Bridges
Mathematical Connections
in Art, Music, and Science

From the Circle to the Icosahedron

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Abstract

The following exercise is based on experiments conducted in circular Origami. This type of paper folding allows for a completely different geometry than the square type since it lends itself very easily to the creation of shapes based on 30-60-90 degree angles. This allows for experimentation with shapes made up of equilateral triangles such as deltahedra. The results of this research were used in Annenberg sponsored activities conducted in a progressive middle school in Houston TX, as well as a workshop presented at the 1999 Bridges Conference in Winfield, KS. Not including preparatory and follow up work by the teacher, the activities in Houston were composed of two main parts, the collaborative construction of a three-yard-across, eighty-faced regular deltahedron (the Endo-Pentakis Icosi-dodecahedron) and the following exercise. The barn-raising was presented last year in Winfield, and the paper folding is the topic of this paper.

1.Introduction

Circular Origami introduces a whole different set of possibilities from the traditional square sort. The geometry of the circle allows for the creation of a convenient regular triangular grid using a few simple steps repeated around the center. These steps are all based on the properties of the circle and of the grid itself. In the first part of the exercise, the lines of the grid will be folded. Later, the grid will be used to move from the 6-triangles-per-vertex flat plane geometry to the 5-triangles-per-vertex geometry of the regular icosahedron. Using this method to create an icosahedron gives insight into its properties by showing how it can grow out of the flat plane in a simple progression. Later, it would be possible to expand this method to other polyhedra made of a single kind of faces (the cube, the tetrahedron, the octahedron and other deltahedra).

2. Preparing the folds.

In the first part of the construction, the disk is folded into a triangular grid by making use of the specific geometry of the circle. All folds are on the same side.

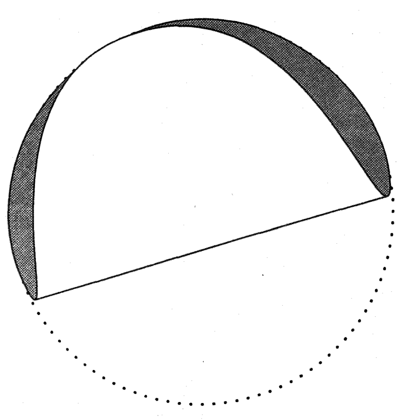

Figure 1: Fold the circle in half by bringing together the opposite edges

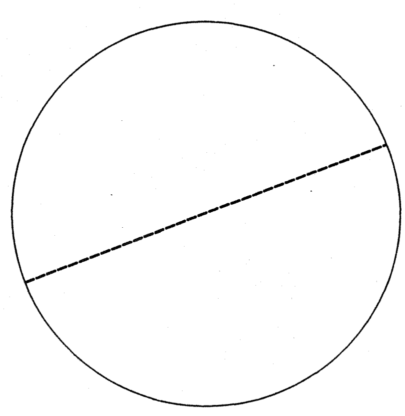

Figure 2: Unfold the paper

The first fold, because it can be made anywhere along the circle, shows the infinite rotational symmetry of the circle. Because of this property, the circle can of course be subdivided into any number of sections, including 2, 3, 4, 6, and 12 which are the ones needed for this exercise. A further advantage of the circle will be exploited here: the fact that the radius of a circle subdivides it into exactly 6 arcs comes into play in steps 5 and 6 (in trigonometric terms, the fact that *sin*(/6) = ½ allows for the subdivision of the circle into 6 without the use of measuring instruments).

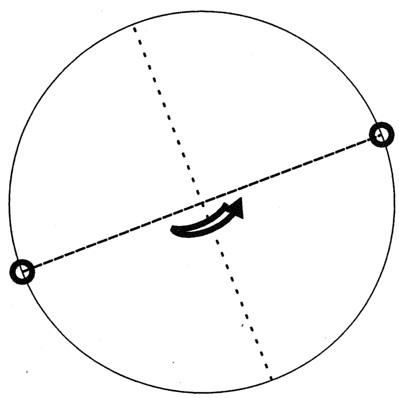

Figure 3: Fold the perpendicular by bringing together the ends of the first fold—unfold

