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Mathematical Connections
in Art, Music, and Science

Generalized Koch Snowflakes

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Abstract

In this paper, we show how a new procedure based on vector calculus and modular arithmatic is used to generate the Koch Snowflake. The procedure is then applied to create more generalized snowflakes. We also use a new method of calculating their fractal dimension based on the scaling factor.

1. Introduction

The fascinating topic of fractal geometry which was popularized by Benoit Mandelbrot, [1], in the seventies remains an intriguing subject. The most intuitive example of a fractal is the boundary of the Koch snowflake which was constructed by the Swedish mathematician Helge von Koch in 1904. The Koch snowflake begins with an equilateral triangle of unit side: the *initiator*. The first step in the construction replaces each side by adjoining a broken line made up of $m = 4$ equal sides of length $r = 1/3$. This broken line is referred to as a *generator*.

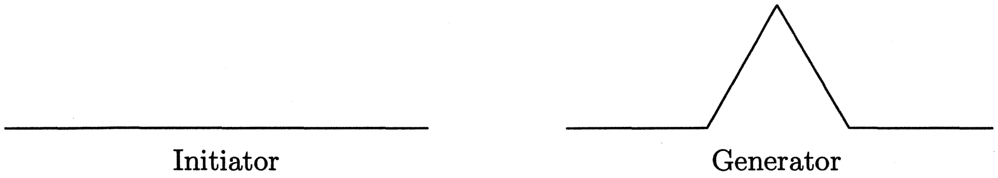


Figure 1. *The Koch Curve's Initiator and Generator.*

Each subsequent stage of the construction proceeds similarly, replacing each line segment with a copy of the generator, reduced in size so as to have the same end points as those of the interval being replaced. The snowflake consists of the curve obtained by continuing the construction indefinitely.

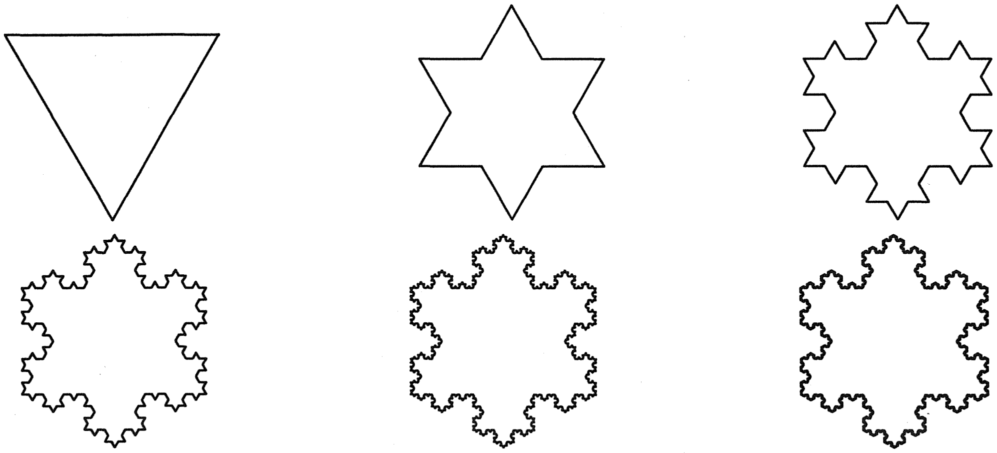


Figure 2. *Koch snowflake.*

Each side of the snowflake is self-similar, that is if we were to zoom in on the side once, twice, or an infinite times, the resulting pattern would be identical to the original generator. As we are using a recursive procedure, we can create a computer program to draw them. However, since it is impossible to run a program indefinitely, we cannot create an actual fractal but only a good approximation. Even though we only generate a fractal to a certain stage of self-similarity, it is not necessarily any less beautiful or interesting than its infinite counterpart. In fact, in most fractals, it would be difficult to tell the difference between the stages after a certain point.

2. Fractal Dimension:

One way in which we can try to "harness" the essence of fractals is by studying their dimension. The dimension of a fractal is a way of describing how densely a fractal occupies a space. Using dimension, we can objectively describe and compare any number of fractals. One way of defining fractal dimension is the method of self-similarity.

