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On Musical Space and Combinatorics: Historical and Conceptual Perspectives in Music Theory *

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Abstract

Music theory has enjoyed a ubiquitous association with mathematics from its earliest beginnings. Among the abundant mathematical models that have inspired theories of music, combinatorics has played a continuous role since the seventeenth century, articulating musical spaces in which relations between elements of a discrete system can be articulated and quantified. This paper explores some representative theories of musical relations expressed in spatial terms using combinatorial techniques, revealing the abstract and profound ways in which mathematics, specifically combinatorics, informs music theory.

1. Introduction

"Nowhere do mathematics, natural sciences and philosophy permeate one another so intimately as in the problem of space" [1]. In this statement, Hermann Weyl depicts space as the primary locus of interconnection among three enormous domains of intellectual and scientific knowledge and endeavor. Weyl's designation of space as a problem reflects the difficulty of articulating the ontology of space, since the concept of space embraces both concrete and abstract or metaphorical meanings. Space may be broadly defined as an extent in which objects or phenomena exist in relative positions, relations, or distances from one another. This expansive definition applies both to concrete spaces in which physical objects or beings reside, as well as to abstract or metaphorical spaces constructed in the human imagination, in which reside concepts, ideas, and abstractions of physical objects or phenomena.

While concrete space is profoundly important to the study of musical acoustics and cognition, which bear close ties with music theory, it is primarily in the realm of compositional and speculative music theory that metaphorical space as a component or product of rationalism emerges as intrinsic to the discipline. This paper explores the pervasiveness of abstract or metaphorical space in music theory through the frame of the mathematical subfield of combinatorics, the study of enumeration, groupings, and arrangements of elements in discrete systems. This brief study begins at the time of the incursion of mathematical combinatorics into music theory in the seventeenth century, almost immediately upon its debut in mathematical discourse, and surveys combinatorial techniques and attitudes in some seventeenth- and eighteenth-century writings on musical composition. From there the discussion turns to two little known theorists of the late nineteenth and early twentieth century, whose theories inductively develop algorithms from combinatorial techniques within the discrete, modular system of twelve pitch classes. These writings demonstrate an important shift in music-theoretical thought, in which combinatorial techniques, applied in the seventeenth and eighteenth centuries to surface musical relations and

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configurations, later become hidden from view, and applied not to surface configurations, but to the elements from which those configurations—melodies and chords—are made. The paper concludes with some reflections on the place of combinatorics and the significance of metaphorical space in contemporary music theory.

2. Marin Mersenne's *Harmonie universelle*

Numerical models and rational systems of classification formed the infrastructure of speculative music theory from its earliest Pythagorean and Platonic sources to the Renaissance, reflecting the status of music as one of the four sciences of the medieval quadrivium (along with arithmetic, geometry, and astronomy). In the seventeenth century, the Scientific Revolution and the efflorescence of new branches of mathematics, especially probability theory and analytic geometry, irrevocably altered the nature of the interface between mathematics and music theory. Probably the earliest conspicuous manifestation of the new mathematical models in music theory was the adoption by renowned mathematician and music theorist Marin Mersenne (1588-1648) of mathematical processes informed by combinatorics in his landmark treatise, *Harmonie universelle* (1636-37). In the section on melody ("Livre second de chants" or "Book Two on Melodies"), he applied now classic combinatorial formulas to perform such operations as tabulating all 720 (= 6!) permutations of a hexachord (a collection of six notes). A portion of Mersenne's complete table, which occupies twelve pages of text, is shown in Figure 1. Mersenne also recorded the number of permutations of from 1 to 22 discrete pitches in the three-octave range of the diatonic gamut of his time (the last resulting in a colossal number of 22 digits), and performed more complex calculations such as the partitions of 22 or the multisets within the 22-pitch gamut comprising melodic figures that included note repetitions [2].

