BRIDGES
Mathematical Connections
in Art, Music, and Science

Polyhedral Models in Group Theory and Graph Theory

Raymond F. Tennant
Department of Mathematics, Statistics, and Computer Science
Eastern Kentucky University
Richmond, KY 40475
Email: tennant@acs.eku.edu

Abstract

A longstanding method for understanding concepts in mathematics involves the creation of two or three-dimensional images which describe a particular mathematical idea. From our earliest learning experiences, we are taught mathematics by appealing to our strong visual and tactile intuition. For students studying mathematics at the college or university level, the use of polyhedral models and graph theoretic constructions may be a valuable tool for gaining insight into abstract areas such as group theory and topology.

This investigation focuses on the use of Platonic and Archimedian solids to describe ideas in abstract algebra and to understand the concepts such as duality and symmetry subgroup. The reasoning behind several proofs of Euler s Formula are explored with the use of models. For the most part, planar graphs of polyhedra are used in place of actual three-dimensional models. This has the advantage of allowing for all of the vertices, edges, and faces to be viewed at the same time.

Planar Graphs

The notion of graph in graph theory is simply a diagram consisting of points, called vertices, joined together by lines, called edges. Each edge joins exactly two vertices or in the case of a loop, joins one vertex to itself. A graph is called planar if it can be drawn in the plane in such a way so that no two edges meet each other except at a vertex to which they both connect. The regions bounded edges are called faces.

Each of the Platonic Solids may be expressed as drawn as planar graphs. The faces of the polyhedron correspond to the faces of the graph including the unbounded region, called the face at infinity. In fact, any convex polyhedron may be drawn as a planar graph. To visualize this, imagine a convex polyhedron with glass faces. Placing your eye close to one of the faces and peering inside will give you a clear vision of all of the vertices and edges of the polyhedron. This image you see could be projected onto a planar graph.

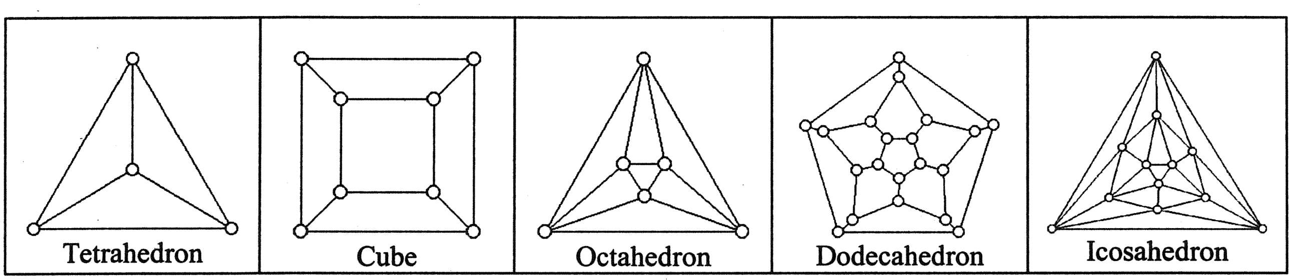


Figure 1

Duality

The concept of duality of polyhedra may be understood through a sequence of graphs. If P is a connected planar graph and P^* then the dual graph of P, call it P^*, can be constructed from P in the following manner. First, choose one point inside each face, including the face at infinity, of the planar drawing of P. These points are the vertices of P^*. Next, for each edge of P, draw a line connecting the vertices of P^* which lie on each side of the edge. These new lines are the edges of P^*. That a tetrahedron is dual to itself is seen in the graphs below.