Definition: [2] For a self-similar figure subdivided into m similar pieces, each of which can be magnified by a factor of f to yield the whole figure, the *fractal dimension* is

$$D = \frac{\log(m)}{\log(f)} = \frac{\log(\text{number of pieces})}{\log(\text{magnification})}$$

The fractal dimension of a fractal exceeds its topological (Euclidean) dimension. The dimension of the Koch snowflake can be calculated using the self-similarity formula. We use the sides of the snowflake because no piece of the snowflake may be magnified to look like the whole object; however, pieces of the sides are self-similar. If we proceed to the k^{th} stage of the construction, there are 4^k pieces used, but the magnification factor is 3^k. Therefore

$$D = \frac{\log(4^k)}{\log(3^k)} = \frac{k\log 4}{k\log 3} = 1.2618$$

Now what does this mean for the dimension of the Koch snowflake in its entirety? Well, one might jump the gun and assume since the snowflake is composed of three Koch curves as described above that the dimension of the snowflake would be three times our calculation of 1.2618. However, thinking of dimension as a sort of "density", we can apply the following theorem to find the dimension of the Koch snowflake.

THEOREM: [3] If $dim(B) \leq dim(A)$ for sets A and B, then $dim(A \cup B) = dim(A)$.

Since the Koch snowflake in its entirety is really a composite of three Koch curves as described above, each with the same dimension, $dim(A) = dim(B) = dim(C)$, the snowflake's fractal dimension is 1.2618.

3. Generalized Snowflakes and Their Dimension:

In this paper we will focus on generalizing the Koch snowflake; by creating a new routine that uses simple modular arithmetic and vector algebra to produce beautiful snowflake type fractals. Manipulating the process initiator/generator we described to create the Koch snowflake, we can create an infinite number of new generalized snowflakes. Let us now fully concern ourselves with the initiator/generator method by addressing three crucial questions:

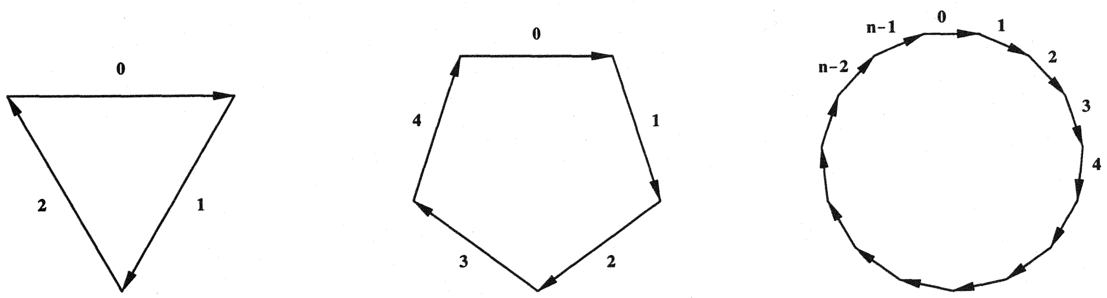
(a) How does the method work on a mathematical (as opposed to simply graphical) level — i.e., what are the main components of the process?

(b) How can we utilize the components of the process to quickly and easily graph a snowflake (without having to graph each generation)?

(c) Can the initiator/generator method make calculating dimension of a snowflake purely routine, based solely on the basic components of the process?

By answering these three questions, in the end, we will be able to more fully understand the beauty that exists between mathematics, art and nature. To begin with, we can break down the initiator/generator process into two main components: vector algebra and modular arithmetic. These two simple subjects of mathematics will give us all that we need in order to understand the process, and later graph many generalized snowflakes created by the process. First, let us consider how vector algebra is used on the definition of the initiator. The only restriction here, for simplification purposes, is that the initiator be a regular polygon - an object of three or more sides of equal length. For the Koch snowflake, our initiator was an equilateral triangle.

3.1. Vector Algebra. For those who have no experience in vector algebra, or for those who are a bit rusty the basic idea centers around the idea of a vector. What is a vector? Think of a line segment with direction, i.e., an arrow. The key is "direction." Since up is entirely opposite from down, and left from right, there needs to be a way to tell one from the other. That's where the vector (arrow) comes in. How do vectors apply to an initiator of a snowflake (a polygon)? The graph must start at a point, which for simplification, we call the origin of the graphing axis. Also, let us have the first side be drawn to the right and be parallel to the horizontal axis. After drawing the first side, one would continue on to the second, third, fourth, etc. Since each side is added by drawing in a certain direction, they are considered to be a number of vectors and not line segments. If one were to never lift the pencil, the resulting graph would really be a sequence of vectors whose direction changed in a clockwise fashion. Consider the examples below.



