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Mathematical Connections
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Subsymmetry Analysis and Synthesis of Architectural Designs

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Abstract

This paper presents an analytic and synthetic method founded on the algebraic structure of symmetry groups of a regular polygon. With the method, an architectural design is analyzed to demonstrate the use of symmetry in formal composition, and then a new design is constructed with its hierarchical structure of the method.

1. Introduction

The approach of subsymmetry analysis and synthesis of architectural designs shows how various types of symmetry, or subsymmetries, are superimposed in individual designs, and illustrates how symmetry may be employed strategically in the design process. Analytically, by viewing architectural designs in this way, symmetry which is superimposed in several layers in a design and which may not be immediately recognizable become transparent. Synthetically, architects can benefit from being conscious of using group operations and spatial transformations associated with symmetry in compositional and thematic development. The advantage of operating the symmetric idea in this way is to provide architects a method for analysis and description of sophisticated designs, and inspiration for the creation of new designs.

The objective of the research resides in searching out the fundamental principles of architecture. A study of the fundamental principles of spatial forms in architecture is an essential prerequisite to the wider understanding of complex designs as well as the creation of new architectural forms. In this I stand by the Goethe's theory of metamorphosis in *The Metamorphosis of Plants*. His theory centered in the notion that there may be an ideal form with what he called *urform*, a key to understanding the development of forms. Then, based on the *urform*, a variety of new designs can be developed. An architect, Frank Lloyd Wright in his article, "In the Cause of Architecture: Composition as Method in Creation (1928)", laid emphasis on the study of the principles of composition, in his words, 'geometry at the center of every Nature-form we see'. By looking into nature and grasping the fundamental principles at work, architectural forms that are not imitative but creative are developed with the creative endeavor.

In this paper, the methodology employed in my previous papers (Park, 1996, 2000) is recounted, but a new analytic and synthetic design has been added. The study of the subsymmetries begins by characterizing three categories of process:

1. to outline the important properties of the subsymmetries of regular polygons;
2. to analyze an existing architectural design; and
3. to synthesize an abstract design with respect to the subsymmetries associated with the symmetry of the square.

2. Subsymmetry Methodology

Symmetry operations are concerned with spatial displacements which take a shape and move it in such a way that all the elements of the shape precisely overlay one another, so that, despite the displacement, the shape retains the appearance of the shape before displacement. In two dimensions, there are two symmetry groups of plane symmetry: finite group and infinite group. The finite group of plane symmetry is called the point group. Spatial transformations take place in a fixed point or line. The transformations involve rotation about the point and reflection along the lines, or the combination of both. In the point symmetry group, no translation takes place. In the infinite symmetry group, spatial transformations occur where the basic movement is either translation or a glide translation (the composite movement of a reflection and a translation). In this group, designs, which are invariant under one directional translation, are called the frieze group, and designs under two directional translations are called the wallpaper group.

Subsymmetries arise from a curtailment of some of these operations: formally, selecting subgroups from the group of symmetries. Symmetry applies to a shape as a whole, but it may also apply to components (not necessarily discrete) which make up the form, and also to the consequences of multiplying the form in some larger assembly. Symmetry may be local, or global. In Shape Grammar applications (Stiny, 1980), the local symmetry of individual shapes in a shape rule effects the number of distinct applications, or the number of colorings of the shapes. Equally, simple set grammar rules applied in parallel are sufficient to derive the standard spatial groups of global symmetry. Since most of these are infinite groups they have little relevance for material form-making: only the point groups apply to finite objects. Thus, we limit our focus on finite point group symmetry.