Mersenne's adoption of combinatorial methods reflected his mathematical expertise, and was motivated by his desire to demonstrate objectively the vast, yet computable number of possibilities for melodic construction. As a composer and musician, he found combinatorial means of explaining the rejection of some permutational possibilities on aesthetic grounds; for example, the melodic interval of the major sixth was regarded as improper, both ascending and descending, so among the 720 permutations of the major hexachord, he calculated that 1/3 (= 240/720) were syntactically incorrect, because they included the notes *ut* (C) and *la* (A) in succession. That is to say, using combinatorics, he found a means to quantify with precision fundamental principles of melodic syntax.

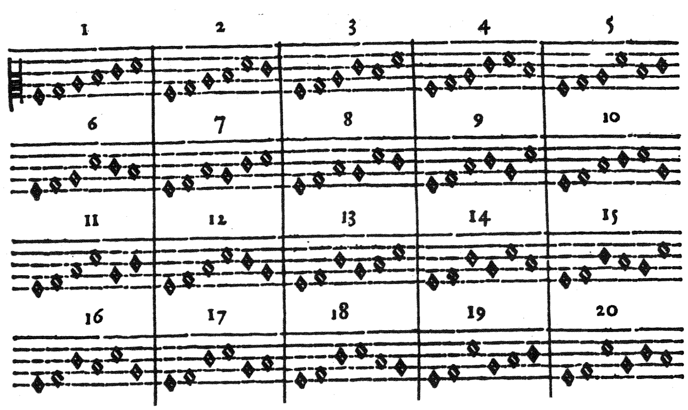

Figure 1: tabulation of the 720 permutations of 6 notes, excerpt (Mersenne, *Harmonie universelle*, 1636-37)

3. *Ars combinatoria*

The view of musical materials as a finite set of elements from which combinations are selected inspired a number of treatises and practical manuals on rational methods of musical composition in the later seventeenth and eighteenth centuries. Figure 2 illustrates an example of the 24 (=4!) permutations of a four-note melodic-rhythmic figure from Joseph Riepel's 1755 treatise *Grundregeln zur Tonordnung* [3]. Allusions to or explicit techniques derived from mathematical *ars combinatoria* became a familiar feature in eighteenth-century composition treatises, and were valued for the pedagogical benefits they offered to students of composition. Although the approach is mechanistic and the literary tone of these treatises can be light and diversionary, the underlying serious objective was to stimulate the musical imagination and transmit knowledge and skill in manipulation of musical materials.

Figure 2: melodic-rhythmic permutations (Riepel, *Grundregeln zur Tonordnung*, 1755)

4. Musical circles

A more abstract expression of the combinatorial disposition of musical space in eighteenth-century treatises is found in the circular diagrams used to depict relations of proximity and remoteness of keys for modulation within the system of 24 major and minor keys. While differing from each other in substance and presentation, such diagrams evince principles of Cartesian rationalism by identifying spatial relations in terms of relative distance from a referential point. Three "musical circles" from composition treatises of the first half of the eighteenth century are given in Figure 3. Johann David Heinichen was the first to discuss the theoretical implications of the musical circle. In his circular diagram from *Neu erfundene und gründliche Anweisung* (1711), the fifth-related major (*dur*) keys proceed in counterclockwise motion in alternation with the fifth-related minor (*moll*) keys so that each major key is followed (counterclockwise) by its relative minor [4]. Johann Mattheson, in *Kleine General-Bass Schule* (1735), presents a more elaborate pattern of interweaving of the major and minor keys in his musical circle. (The text in the interior of Mattheson's circle reads "improved musical circle, which can lead around more easily through all keys than those previously invented.") Georg Andreas Sorge's multi-layered model of concentric circles in *Vorgemach der musicalischen Composition* (1745-47) separates the fifth-related major and minor keys, enumerating the two underlying circles of twelve fifths, and in the outer circles incorporates both Heinichen's and Mattheson's pairings of relative major and minor keys [5].

