BRIDGES
Mathematical Connections
in Art, Music, and Science

What Do you See?

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Abstract

In this paper I first describe a kindergarten workshop experience which opened my eyes to the fact that children in kindergarten already possess a certain visual literacy. The giant error in education is that explicitly developing this visual literacy is not considered, except in art courses. An example in the form of a checkerboard is then discussed as a source of developing children's visual literacy in mathematics with applications to the arts. Thus one can teach children to see both as a mathematician and as an artist.

1. Kindergarten and Visual Literacy

I recently taught artmath workshops in kindergarten classes. I started by passing around a variety of seashells that included both complete shells as well as sliced shells and broken shells where the inner structure was visible. The two-dimensional spiral at the end of the shell was pointed out as well as the three-dimensional spiral structure of the shell itself. Thus the children experienced the spiral structure as well as the form-space sculpture of these ingenious early examples of architecture.

The second exercise was making Geos (geometric sculptures) using foam core shapes such as squares, triangles, and rhombi joined by inserting round wooden toothpicks in the styrofoam edges, as discussed in [1]. Of course each sculpture was different although each child was given the same set of six shapes.

For the third exercise I had brought a variety of wire knots made from clothesline wire covered with blue plastic so that the knot shape is clearly visible. I first discussed the various knot shapes, which may have been slightly beyond them. However, we then dipped the wire knots into a soap-water solution to obtain the soapfilm minimal surfaces. They thoroughly enjoyed seeing the beautiful minimal surface sculptures, which is always the case in all the children's workshops I've taught, as well as with adults.

The kindergarten workshop experience was invaluable because it opened my eyes to the fact that the children can appreciate the above exercises because they touch, see, and feel as they experience the exercises. In particular, they already have a certain visual literacy. The giant error of education is that it does not consider developing this visual literacy in K-12, except in art courses. Hopefully due to the efforts of Bridges, ISAMA, Nexus, and others, this will change.

My purpose in this paper is to present an example that can be introduced to children in order to develop their visual literacy in mathematical concepts, as well as have applications to art. The operational question to ask the children Is What Do You See? This gives them the opportunity to look and to try to see, which is what it is all about.

2. Checkerboard Example

This is a variation and extension of an example that I discussed in [2]. Consider a 4x4 checkerboard, as shown in Figure 1. What do you see? Take your time - it's all yours.

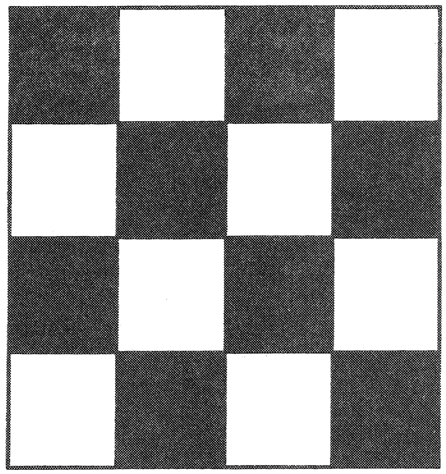

Figure 1

There are at least three ways to see the checkerboard. In [2] I only saw it numerically. However, one can also see it from a geometric viewpoint as well as a topological viewpoint. The geometric and topological viewpoints can usually be understood by kindergarten children whereas the numerical viewpoint requires addition and multiplication.

To begin, it is helpful to rotate the checkerboard 45□ - say clockwise - so that it is on point, as in Figure 2.