Triangle Pentagon n-sided polygon

Figure 3. *Vectors and Polygons.*

Now that we know how vectors are used in defining the initiator of a snowflake, how are they used to define the generator? As can be seen in Figure 3, each vector is given a reference number according to the order in which it was "drawn." Note that we start at zero, although there would be no drastic consequences if we had started at one. If there are n sides in our polygon, our vectors will be designated with numbers ranging from zero to $n-1$. Using these side indications, we can easily define a generator using the sides of the polygon. For example, let us return to the familiar Koch snowflake. The initiator is the triangle shown in Figure 3, with sides zero to two. We can use the negatives of each of those numbers, including negative zero! What this means for the graphical representation of our generator, is the negative of a certain side will have a direction completely opposite from the original side. The vector will in fact pivot one hundred and eighty degrees. So, if we were to consider the vector -**0**, its graphical representation would be identical to that of **0**, but would be pointing to the left instead of the right. We can see from Figure 4, that its generator is the set of sides **0,-1,-2,0**. This means that for every generation of the Koch snowflake, each line segment will be replaced by a scaled-down sequence of the original sides **0,-1,-2,0**.

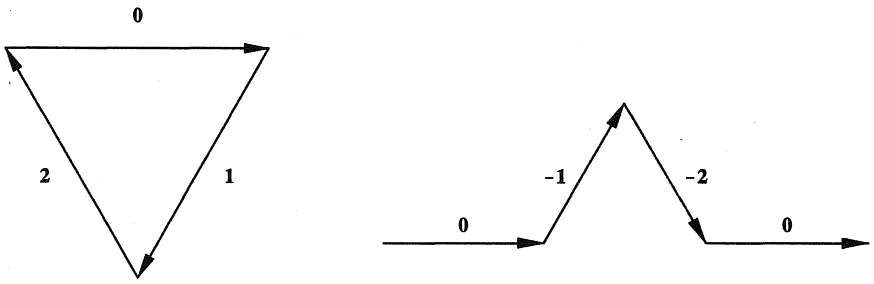


Figure 4. *The Koch Snowflake Generator's Vector Sequence.*

To graph the entire snowflake, we must also know the coordinates of each vector. The vectors are numbered zero to $n-1$. Using those numbers, the coordinates of the vector i equals $A_i = (r\cos\theta_i, r\sin\theta_i)$, where r is the scaled vector (1/scaling factor), and $\theta_i = -2\pi\, i/n$ for $i = 0, 1, \ldots, n-1$. Vectors with a negative corresponding number, $-i$, have coordinates such that $A_{-i} = -A_i$. For the Koch snowflake, we start with an equilateral triangle ($n = 3$). If we assume that each side of the triangle has length one ($r = 1$), then:

Vector	i	θ_i	A_i
0	0	0	$A_0 = (\cos\theta, \sin\theta) = (1, 0)$
1	1	$-\frac{2}{3}\pi$	$A_1 = \left(\cos\frac{2}{3}\pi, -\sin\frac{2}{3}\pi\right) = (-.5, -.866)$
2	2	$-\frac{4}{3}\pi$	$A_2 = \left(\cos\frac{4}{3}\pi, -\sin\frac{4}{3}\pi\right) = (-.5, .866)$

Table 5. *Vector Coordinates.*

Using our set of vector coordinates A_0, A_1, A_2, we can now graph the equilateral triangle initiator of the Koch snowflake. This is where we put to work the "algebra" part of "vector algebra." We are going to add the vectors together, which will enable us to find the corner points of the triangle, and finally graph the initiator. It is just as easy, but a little more work to find the graph of the first generation of the Koch snowflake. Since the pattern set of the first generation is $\mathbf{0, -1, -2, 0, 1, -2, -0, 1, 2, -0, -1, 2}$, all we need to do is go through the same procedure as above while remembering to scale down the new vectors. Since the length of the generator is three times that of the original initiator side, our scaling factor is three and we must reduce our new vectors by $r = 1/3$. Therefore, we can still use our old calculations for our A_i's, but we must multiply each one by $1/3$. Once we know how to graph the Koch snowflake's initiator and first generation, it is routine to graph any generation of any snowflake if it weren't for two things. First of all, a lot of calculations are involved, even for first generation curves, secondly, you may be wondering where the pattern set for the first generation of the Koch snowflakes; $\mathbf{0, -1, -2, 0, 1, -2, -0, 1, 2, -0, -1, 2}$ came from. This is the part where our second basic mathematical subject comes into play. Of course, we're talking about modular arithmetic!