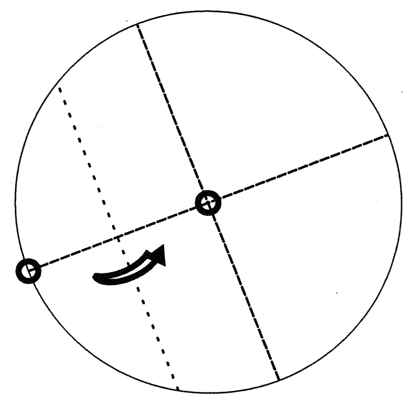

Figure 4: Fold one extremity of the first fold towards the middle as shown—unfold

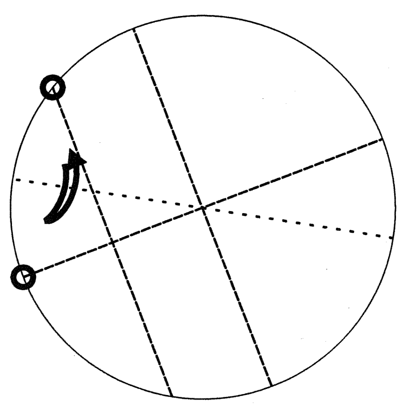

Figure 5: Fold one end of the last fold towards the nearest end of the perpendicular fold—unfold

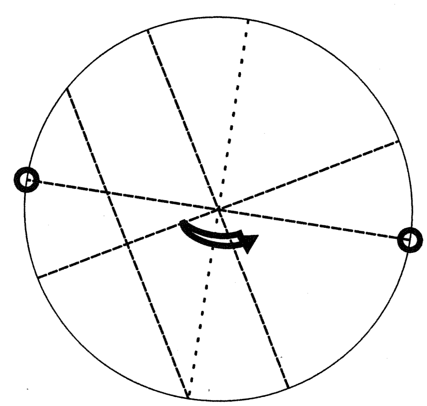

Figure 6: Fold the perpendicular of the last fold by bringing its ends together—unfold

The fold of step 4 not only subdivides the perpendicular radius in half, but also subdivides the quarter arcs into 2/3 and 1/3 on either side. This introduces the factor 3 into the subdivisions of the entire circle. In step 5, the new points are used to begin transferring this 30° angle all around. When the exercise was performed with the children, the teacher pointed out angles of 90°, 60° and 30° as they appeared, as well as equilateral and 30-60-