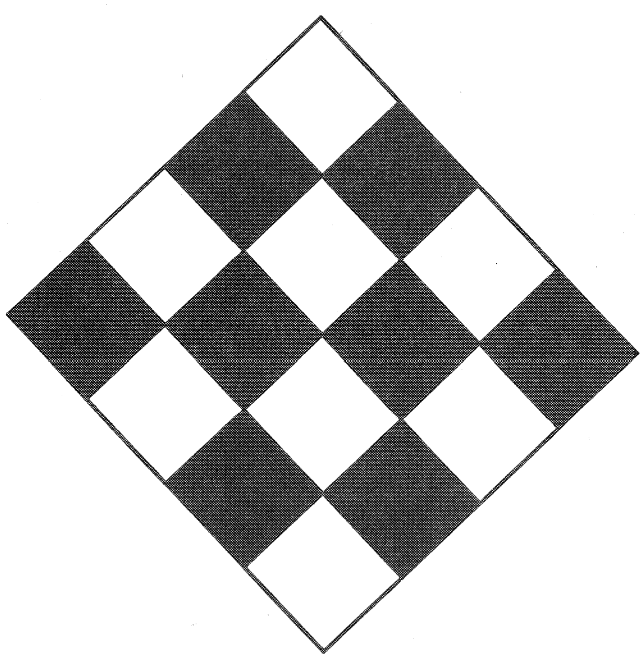

Figure 2

From a geometric viewpoint, one sees that there is vertical as well as horizontal symmetry. Using a mirror, this can be explained as mirror symmetry by placing the mirror on the horizontal and vertical diagonals. One can also point out the half-turn symmetry which is rotational symmetry by a rotation of 180□ about the center point.

To extend this exercise, one can image a checkerboard extended infinitely in all directions. Why not introduce them to infinity early in life? Thus one has an infinite checkerboard, which is a first wallpaper pattern for them, as approximated in Figure 3. Now what else do you see.

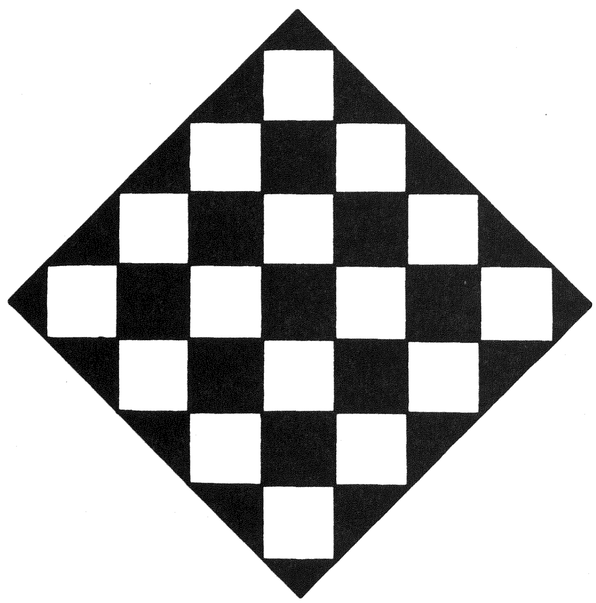

Figure 3

For the infinite checkerboard, there are infinitely many lines of reflection. There are the previous diagonal lines of reflection. There are also horizontal and vertical lines of reflection dividing the squares in half. There are centers for half-turns at the corners of squares. There are also now centers for half-turns at the center of each square.

One can now also see translation symmetry. There is diagonal translation symmetry, where the translation distance is □2 units. There is also horizontal and vertical translation symmetry, where the translation distance is 2 units.

Finally there is also glide-reflection symmetry. For the diagonal line axis one translates □2 units and then reflects. For horizontal and vertical lines through the centers of squares, one translates 2 units and then reflects. For horizontal and vertical lines along the sides of squares, one translates 1 unit and reflects.

This example introduces children to symmetry. In fact one can visually present all frieze patterns and wallpaper patterns to children. Exercises are then to classify frieze patterns and wallpaper patterns. This is strictly a non-trivial visual exercise. For example, in the following wallpaper patterns, can you see centers of half-turns and glide reflections.