3.2. Modular Arithmetic. We can use modular arithmetic "$(x + y) \bmod z$". Our x, y, and z come from the initiator side presently being worked on, the generator pattern set, and the number of sides in the initiator. The results of a set of equations like this will give us our snowflake pattern set.

If we consider a generator with m elements that is applied to a n-sided polygon, we will produce a pattern of $n \times (m^k)$ elements for the k^{th} generation. For the Koch snowflake, which has a three-sided initiator and a four-element generator, the pattern set will contain twelve elements for the first generation, forty-eight elements for the second generation and so on. The pattern set of a k^{th} generation snowflake depends directly on the pattern set of the $(k-1)^{st}$ generation.

If we let the initiator be generation zero, we know that the first generation of a snowflake must depend on the pattern set of initiator. Thus, the first generation of the Koch snowflake builds on the equilateral triangle initiator, which has three sides. First we apply the generator to side 0. In our modular arithmetic equation $(x+y) \bmod z$, x is the side we are presently applying the generator to. For now, $x = 0$, but it will later also equal 1 and 2. We let y vary, in that it can equal each element of the generator. Finally, z is the number of sides in the initiator $(z = 3)$. We need to keep track of all of the negative elements, since those will ultimately influence our pattern set. Remembering to do so is easiest if we adjust our modular arithmetic formula to become: $[(\text{sign of } x) \times (\text{sign of } y)] \times [(|x| + |y|) \bmod z]$. In this way, we first would calculate the absolute value of x plus the absolute value of $y \bmod z$, and then multiply that answer by the signs of x and y. We start with side 0 and the first four of twelve pattern elements.

G_0	G_1	G_2	G_0	G_1	G_2	G_0	G_1	G_2
0	0	0	1	1	1	2	2	2
		-1			-2			-0
		-2			-0			-1
		0			1			2
	-1	-1		-2	-2		-0	-0
		2			0			1
		0			1			2
		-1			-2			-0
	-2	-2		-0	-0		-1	-1
		0			1			2
		1			2			0
		-2			-0			-1
	0	0		1	1		2	2
		-1			-2			-0
		-2			-0			-1
		0			1			2

Table 6. *Sequence of Pattern Sets for the Koch Snowflake's Initiator G_0, First Generation G_1 & Second Generation G_2.*

Due to the construction of the generator, for any generation, a positive side i will return a pattern set of $i \bmod 3, -(i+1) \bmod 3, -(i+2) \bmod 3, i \bmod 3$. On the other hand, a negative side $-i$ will result in a pattern set of $-i \bmod 3, (i+1) \bmod 3, (i+2) \bmod 3, -i \bmod 3$. Using this shortcut, it is easier to calculate the second generation, which is summarized in the table on the preceding page.

A more efficient method of graphing involves finding the orientation of each vector in any generation of a snowflake individually, i.e., doing the calculations independently at each stage of a snowflake. In order to do this, we must first express the vector in question, w, in terms of a base m. As before, m is the number of elements in the generator,n is the number of vectors in the initiator, and k the generation of snowflake. Remember that for any generation k, where k ranges from one to infinity, the number of vectors in that particular snowflake equals to $n \times (m^k)$. To express the vector in question in base m we have to fill in the blanks "_" in the following equation:

$$w = \text{"\_"} \times m^k + \text{"\_"} \times m^{k-1} + \ldots + \text{"\_"} \times m^1 + \text{"\_"} \times m^0$$

Beginning with the space to the far left and working to the right. The solution for a space is the greatest number that does not force the solution to exceed w. For example for the 35^{th} vector in the second generation of the Koch snowflake, since $m = 4$, $n = 3$, and $k = 2$, we get

$$35 = \text{"2"} \times 4^2 + \text{"0"} \times 4^1 + \text{"3"} \times 4^0$$

The actual representation of our vector 35 in terms of base 4 is in the form of the three integers that filled the spaces written together. Thus, 35 in base 4 equals 203_4. The subscript 4 indicates the base. Using the number 203_4, we can now find the orientation of vector 35. The first digit describes the component vector of the initiator on which vector 35 lies. Thus, the digit two tells us that vector 35 lies on vector two of the initiator, as shown in Figure 7(A) by the arrow. The second digit indicates which part of the generator applied to the previous component vector (two), vector 35 lies on. Thus, as the second digit is zero, we know that the second component describing vector 35 is the very first vector in the generator applied to initiator vector two. This vector is shown by the arrow in Figure 7(B). Lastly, our third digit is three and according to above, describes the vector in the generator applied to zero (the previous component vector) on which vector 35 lies. Three, indicating the last part of the generator (see the arrow in Figure 7(C)), gives us our final component for "building" the 35^{th} vector of the second generation of the Koch snowflake.