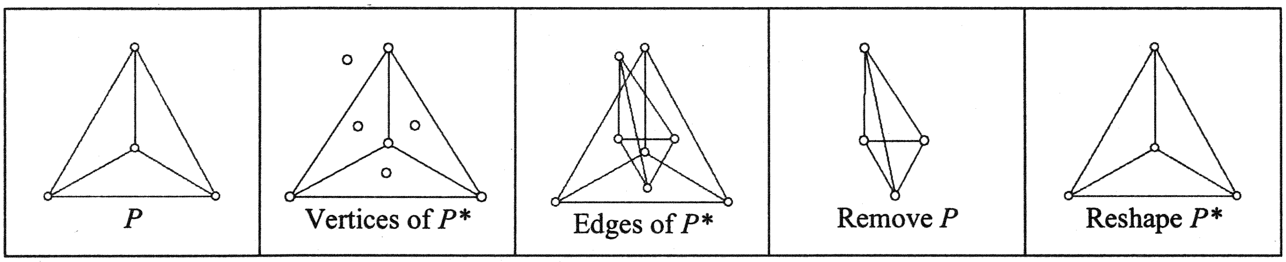


Figure 2

The process works for any convex polyhedron as is shown in the case of the truncated cube.

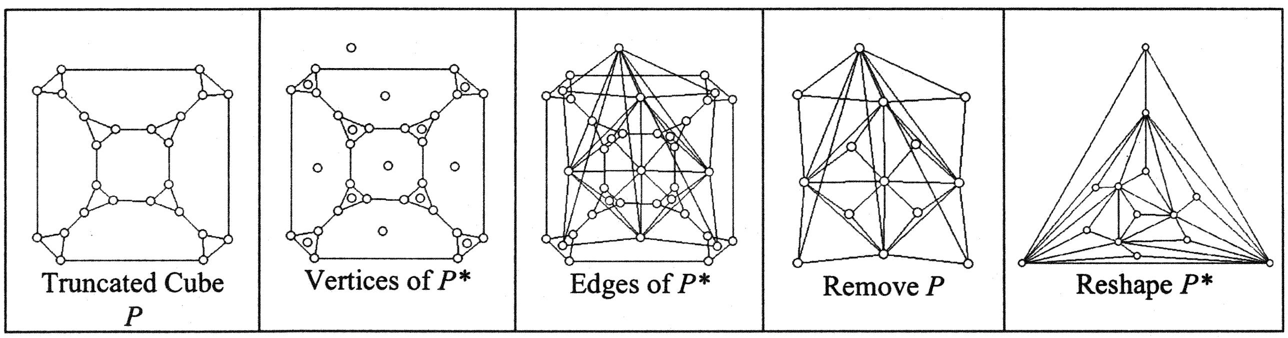


Figure 3

Symmetry Groups

Definition A symmetry of a polyhedral model is a rotation or reflection, which transforms the model so that it appears unchanged. The rotational symmetries along with the identity transformation form the symmetry group of rotations of a polyhedral model.

A number of classical groups may be represented as symmetry groups of rotations of polyhedra. A cyclic group of order n may be represented by a pyramid with a regular n-sided polygon for a base. A dihedral group with $2n$ elements may be represented by a prism or antiprism with n-sided regular polygons.

The alternating groups, A_4 and A_5, and the symmetric group, S_4, may be viewed as the symmetry group of a tetrahedron, icosahedron (or its dual, the dodecahedron), and octahedron (or its dual, the cube), respectively. Historically, the groups A_4, S_4, and A_5 were referred to as the tetrahedral, octahedral, and icosahedral groups.