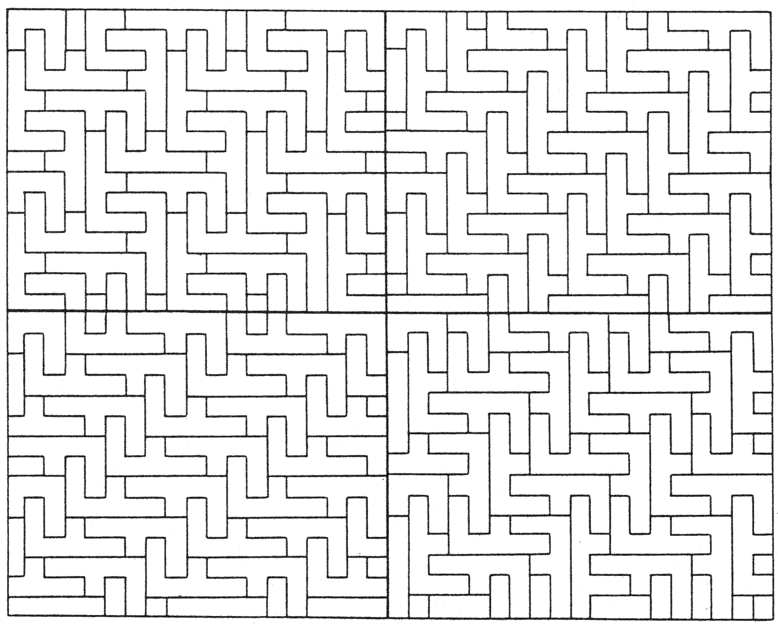

Figure 4

We will now consider the checkerboard in Figure 1 from a topological point of view. If we regard the small squares as countries, then the black-white coloring distinguishes the countries. Thus we can consider the checkerboard pattern as a simple two-color map. Note that the checkerboard pattern can be generated by equally spaced lines connecting opposite sides of the square. Can you see a general Two-Color Theorem? Can you see a generalization that would lead to a Three-Color Theorem?

A possible Two-Color Theorem is any map generated by lines that connect points on different sides of the square is two-colorable. The proof is inductive. Each time a new line is drawn, the colors of the regions on one side of the new line are switched. Note that this distinguishes any two regions having a boundary on the new line. Also regions previously distinguished will remain distinguished. Thus the two-coloring is preserved at each stage. The first step is obvious.

From an artistic point of view, it is interesting to present the development of the map one stage at a time. For example, if the map is generated by four lines, we consider four squares and add one line at a time, as shown in Figure 5.

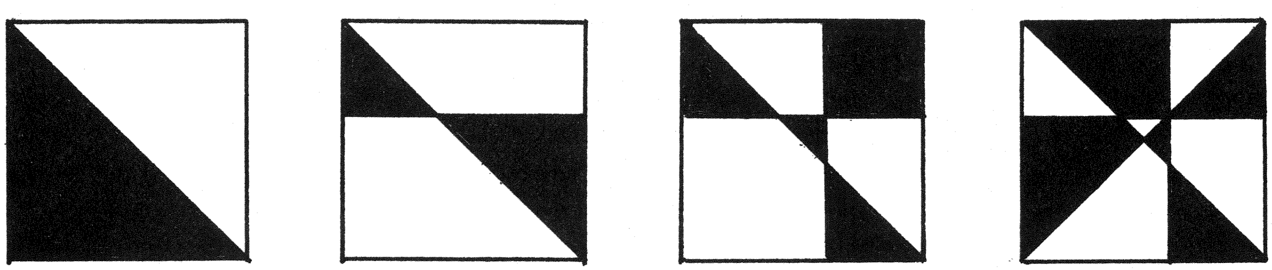

Figure 5

We refer to the four images in Figure 5 as an example of sequential art. Here the artwork consists of all four stages so that one can appreciate the development from minimal to more complex. Note that the sequence of 4 lines can be introduced in 4! = 24 possible permutations.

One can also introduce curved lines as well as closed curves such as squares, triangles, and circles. In this case one switches the colors inside or outside the closed curve in order to preserve the two-color map. An example of sequential art is shown in Figure 6, starting with a circle.

