfamiliar Dodecahedron, *Mathematics Teaching* 83 (Jun.78), pp.46-7.
[2] Johnson N.W. Convex Polyhedra with Regular Faces, *Canadian Journal of Mathematics* 18 (1966), pp.169-200.
[3] I was first made aware of this polyhedron by Adrian Pinel
[4] Jessen B. Orthogonal Icosahedron, *Nordisk Matematisk Tidskrift* 15 (1967), pp.90-96.
[5] Stewart B.M. *Adventures Among the Toroids* published privately (1980), pp.125-135.
[6] www.atm.org.uk