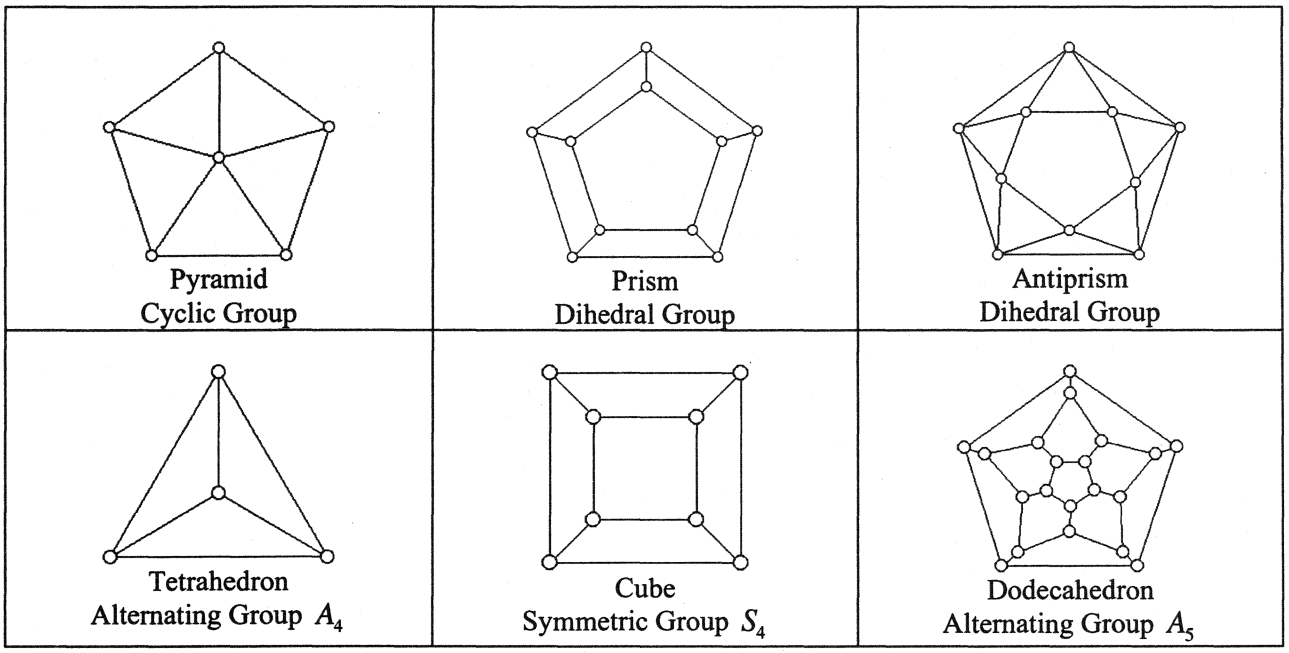


Figure 4

It turns out that Z_n, and, D_n, for any positive integer n, along with A_4, S_4, and A_5 make up a complete list of the groups which can be described as the rotational symmetry group of a convex polyhedron.

Sylow p-Subgroups of Symmetry Groups

Subgroups of symmetry groups may be described by looking at substructures of the polyhedron. For example, the group of rotations of the dodecahedron is A_5, group with $60 = 2^2 \cdot 3 \cdot 5$ elements. By this factorization, we see that A_5 has Sylow p-subgroups for p = 2, 3, and 5. Each of these subgroups may be thought of as acting on a substructure of the dodecahedron.

Definition Let G be a finite group and let p be a prime which divides $|G|$. If p^k divides $|G|$ and p^{k+1} does not divide $|G|$ then any subgroup of G of order p^k is called a Sylow p-subgroup of G.