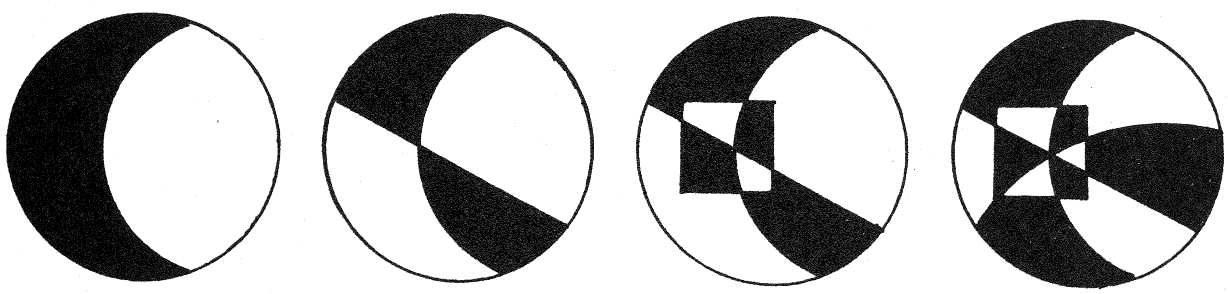

Figure 6

One can also ignore the 2-color map restriction and simple use the grid to generate a 2-color design. Moreover, in the work of Douglas Peden [3] he uses a grid of sinusoidal lines as a generalization of the usual grid. In this case he is producing optical art, where the surface appears to oscillate. An example is shown in Figure 7 (courtesy of Douglas Peden).

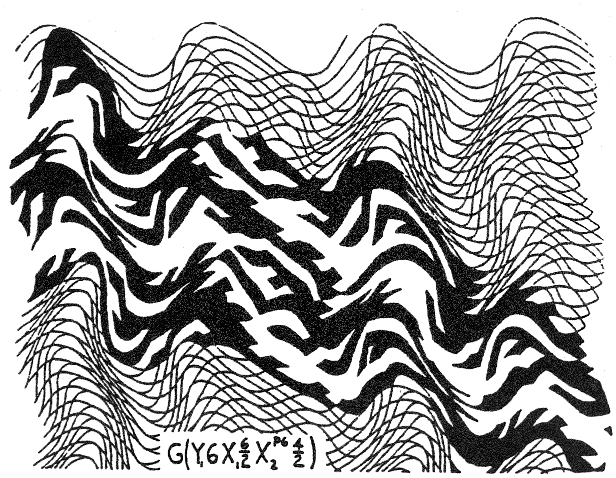

Figure 7

For a three-color map, one allows the line to stop at a previously drawn line. Two examples are shown in Figure 8. The proof of the three-color property was described to me by Richard Steinberg [5].

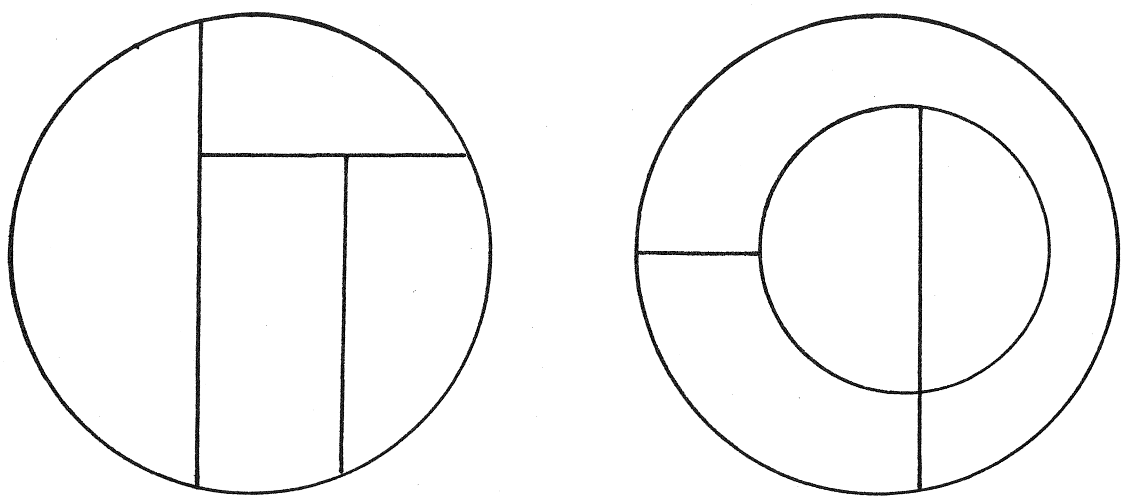

Figure 8

A large number of examples of geometric art appear in [4]. In particular, there are excellent examples of optical art in black and white by Victor Vasarely and Bridget Riley.