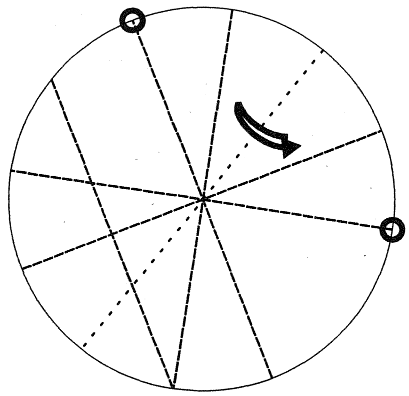

Figure 7: Fold the bisector as shown—unfold

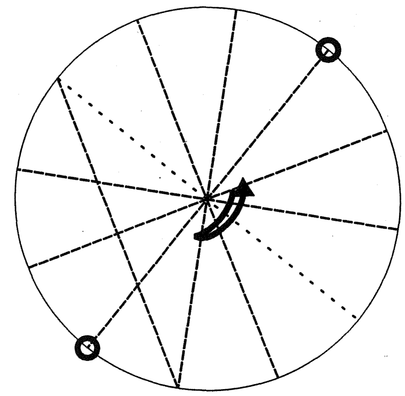

Figure 8: Fold the perpendicular of the last fold—unfold

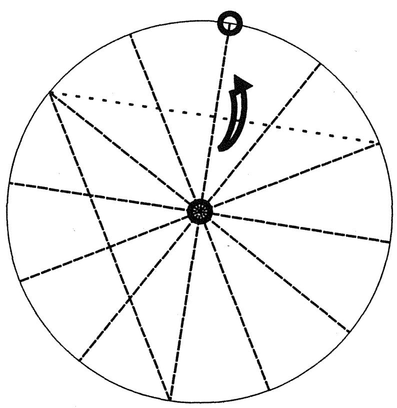

Figure 9: Fold in the end of the fold shown —unfold

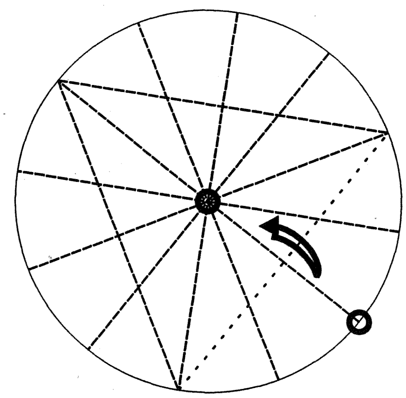

Figure 10: Fold in the last side of the inscribed triangle as shown—unfold

In step 9, we build the grid itself. This begins with an inscribed triangle started in step 4 (where it served to subdivide the circle) and ending in step 10. Note here not only the presence of several 30-60-90 triangles but also the reflectional and 3-fold rotational symmetries. Then a star of David is folded using the existing inscribed triangle. Note the new 6-fold rotational symmetries. This is a good time to stop and open a discussion with the children about visible symmetries. Finally, in steps 12 and 13, the smaller grid is folded, giving us enough triangles to construct the icosahedron.

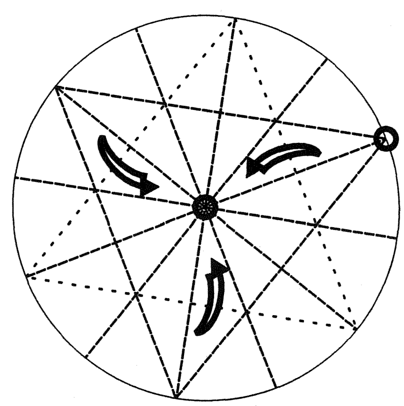

Figure 11: Fold all three points of the inscribed triangle towards the center—unfold

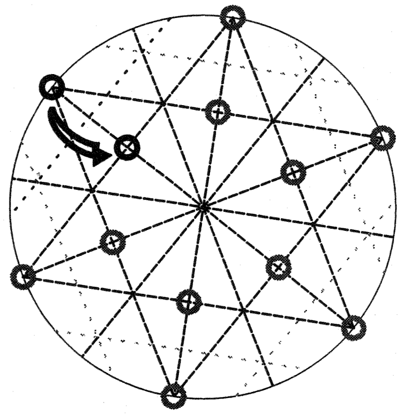

Figure 12: Fold all six vertices of the two inscribed triangles to the middle of the closest edge of the other triangle as shown—unfold

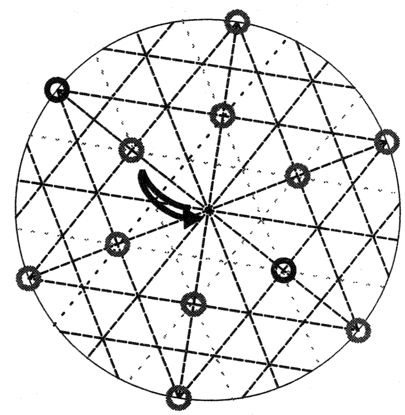

Figure 13: Fold all six vertices of the two inscribed triangles to the middle of the opposite edge of the same triangle as shown—unfold

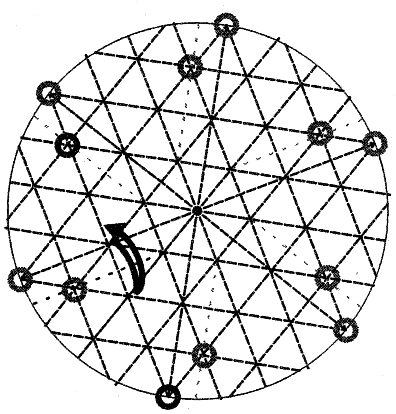

Figure 14: Fold six partial folds as shown—unfold

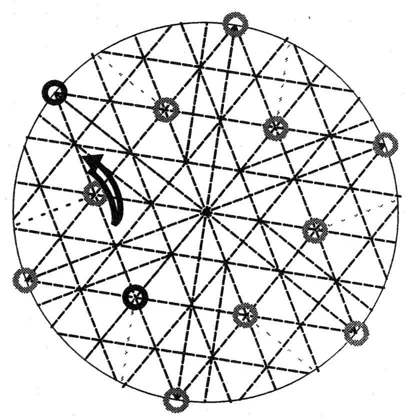

Figure 15: Fold six partial folds as shown —unfold

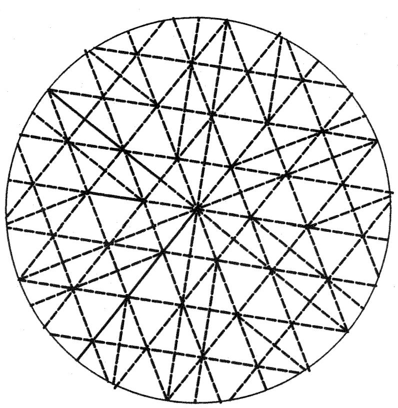

Figure 16: At this point your paper should look like this

An advantages of this method is that children can create their own regular triangular grid without measuring instruments (including a compass), and without difficulty (the most challenging fold is in step 4!) Once the grid is ready, any deltahedron can be constructed, provided there are enough triangles in the circle. See polyhedron being built up from the plane