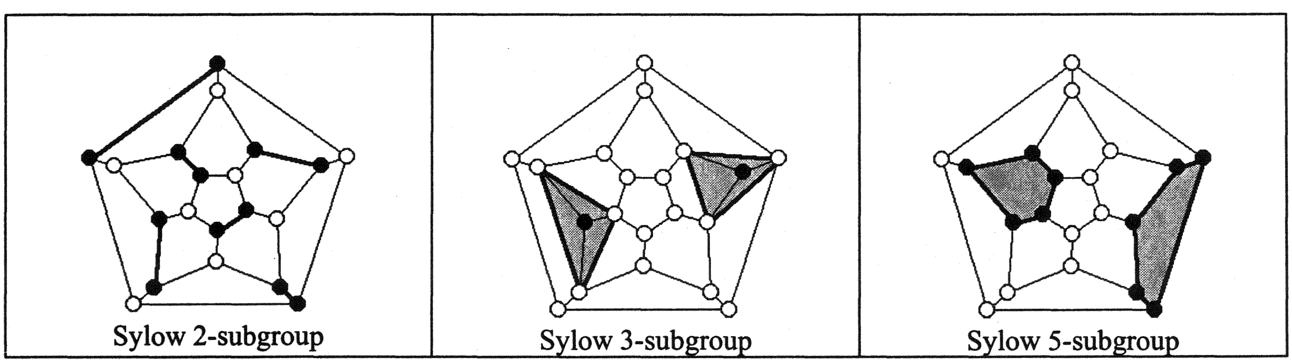


Figure 5

The Sylow 2-subgroup may be seen as acting on 6 edges lying in 3 perpendicular planes. The Sylow 3-subgroup acts on two vertices, which are diametrically opposed. The Sylow 5-subgroup acts on two faces, which lie in parallel planes.

By counting the rotational axes of various orders, it is possible to determine the numbers Sylow p-subgroups.

p	number of axes of order p on a dodecahedron	number of elements of order p in A_5	number of elements in a Sylow p-subgroup	number of Sylow p-subgroups
2	$\frac{e}{2}=\frac{30}{2}=15$	$15\cdot 1=15$ *	4	$\frac{15}{4-1}=5$
3	$\frac{v}{2}=\frac{20}{2}=10$	$10\cdot 2=20$	3	$\frac{20}{3-1}=10$
5	$\frac{f}{2}=\frac{12}{2}=6$	$6\cdot 4=24$	5	$\frac{24}{5-1}=6$

Figure 6

* Note that A_5 has no elements of order 4 since this would be represented by an odd permutation. This means the Sylow 2-subgroups are isomorphic to $Z_2 \oplus Z_2$.

Students may easily verify that this calculation is correct by applying Sylow s Third Theorem.

Sylow s Third Theorem

Let n_p denote the number of Sylow p-subgroups.

Then $n_p \equiv 1 \pmod{p}$ and n_p divides $|G|$.

Orbit-Stabilizer Theorem

Creating new regular polyhedra from old ones by truncating or stellating may result in the new polyhedron having the same rotational symmetry as the original. For example, consider the icosahedron and the truncated icosahedron.

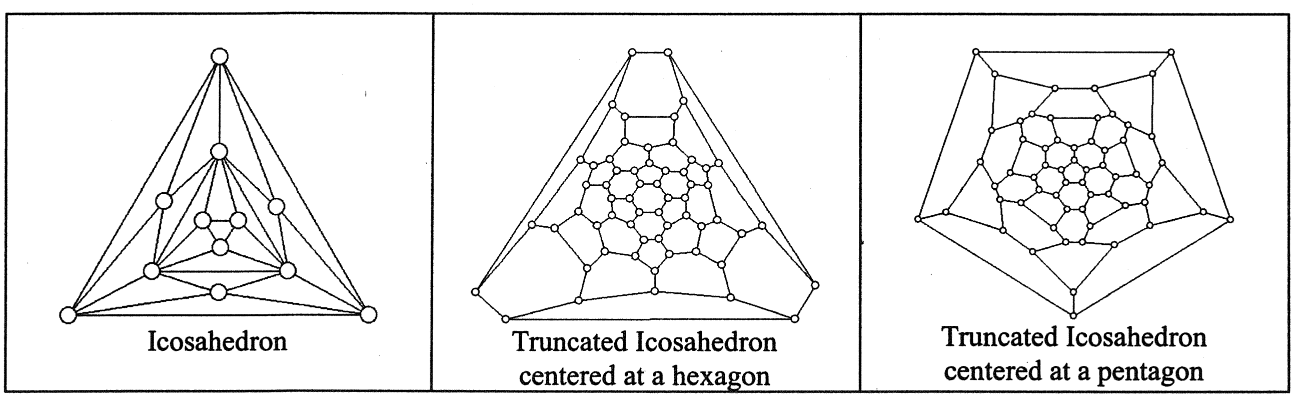
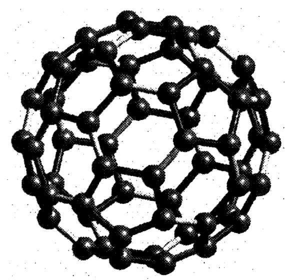