We will now look at the checkerboard in Figure 1 numerically. Since the checkerboard is 4 by 4, we see 16 squares. There are 2 white and 2 black in each row and each column so we see 2 □ 4 + 2 □ 4 = 8 + 8 = 16. What other numerical relationship do you see? Consider the position in Figure 2. Take your time - it's yours.

The diagonals in Figure 1 appear vertically in Figure 2 and are more obvious in this position. The number of squares in each vertical diagonal starting from the left are 1,2,3,4,3,2,1. Thus we see the symmetric sum

(1) $$1 + 2 + 3 + 4 + 3 + 2 + 1 = 16 = 4^2 .$$

If we consider a 5 by 5 square, we will se

(2) $$1 + 2 + 3 + 4 + 5 + 4 + 3 + 2 + 1 = 25 = 5^2 .$$

Now can you guess the following symmetric sum?

(3) $$1 + 2 + 3 + \cdots + 9 + 10 + 9 + \cdots + 3 + 2 + 1 .$$

What is the general formula?

In an n by n square we see the general formula

(4) $$1 + 2 + 3 + \cdots + (n - 1) + n + (n - 1) + \cdots + 3 + 2 + 1 = n^2 .$$

From (4), one can derive the formula

(5) $$1 + 2 + 3 + \cdots + (n - 1) + n = n (n + 1)/2 .$$

Lastly, here is the formula for the sum of the first n odd integers.

(6) $$1 + 3 + 5 + \cdots + (2n - 1) = n^2 .$$

Can you find (6) in an n by n square? Consider n = 3 to start.

Lastly, we will view the checkerboard in Figure 1 from one other geometric viewpoint, which in a sense is the most obvious. Namely, a tiling of the plane by squares. A first generalization is to present children with cardboard four-sided shapes, as in Figure 9. An exercise is to have them trace the shape on paper in order to construct a tiling.

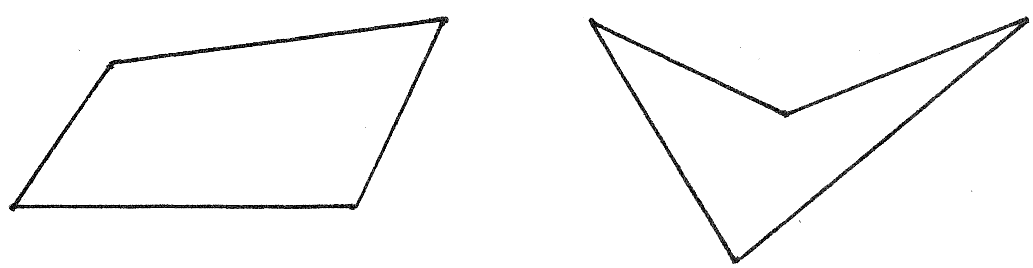

Figure 9

One can then tell the children that any four-sided shape will tile. For older children, one might even attempt to explain a proof.

The main point is that a checkerboard is a simple example of tiling that leads to the vast subject of tiling in general. In particular, there are many beautiful examples of colored tilings such as Islamic tilings.

The MAA publication Mathematics Magazine contains a large number of visual examples under the title Proofs Without Words. These are excellent exercises in developing visual literacy. It all comes down to What Do You See? There is a lot out there to look at and when you look long enough, you start to see.

References

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[2] N. Friedman, "Hyperseeing, Hypersculptures, and Space Curves", Conference Proceedings, 1998 Bridges: Mathematical Connections in Art, Music, and Science, Reza Sarhangi, Editor, Winfield, Kansas, USA.

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[4] G. Rickey, Constructivism, Origins and Evolution, George Braziller, Inc. New York, 1967, revised 1995.

[5] R. Steinberg, private communication, 1993.