Let us provide an elementary account of the mathematical structure of a symmetry group, in particular, the *point groups* in two dimensions. There are two finite point groups: the *dihedral group* denoted by D_n for some integer *n*; and *cyclic group* denoted by C_n. The spatial transformations of the dihedral group comprise rotation and mirror reflection; yet the cyclic group contains rotation only. The point groups have no translation. The number of *elements* in a finite group is called its *order*. The symmetry group of D_n has order $2n$ elements, while C_n has order n elements. For example, the symmetry of the square which is the dihedral group D_4 of order 8 has eight distinguishable spatial transformations which define it: four quarter-turns; and four reflections, one each about the horizontal and vertical axes and the leading and trailing diagonal axes. C_4 has four spatial transformations: the four quarter-turns. In this, it may be a noticeable fact that C_n is a subgroup of D_n. Also, by computing symmetry groups of a regular polygon, it is possible to generate the entire group as well. Let us begin by examining the lattice of subsymmetries of a square.

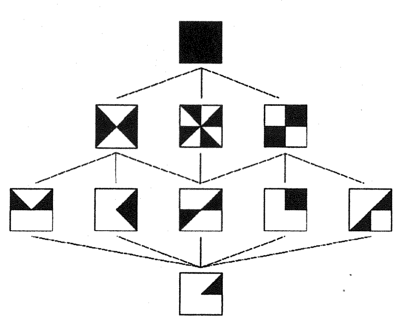

Figure 1: *The lattice of subsymmetries of the square: At the top is D_4 of order 8, below are subsymmetries of order 4, then below that again of order 2, and finally C_1 of order one, the unit element.*

The symmetry group of a square D_4 includes not only reflections in its four axes but also rotations through 0°, 90°, 180°, 270° respectively. Thus, the symmetry group of the square contains eight transformations, and these are the elements of the group. The diagram illustrates all possible subsymmetries; some with four elements, some with two, and just one, the identity or asymmetry, with one element. The structure of the diagram can be accounted for in two ways: from top to bottom, symmetries are 'subtracted' from the full symmetry of the square; and conversely, from the bottom to the top, subsymmetries are ‘added’ to achieve higher orders of symmetry. Starting from the top of the diagram, level 1 represents the full symmetry of the square D_4 with four rotations and four reflections. Level 2 consists of two reflexive subsymmetries D_2, one shows two orthogonal axes, and the other shows two diagonal axes at 45° to the orthogonal. Both of these subsymmetries exhibit a half-turn through 180°. The third subsymmetry shows four quarter-turns C_4, or 90° rotations. At level 3, there are five subsymmetries. Four with reflective symmetry D_1, two subsymmetries with a single reflective axis on the orthogonal, simple bilateral symmetry, and two subsymmetries with a single reflective axis on the diagonal. The fifth subsymmetry C_2 at this level has the half-turn rotation only. At the bottom level is the *unit element* or the identity of the group C_1. This element has no reflection axes, and no rotation less than the full-turn through 360°. The lattices of subsymmetries of other polygons such as an equilateral triangle, pentagon, etc. can be considered as well in its hierarchical order.

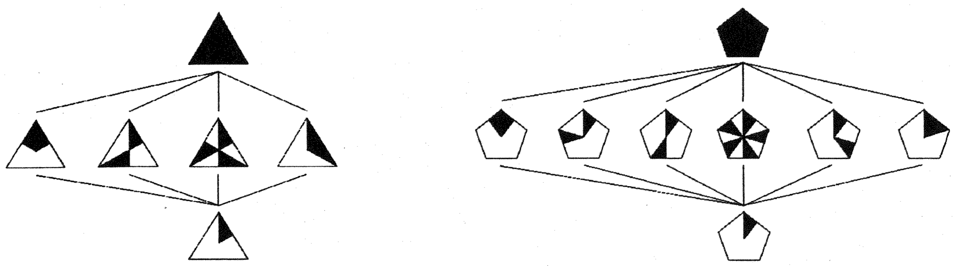

Figure 2: *The lattice of subsymmetries of an equilateral triangle, and a regular pentagon*

As with the examples of the regular polygon, above, the subgroups may be further differentiated according to axes into what we are calling here its subsymmetries. A polygon with *n* edges has at most dihedral symmetry of order 2*n*, where the order of a finite group is the number of elements. The subgroups of the symmetry group of a regular *n*-gon are perhaps ordered in the lattice diagram. For instance, D_3 is the group of symmetries of an equilateral triangle, which has order 6 with its D_1, C_3 and C_1 subsymmetries. Furthermore, we can generalize the lattice diagram of the regular polygon, which shows its hierarchical order of subsymmetries.