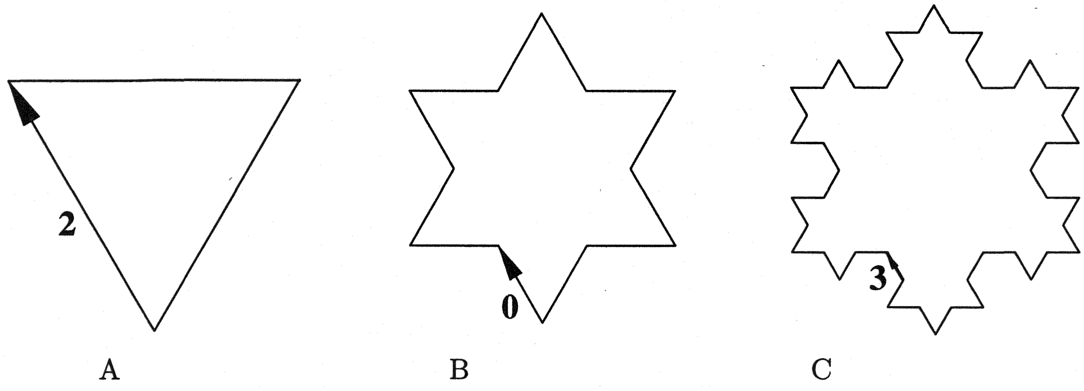


Figure 7. *Composing Side 35.*

This method was applied to create a Mathematica program "Fractastic", available at

(`http://webcampus3.stthomas.edu/c9shakiban/`).

It uses three inputs — the number of sides in the initiator, the pattern set in the generator, and the number of desired generation — to compute and then output a graph of the initiator and a sequence of snowflake generations.

C. Snowflake Dimension:

As it turns out, we can also find the fractal dimension of the snowflake as a straightforward calculation, which comes in the form of a formula:

$$D = \frac{\log m}{\log f}$$

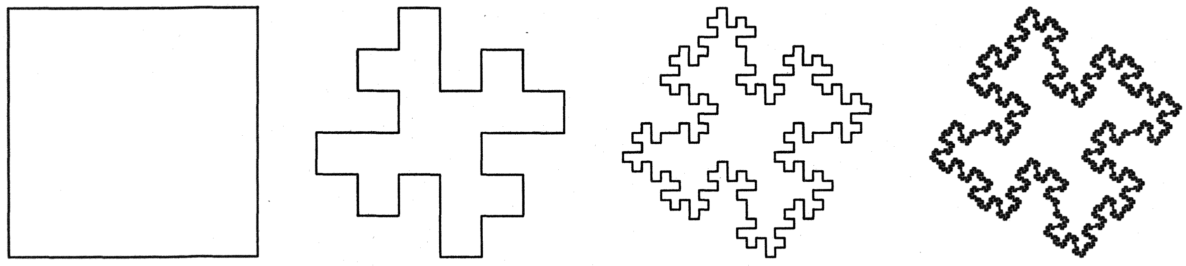
where m is the number of elements in the generator, and f is the scaling factor. If the generator is not listed, it is certainly possible after inspection of the first generation of the snowflake to determine the generator and continue with calculating the dimension. The variable f, on the other hand, is not as simple to find. It must be calculated and follows the formula below where $v_1, v_2, \ldots, v_N$ are the vector coordinates of the generator: $f = v_1 + v_2 + \ldots + v_N$ This procedure for finding snowflake dimension is far from being a perfect tool for all snowflakes. It works perfectly if the next generation of the snowflake does not overlap with the previous generation. When the snowflake generator crosses itself, goes back along the same line segment more than once, or lacks continuous "forward motion", the chances are that the snowflake, in its limit, is space filling. There is still innumerable details we don't know or understand about their fractal dimension.