Figure 7

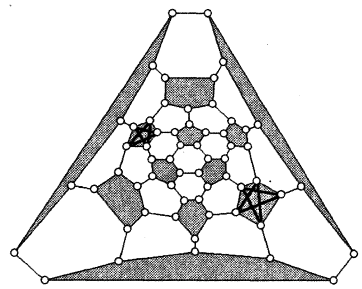
A useful tool for determining the size of a finite symmetry group involves looking at orbits and stabilizers as the symmetry group acts on the polyhedron. The group of rotations of a polyhedron may be thought of as permuting around some geometric set of the polyhedron. We say the group is acting on the vertices, edges, faces, or some other set of components.

Definition Let G be a group of rotations acting on the set I of components of a polyhedron.
For each $i \in I$, the orbit of i under G is defined by $orb_G(i) = \{\varphi(i): \quad \varphi \in G\}$.
For each $i \in I$, the stabilizer of i in G is defined by $stab_G(i) = \{\varphi \in G: \quad \varphi(i) = i\}$.

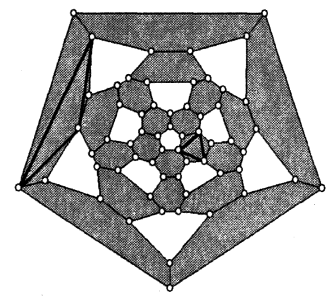
Orbit-Stabilizer Theorem
Let G be the group of rotations acting on the set I of components of your model. For any $i \in I$, $|G| = |orb_G(i)|\,|stab_G(i)|$.



3-Dimensional truncated icosahedron Also known as a Buckyball, or a soccerball



Acting on faces
i = a pentagonal face with a star
orbit(i) = 12 pentagons
stabilizer(i) = 5 rotations



Acting on faces
i = a hexagonal face with a triangle
orbit(i) = 20 hexagons
stabilizer(i) = 3 rotations

Figure 8

Whether viewing from pentagonal faces or hexagonal faces, the Orbit-Stabilizer Theorem gives us that there are $12 \cdot 5 = 20 \cdot 3 = 60$ elements in the rotational symmetry group of the truncated icosahedron. Since no new rotations have been introduced in truncating, the rotational symmetry group is A_5, same as the icosahedron.

Proofs of Euler s Formula

Euler s Formula (for Convex Polyhedra)
Let P be a convex polyhedron, and let v, e, and f denote, respectively, the numbers of vertices, edges, and faces of P.
Then $v - e + f = 2$.

Euler s first strategy for a proof (c. 1751) of his formula involved starting with a convex polyhedron and removing a vertex along with all of the edges and faces, which adjoin it. New triangle faces are added over the hole that has been created. With each step in this process, the value for $v - e + f$ stays the same. The desired result is to continue the process until a tetrahedron is reached and since $v - e + f = 2$ for a tetrahedron, then this must be the case for the original polyhedron. This process has a flaw in that at a given stage you may not be left with a polyhedron.

In 1813, Cauchy gave a proof of Euler s formula, which involved projecting a convex polyhedron onto the plane in the manner used in this discussion. He argued that the value of $v-e+f$ is the same for both the original polyhedron and its projection in the plane. Further, it is possible to add edges to the planar graph so that all the faces are triangles and $v-e+f$ remains the same. Finally, he showed that $v-e+f=2$ for the planar graph with triangle faces.

This method is exhibited for the icosahedron below.