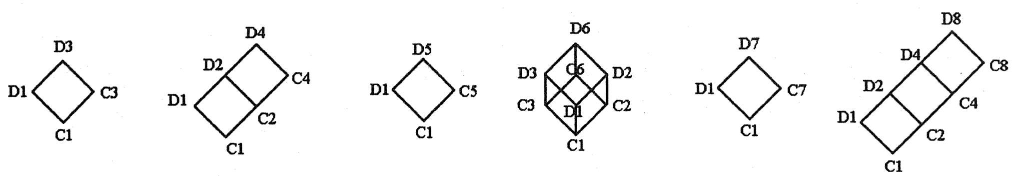

Figure 3: *The lattice of subsymmetries of regular polygons: equilateral triangle, square, pentagon, hexagon, heptagon, and octagon (from the top left)*

3. The Analytic Example

While innumerable examples could be quoted to illustrate this analysis, we take an architectural plan, which appears to use symmetry in the project, and scrutinize the plan to discover its symmetrical structure. We isolate partial elements of the design relying on its symmetrical order till the symmetry exhausts to identify its overall symmetry. Although, in most cases, the whole design is seemingly asymmetrical despite an almost obsessive concern for symmetry in the parts of the design, the analytic approach demonstrates how various symmetric transformations may be involved in each of the parts of the design, exposing the underlying structure of its spatial order. By doing so, the architect's conscious use of the symmetrical idea in the design becomes clearer.

Rudloph Michael Schindler's Popenoe House, designed in 1922 but demolished, is an example for our analysis. Schindler's debt to symmetry, particularly the hybrid use of various subsymmetries in the project, is astonishing. The plan of a single story desert cabin on a rectangular site is extremely simple, structurally and spatially; however, it is one of the most striking examples in its transparent interplay of his lifelong proportional method, "Reference Frames in Space" (Schindler, 1946), and symmetry. At large, D_4 subsymmetries, including both reflective and rotational symmetry, and his fundamental 48-inch (4') unit system guide all the major decisions of the spatial composition as well as details. The interweaving of both proportional and symmetrical ideas in his design is one of his major compositional tools throughout his career (Park, 1999). Although the use of the proportional method is consistent among his designs, there are minor differences in their symmetric application. For example, the spatial composition of the Free Public Library Project (1920) is determined by the various reflective subsymmetries without using the pure rotational subsymmetries such as C_4 and C_2, and that of the How House (1925) mainly by a reflective symmetry along a diagonal axis.

The primary layout of the house underlies in a 22' by 22' square, overlaying a 48-inch (4') unit system. Then, the square is subdivided into 6', 10', and 6', concentrically, which produces A B A rhythm. The spatial interval presents its sequential ratio to 3:5:3. The concentric spatial schema forms the absolute four-fold symmetry with the square. The *parti* is more of an underlying tool that governs spaces with regularity and shows an extreme clarity of its geometrical origin.