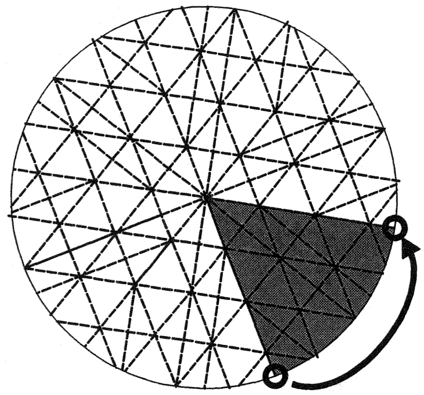

Figure 17: Join up two ends of radii as shown tuck the extra material under

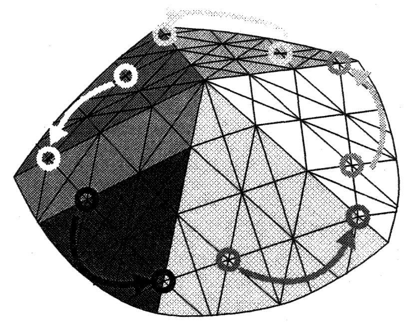

Figure 18: Join five pairs of points as shown tucking the material under

3. Cutting out the net

After step18, you would normally repeat that same procedure again for the third row of 5-vertices, but the whole process becomes complicated by the fact that there begins to be too much material tucked in. At this stage, it becomes necessary to remove some of the excess material that would otherwise get tucked under. To this end, it is best to flatten out the circle again and make a few strategic cuts. Although this is counter to traditional Origami practice, it is necessary for the successful closing of the shape. Furthermore, it will help emphasize some of the properties of the icosahedron and its relationship with the net. As we know, an icosahedron is composed of exclusively 5-vertices. So the first cut needs to reduce a 6-vertex to a 5-vertex. In an ideal world, starting in the middle, we would therefore cut a 60° wedge into the circle (represented by the shaded are of figure 19). This way, we are left with only 5 triangles at the first vertex. For practical purposes, however, we will only cut out half of that wedge so as to keep some material as a tab. Repeating the exercise for the next 5 vertices (the first row around the center), we get 6 new cuts as shown in figure 20 (there are 6 cuts because 2 of them will in fact overlap when the shape is assembled).