(a) (b) (c)

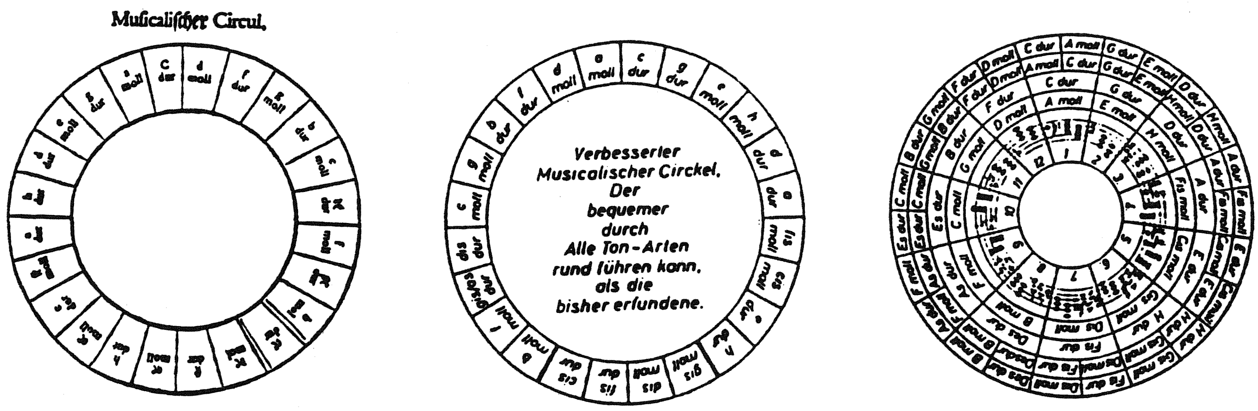

Figure 3: musical circles of (a) Heinichen (1711); (b) Mattheson (1735); (c) Sorge (1747)

5. Modular arithmetic and abstraction

The eighteenth-century musical circles represent abstract relations within a discrete system (of 24 major and minor keys) that was implicitly modular, yet were still intended as a practical aid for musical composition by rendering visible the entire range and quality of relational possibilities for modulation. The formal entry of modular arithmetic into mathematics early in the nineteenth century, in conjunction with the widespread (though not universal) acceptance of equal temperament, in theory if not practice, initiated a line of speculative thought in music theory that extends through the twentieth century and continues to occupy a central position as a component of contemporary music theory. The combined agency of modular arithmetic and equal temperament enabled the formulation of theories of pitch structures based on algebraic methods and a recovery of pure speculation in music theory, that is, theory removed from the immediate concerns of musical practice and style. The twelve pitch classes of the equal-tempered system—in which all pitches are assigned to one of twelve classes based on octave and enharmonic equivalence—are strongly affiliated with theories of atonal and serial music of the twentieth century; their manifestation in the work of earlier authors not implicated in the revolution of harmonic language in the twentieth century reveals important aspects of the generality of abstract musical space.

The earliest proponents of this new line of thought worked independently of each other, and were separated, voluntarily or involuntarily, from the mainstream of music theory and practice of their time. The explicit combinatorial mathematics that was integral to their theories did not conform to the conservatory-based norms and institutions of nineteenth- and early twentieth-century music instruction, so their work has generally not received much attention, and they remain rather obscure even among specialists in the discipline of music theory. Two of these independent thinkers will be discussed here: Anatole Loquin, writing in Bordeaux, France, and Ernst Bacon, writing in Chicago, U.S.A.

5.1 Anatole Loquin (1834-1903). Anatole Loquin, in 1871, described his objective of calculating a complete inventory of all possible combinations of notes—triads and seventh chords, as well as formations resulting from the addition of non-harmonic tones. Loquin first classified all pitches into twelve congruence or equivalence classes (commonly known since the mid-twentieth century as *pitch classes*), using the modular analogy of a circle. (Loquin's concept of the circle was intrinsically different from the eighteenth-century musical circles of key relations discussed earlier, as his represented the twelve pitch classes, devoid of triadic or functional associations.) Treating the twelve pitch classes as a discrete system, he used algebraic methods to determine the number of combinations of pitch classes in all cardinalities (the number of notes in a harmony) from 1 through 12, starting from one fixed, referential note or pitch class. Figure 4 shows Loquin's triangular table of combinations. (This table, as Loquin notes, is a reorientation of Pascal's famous triangle.) The total number of combinations of each cardinality from 1 to 12 is given along the right and bottom edges of the table. Along left edge are indicated the ordinal numbers assigned to the chords, from 1 to 2048 [6].