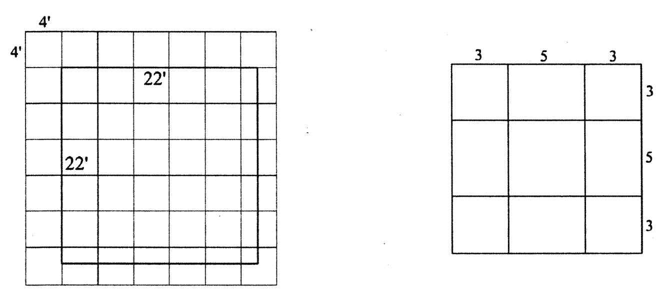

Figure 4: *Basic **parti** of the plan (Top left, basic **parti** with the unit system; Top right, the subdivision of the **parti**)*

Additional screened four porches, including Living, Sleeping, Kitchen, and Dining, are disposed in the pinwheel types of C_4 symmetry around the central square plan. These wrap around the primary square. The length of the porch wing increases successively, clock-wise in increments of 3', 4', 6', and 10'. It forms a spiral shape, which reinforces the rotational character. Whereas the basic composition of additional porches seems to derive from the absolute pinwheel type of C_4 symmetry, the final design is asymmetrical. If you see the architect's initial scheme as shown below right, his original idea of the composition becomes clearer. In the initial scheme, four porches, including four entrance doors and window openings, are set along the C_4 cyclic symmetry. It means that the asymmetric design derives from a disciplined understanding of the principle of rotational symmetry rather than merely being arbitrary.

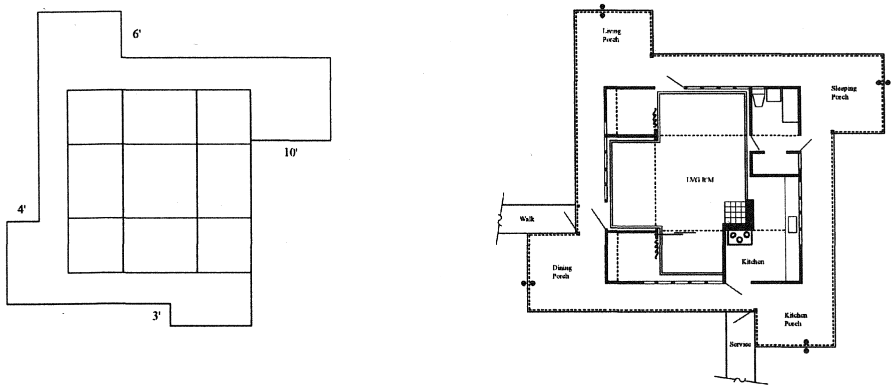

Figure 5: *Top left, the basic parti of the project; Top right, the initial scheme*

The fireplace is set along the diagonal axis of D_1 symmetry. The details of the fireplace reinforce the architect's conscious use of the diagonal symmetry as shown below right.

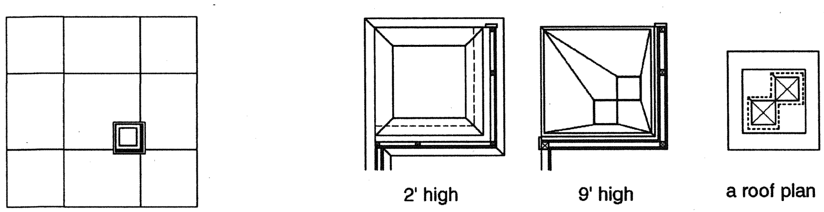

Figure 6: *Top left, the fireplace set along the diagonal axis; Top right, the fireplace horizontal sections cut 2' and 9' from the floor, and a roof plan*

Based on the *parti*, the major spaces of the house are juxtaposed, making its overall spatial configuration asymmetric. It breaks the strict symmetrical order for adjusting functional requirements. The living room is located in the center of the house. Each room is adjacent to the central living room. A sliding door divides the rooms, providing spatial flow as well as spatial flexibility in a minimum space.

The disposition of the raised high ceiling above part of the living room, kitchen and clothes closet is set along an orthogonal axis of D_1, allowing clerestory windows to bring light into the center of the building (below left). The symmetric juxtaposition of the ceiling gives emphasis to his ingenious use of the subsymmetries.