4. Conclusion

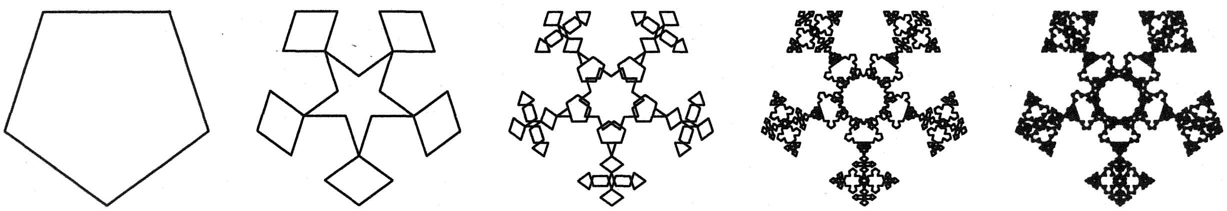
In closing, we want to leave you thinking about the beautiful way in which mathematics and art are connected and though they seem so different, yet are often the same. In the following pages you can find a number of beautiful generalized snowflakes for your viewing. We would like to encourage you to try to look at these snowflakes not only as subjects of mathematics, but also as works of art, and in particular, as part of the mystifying beauty of the universe that appears even its smallest elements.

5. Examples

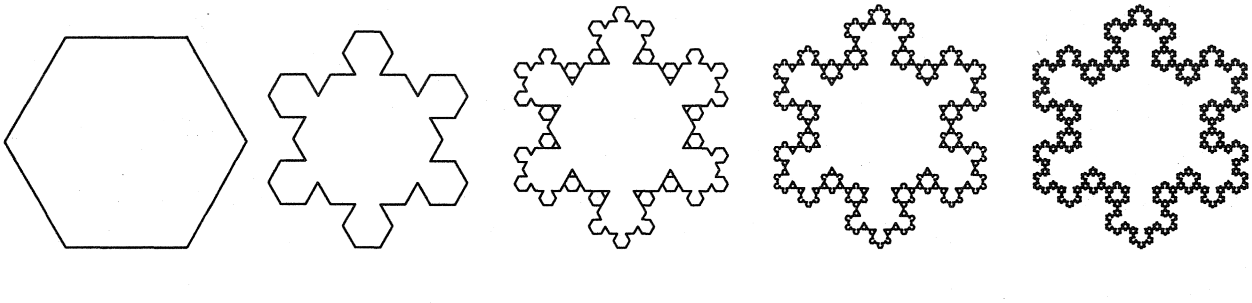
Initiator: square Generator Pattern: $\{\mathbf{0, 3, 0, 1, 1, 0, 3, 0}\}$
Snowflake dimension $= \log 8 / \log 4 = 1.5$



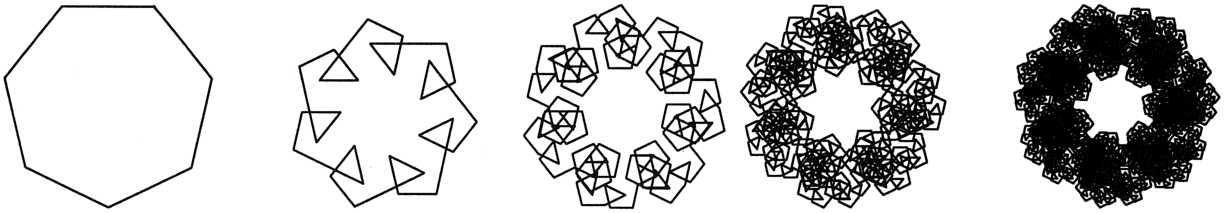
Initiator: pentagon Generator Pattern: $\{\mathbf{0, -4, -3, -2, -1, 0}\}$
Snowflake dimension $= 1.6309$



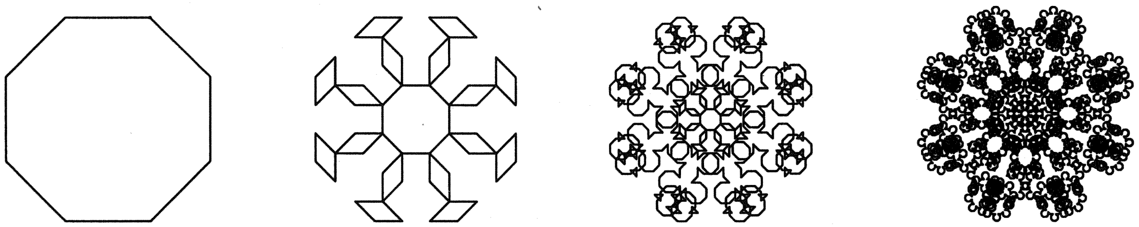
Initiator: hexagon Generator Pattern: $\{\mathbf{0, -1, -2, -3, -4, -5, 0}\}$
Snowflake dimension $= \log 7 / \log 3 = 1.77124$