Accords. Nos			
1.	1	=	1 accord d'une seule note.
2 à 12.	1+ 1+ 1+ 1+ 1+ 1+ 1+ 1+ 1+ 1+ 1	=	11 accords de deux notes.
13 à 67.	10+ 9+ 8+ 7+ 6+ 5+ 4+ 3+ 2+ 1	=	55 accords de trois notes.
68 à 232.	45+ 36+ 28+ 21+ 15+ 10+ 6+ 3+ 1	=	165 accords de quatre notes
233 à 562.	120+ 84+ 56+ 35+ 20+10+ 4+ 1	=	330 accords de cinq notes.
563 à 1024.	210+126+ 70+ 35+15+ 5+ 1	=	462 accords de six notes.
1025 à 1486.	252+126+ 56+21+ 6+ 1	=	462 accords de sept notes.
1487 à 1816.	210+ 84+28+ 7+ 1	=	330 accords de huit notes.
1817 à 1980.	120+36+ 8+ 1	=	165 accords de neuf notes.
1981 à 2035.	45+ 9+ 1	=	55 accords de dix notes.
2036 à 2047.	10+ 1	=	11 accords de onze notes.
2048.	1	=	1 accord de douze notes.

1	11	55	165	330	462	462	330	165	55	11	1	2048 accords.
accords de douze notes.	accords de onze notes.	accords de dix notes.	accords de neuf notes.	accords de huit notes.	accords de sept notes.	accords de six notes.	accords de cinq notes.	accords de quatre notes.	accords de trois notes.	accords de deux notes.	accord d'une seule note.	

Figure 4: table of combinations of pitch classes (Loquin, *Apperçu sur la possibilité d'établir une notation répresentant...les successions harmoniques*, 1871)

Loquin outlined a manual process for computing the figures in the rows and columns of the table; the referential note is assigned the number 1; the number of 2-note combinations, 11, is computed by adding successively each of the remaining notes; the number of 3-note combinations, 55, is computed by adding, to each of the 11 2-note combinations in succession, each of the remaining notes above the highest note of each 2-note combination. The numbers of combinations for the remaining cardinalities are computed in the same way, each row and column of the table comprising an arithmetic progression. By speaking of summing "notes" rather than numbers, and by not providing the algebraic equations to symbolize the calculations of the entries on his table, Loquin's terminology and methodology may seem to lack mathematical rigor, but it is likely that he attempted to simplify the information, suppressing the extent of its mathematical foundation in arithmetic progressions, in order to include mathematically untrained musicians and musical scholars in his readership.

Loquin's total of 2048 combinations is the result of a recursive algorithmic process in which combinations of each cardinality are generated by systematic accretion to the already computed combinations of the next smaller cardinality. His results, although internally consistent, reveal that he was unaware of the group-theoretic basis of the system. For example, his table omits the important cardinality of 0, the empty set (ι), that is required to balance the complementary relationships of the system; the identical totals of combinations in pairs of cardinalities in Loquin's table do not match up with complementary cardinalities within the aggregate of 12. With a little manipulation, however, Loquin's numbers of combinations within each cardinality can be shown to correlate to the correct figures, computed without recourse to a referential pitch class. His grand total of 2048 combinations is exactly half of the total number of pitch-class (pc) sets, 4096 (=2^{12}), in the universe of 12 pitch classes. Figure 5 shows the number of sets in each cardinality from 0 to 12 (the number of combinations of 12 elements taken k at a time, where k is the cardinality) as well as the number of equivalence classes under transposition within each cardinality. The number of sets of each cardinality can be calculated by summing adjacent totals in Loquin's table, counteracting the effect of the referential pitch class. That is, the total of 12 monads (single pitch classes) result from summing Loquin's figures 1 and 11; the total of 66 dyads (2-note sets) results from summing 11 and 55; the total of 220 trichords (3-note sets) results from summing 55 and 165; the 495 tetrachords (4-note sets) from summing 165 and 330; and so on.