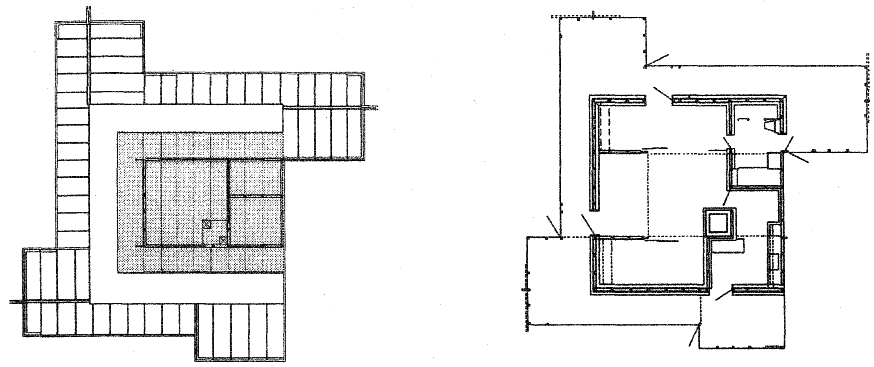

Figure 7: *Top left, D_1 subsymmetry with the ceiling design; Top right, First floor plan of the final design*

The final floor plan makes the example of the 'identity' element, which is equivalent to C_1. It can be said that in the final design, there is an abundance of symmetries within the parts while negating the strict symmetry of the whole. It is clear that the combination of the local and global symmetry is the driving force for the organization of the design. As clearly demonstrated, although the plan never used up all the possibilities of the subsymmetries of the square, various subsymmetries with rotation and reflection are superimposed into a single story design, which is an extremely rare example in architectural design. Also, it seems that Schindler sets up the symmetrical frame for the project, then he breaks it to come up with a strong asymmetrical design with functional necessity.

4. The synthetic example

Now, we build up a single building design, making use of the previous method. In a new design, all the subsymmetries are made evident including the group itself and the identity.

First of all, we need to choose a minimum building element. The main reason to choose the minimum element is to give the clearest possible picture of the design as a whole. The minimum element is composed of dots as columns with rectangles as floors. And then, as seen in the analytic example, we take the subsymmetries of the square with a grid system. The subsymmetries of the square are used as compositional tools and a grid system as an underlying *parti* to juxtapose building elements in order. Any regular polygonal grids or tessellation can be used but a rhythmic grid is implemented in our exercise. March (1981) in his paper, "a class of grids", defines and catalogues a series of grids. Among them, the permutation of the 3, 4, and 5 linear elements are chosen in this exercise. The grid itself forms D_4 symmetry.

Using the element on top of the grid, we arrange columns and floors to form a symmetric design with respect to the subsymmetries. Each floor level is designed in terms of the distinctive subsymmetry. Thus, each level defines a certain type of subsymmetry of the square. Then, we stack them up floor by floor in a standard height, creating a 3-D schematic building design.

We start at the lowest level of the symmetrical order. The first floor consists of the full symmetry of the square, which is D_4 symmetry. Columns and floors are set along the full symmetry of the square with four rotations and four reflections. The second floor illustrates the pinwheel types of rotational design of

C_4 symmetry. Column positions are the same as the previous D_4 configuration, but the floors are arranged based on the four quarter-turns C_4 symmetry. Compared to the level one, floor planes are shifted to form the pinwheel type of symmetry while preserving the column position the same as before. The third floor represents the half-turn C_2 symmetrical design. Although columns are set as D_2 symmetry, two floors are set along the half-turn rotation. The fourth floor consists of two reflexive subsymmetries of D_2. Columns and floors are set along two reflexive axes on the orthogonal. The fifth floor is composed by a single reflective axis on the orthogonal. The design looks similar to a level below, but upon a closer inspection, it shows that columns are set along the D_2 orthogonal axis, and floors are shifted from the central axis. The sixth floor illustrates the D_1 diagonal symmetry where elements and the grid are set along the single reflective diagonal axis. At the top floor is the identity, which is C_1. This is the minimum element without symmetry.