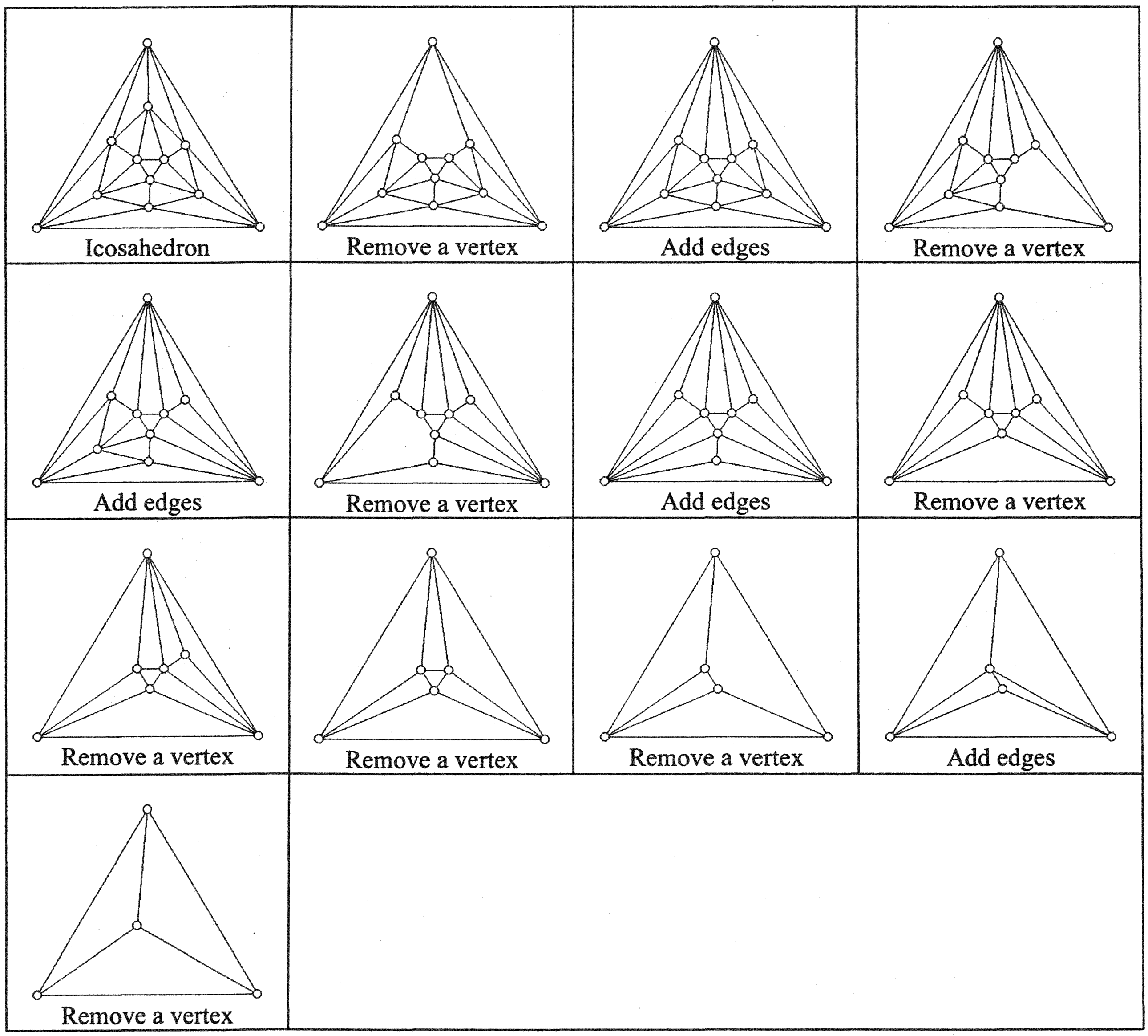


Figure 9

A novel proof of Euler s Formula, which is suitable for middle school students, describes a planar graph whose edges are dams, vertices are posts holding the dams together, the bounded faces are dry chambers and where the face at infinity is the ocean. The idea is to remove a dam (edge) so that the ocean rushes in and fills a chamber (face) with water. In this fashion, one dam (edge) is removed and one chamber (face) is flooded so that $v-e+f$ remains the same. We continue removing one dam (edge) so one chamber (face) is flooded until we have the ocean filling all of the chambers. At this final stage, we have all v posts (vertices) intact and 1 ocean (face). By this process, the posts have stayed connected by dams so there are v-1 dams (edges).