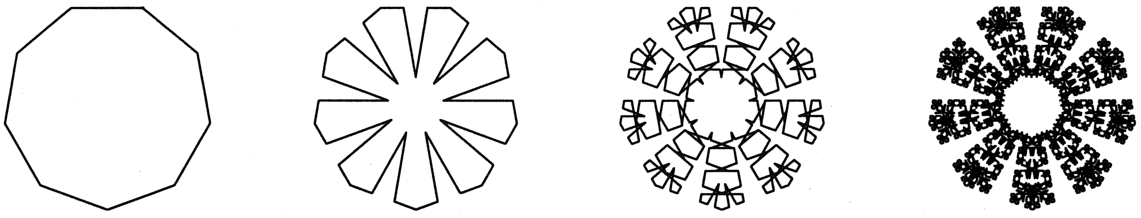
Initiator: heptagon Generator Pattern: $\{\mathbf{0}, -\mathbf{5}, \mathbf{3}, -\mathbf{2}, \mathbf{1}\}$
Snowflake dimension = 2; Space filling curve.



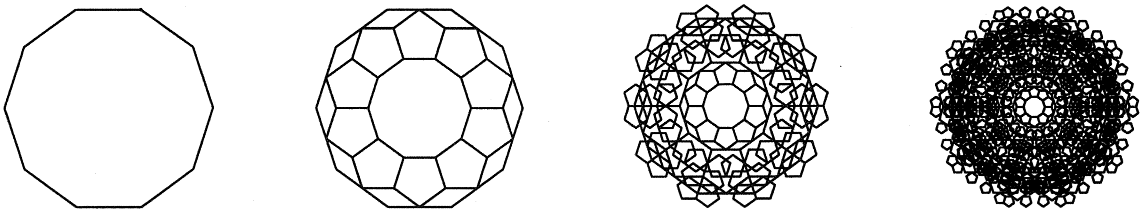
Initiator: octagon Generator Pattern: $\{\mathbf{0}, -\mathbf{7}, -\mathbf{6}, -\mathbf{5}, -\mathbf{4}, -\mathbf{3}, -\mathbf{2}, -\mathbf{1}, \mathbf{0}\}$
Snowflake dimension = $\log 9/\log 3 = 2$; Space filling curve.



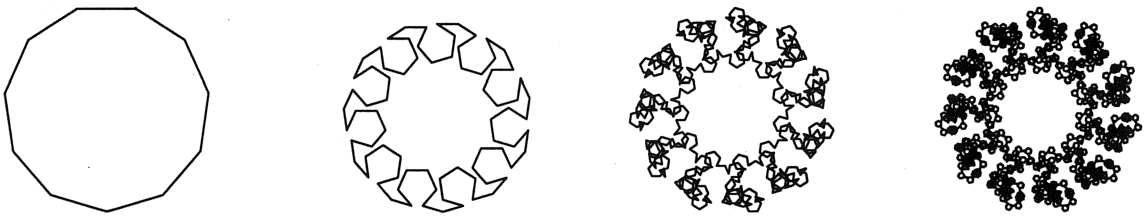
Initiator: nonagon Generator Pattern: $\{\mathbf{0}, \mathbf{2}, \mathbf{2}, \mathbf{2}, \mathbf{7}, \mathbf{7}, \mathbf{7}, \mathbf{0}\}$
Snowflake dimension = 1.8692



Initiator: decagon Generator Pattern: $\{\mathbf{0}, -\mathbf{9}, -\mathbf{7}, -\mathbf{5}, -\mathbf{3}, -\mathbf{1}, \mathbf{0}\}$
Snowflake dimension = 2



Initiator: 11–gon Generator Pattern: $\{\mathbf{0}, -\mathbf{9}, -\mathbf{7}, -\mathbf{5}, -\mathbf{3}, -\mathbf{1}, \mathbf{0}\}$
Snowflake dimension = 2



References

[1] B. Mandelbrot, *The Fractal Geometry of Nature*, Freeman, 1982.

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