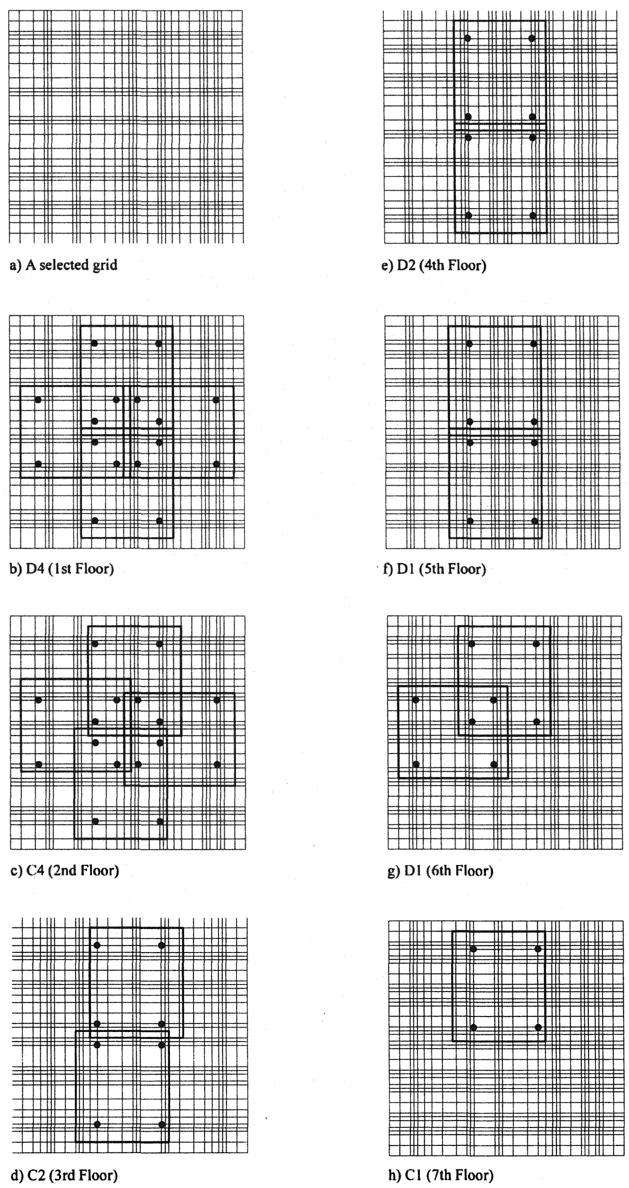

Figure 8: *Each floor with the distinctive subsymmetries*

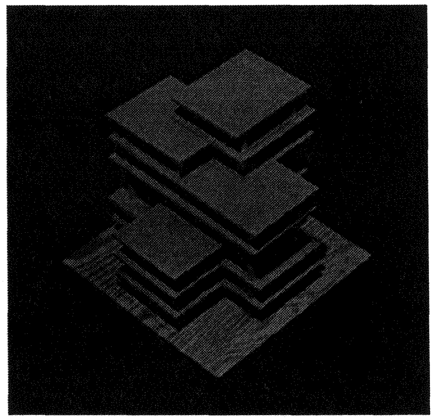

Figure 9: *Final design in a 3-D building form*

5. Conclusion

An analytic and synthetic method founded on the algebraic structure of symmetry groups of a regular polygon has been described to demonstrate the uses in architectural composition. In the Popenoe House, R. M. Schindler explores the various possibilities of the subsymmetries of the square where these are articulated around a single central point. A similar technique is applied for the generation of a new design. However, the synthetic process differs a little from the analytic one. Rather than a regular rectangular grid, we have used a rhythmic grid. Also, whereas the analytic example shows the superimposition of various subsymmetries in a single floor plan, the synthetic design has different types of subsymmetries in each floor plan. In the study, we have shown that symmetry is one of the effective methods not only for reading spatial order of complex designs but also for constructing new designs in architecture.

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