Since $v-e+f$ has stayed the same, we have $v-e+f=v-(v-1)+1=2$.

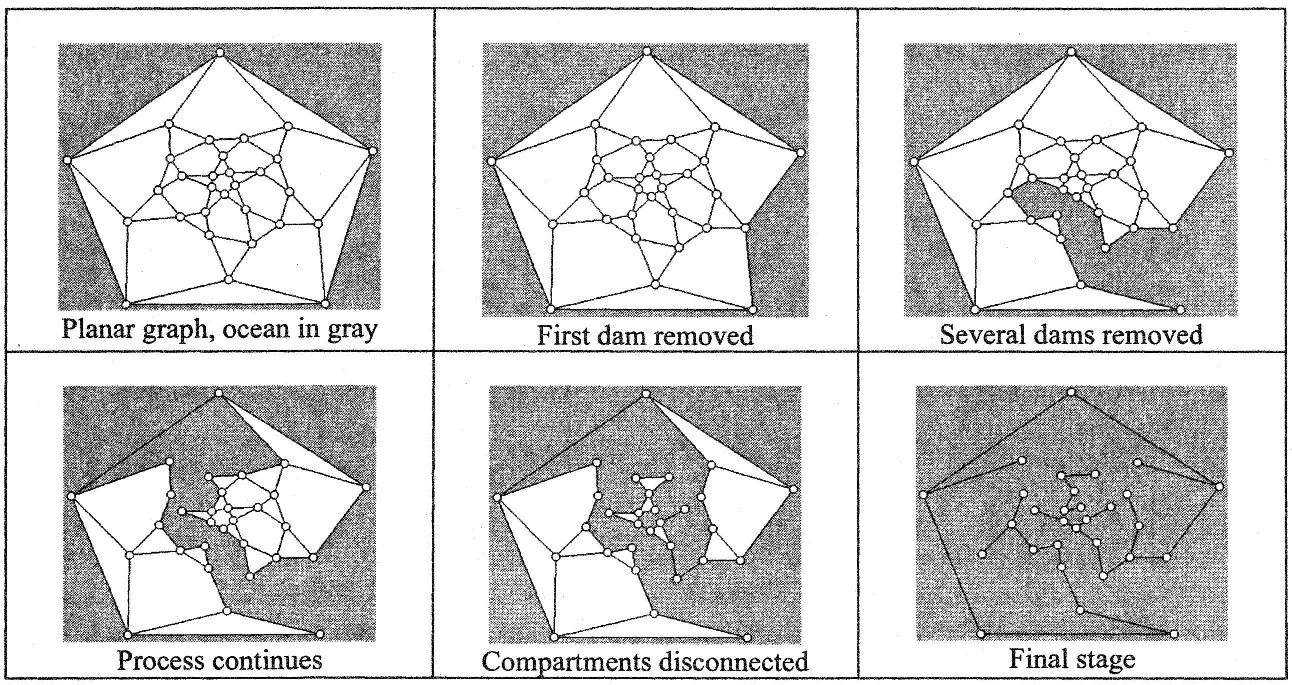


Figure 10

One final note is in order. Verifying that a property holds for a particular polyhedron obviously is no guarantee that the property holds for all polyhedra. Also, a student s verification that a conjecture holds with models should not replace a rigorous proof but rather help in gaining confidence that a conjecture is true while the student struggles with a proof.

Questions for Students to Ask when Testing a Conjecture with a Model

1. Is the model on which I am focused represent the properties of all or most of the polyhedra in the category of the conjecture?

2. Can the steps involved in verifying the conjecture is true for the model be translated into logical steps in a rigorous proof for all polyhedra in a particular category?

References

[1] Peter R. Cromwell, *Polyhedra*, Cambridge University Press, 1997.

[2] Joseph A. Gallian, *Contemporary Abstract Algebra*, Houghton Mifflin College Publishers, 4th edition, 1998.

[3] Robin J. Wilson, John J. Watkins, *Graphs: An Introductory Approach*, John Wiley and Sons, 1990.