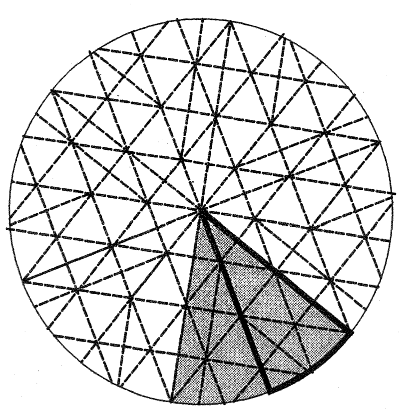

Figure 19: 60° wedge and cutting line

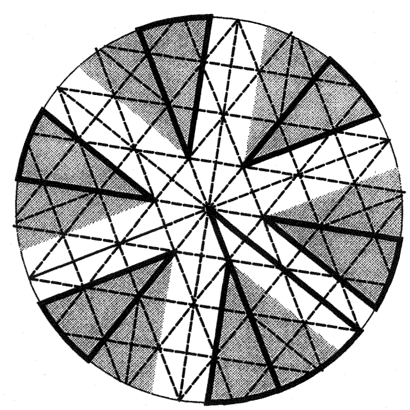

Figure 20: the second row of wedges and cuts

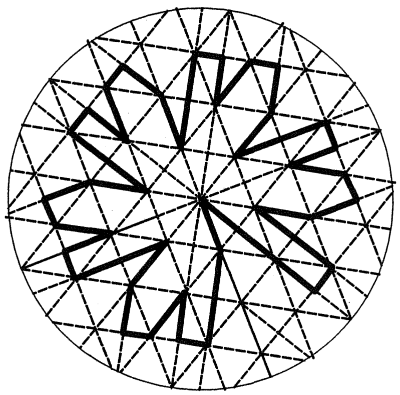

Figure 21: The cutout shape

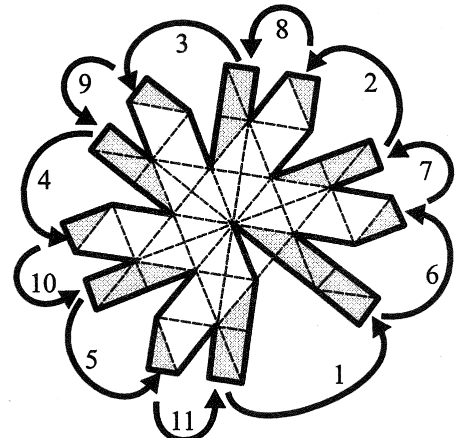

Figure 22: Assembling the icosahedron using the tabs.

Continuing the same process will yield the cutout of figure 21 (some additional cust have to be made due to overlapping areas). It is interesting to note that the cutting exercise itself emphasizes the shape of the icosahedron: the cuts are performed in sets of 1, then 5 (6 with the overlap), then 5, just as the vertices on the icosahedron are, starting with a vertex, in a sequence of 1, then 5, then 5 and then 1 (the last vertex doesn't have a corresponding cut because it is the point of joining of the icosahedron).

Experience has shown that the best way to assemble the shape is using 'blue tack' to stick the tabs to the backs of the triangles. This is particularly effective if the children are going to disassemble and reassemble the net into the deltahedron for further exercises and to experience the process.

4. Conclusion

This exercise is great for demonstrating the symmetry of the icosahedron based on the structure of the net. In the flat state, all the vertices except the last are visible, clearly showing their position relative to the first one. In fact, the net clearly shows how the polyhedron emerges as well as its internal structure. This particular net has such a close relationship to the closed shape that it was used in the Houston Middle School event, after the children had done the paper folding activity, to plan how to color an icosahedron regularly with 5 colors [Morgan, 2000]. They then used their nets to build giant icosahedra using the modular elements made for the endo-pentakis icosi-dodecahedron [Morgan 2000].

The children achieved a high level of intuitive understanding of the icosahedron based on this exercise. First of all, they could see exactly what happens when a 6-vertex (flat) becomes a 5-vertex (pentagonal-pyramidal) by removing a 60° wedge (1 triangle). This step can then be repeated at every vertex, creating an algorithm for building the shape, which coincidentally does close up! Secondly, the fact that the vertices occur in sets of 1, 5, 5,and 1 show the 5-fold symmetry of the whole shape which holds true no matter which vertex you start with.

At Lanier Middle School, the entire exercise (not including the regular coloring of the icosahedron) was completed in one 90 minute session. However, the process can very easily be separated into 2 components, stopping the first before the cutting of the net. The teacher can then concentrate on 2-dimensional geometry in the first session and 3-dimnesional geometry in the second.

Results attained so far in schools justify large scale implementation of the use of these materials and activities and research into their effectiveness both in terms of geometry teaching and of motivating under-achievers to perform better in mathematics. For school classrooms, we recommend having a central project for each area which has a set of triangles and one or two people to go out to schools to use them with children. For museum displays, we recommend having shapes you can get inside and organized shape building activities for visiting school groups. My partner, Simon Morgan, and I have been developing and implementing lesson plans and exercises both using the circular Origami exercises and the giant triangles. Simon is presenting some results in his paper in the same procedings.

References

[Morgan, 2000] Simon Morgan, *Polyhedra, Learning by Building: Designing a Math-Ed Tool.* Bridges 2000 proceedings.

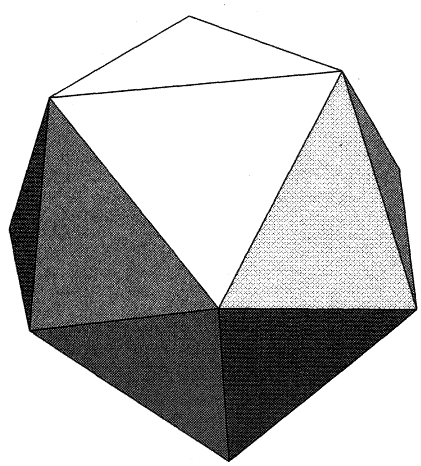

Figure 23: The assembled icosahedron