Cardinality	0	1	2	3	4	5	6	7	8	9	10	11	12
# of pc sets	1	12	66	220	495	792	924	792	495	220	66	12	1
# of equivalence classes	1	1	6	19	43	66	80	66	43	19	6	1	1

Figure 5: numbers of pitch-class (pc) sets and transpositionally equivalent pc-set classes by cardinality

5.2 Ernst Bacon (1898-1990). Across the Atlantic and a few decades later, Ernst Bacon, a young American piano student in Chicago, similarly employed combinatorial methods within the modular system of 12 pitch classes in an unusual monograph entitled "Our Musical Idiom" (1917) [7]. Bacon bypassed the process of unraveling all possible combinations of notes or pitch classes by conceiving of sets of pitch classes in terms of the intervals separating them. He developed an elegant algorithm that efficiently amalgamates combinations of pitch classes into equivalence classes whose members are all related by transposition. (That is, they contain exactly the same succession of intervals, but begin on a different note or pitch class.) The algorithm consists of three steps: (1) the intervals between successive pitch classes are conceptually arranged as if around the perimeter of a circle marked with 12 equidistant points (the distance between adjacent points representing a semitone) in order to render any harmony in a space smaller than an octave; (2) the interval succession, which must always sum to 12 (including the complementary interval that returns to the point of origin), is recorded; (3) cyclic permutations of interval successions are eliminated, as these represent transpositions—reorderings of the same interval succession. Following these steps, each remaining, unique, permutation of the interval succession represents a harmony, a class of transpositionally equivalent pitch-class sets. Interval successions that share the same terms (but are not cyclic permutations) are grouped together as, for example, the successions <1-1-1-4-5>, <1-1-1-5-4>, <1-1-4-1-5>, and <1-1-5-1-4>, representing four five-note harmonies [8]. Bacon's notation of a representative of each of the four harmonies in this combination is shown in Figure 6. (Note that he omits the complementary interval completing the octave.)

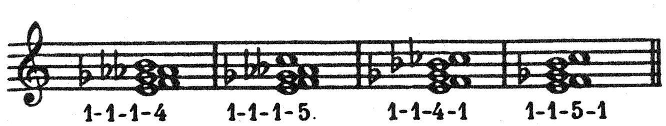

Figure 6: four five-note harmonies formed from non-cyclic permutations of one combination of intervals (Bacon, "Our Musical Idiom," 1917)

Bacon's table showing his computations of the 43 transpositionally equivalent classes of four-note harmonies, or tetrachords, is given in Figure 7. On the left side of the table appears the tabulation of intervals (from 1 to 9 semitones, or from a minor second to a major sixth) comprising each combination.

Just to the right of the middle of the table appears the formula that applies to each combination (depending on such circumstances as whether the component intervals are unique or whether there are repetitions). Finally, the far right column gives the total number of harmonies (unique, non-cyclic permutations) within each combination, and these are shown to sum to 43. Bacon provides analogous tables for all cardinalities from 2 through 10. Although he notes some large-scale symmetries within the system, like Loquin, he does not pursue the group-theoretical implications of his data.

	1	2	3	4	5	6	7	8	9	Calculations of Harmonies	H
1	3								1	H=3!/3!	1
2	2	1						1		H=3!/2!	3
3	2		1				1			"	3
4	2			1		1				"	3
5	2				2					By trial	2
6	1	2					1			3!/2!	3
7	1	1	1			1				3!	6
8	1	1		1	1					"	6
9	1		2		1					3!/2!	3
10	1		1	2						"	3
11		3				1				3!/3!	1
12		2	1		1					3!/2!	3
13		2		2						By trial	2
14		1	2	1						3!/2!	3
15			4							By trial	1
										Total	43

Figure 7: computation of the 43 transpositionally equivalent tetrachord classes (Bacon, "Our Musical Idiom," 1917)

6. Conclusion

Both Loquin and Bacon were driven by an impulse to construct a taxonomy of all possible combinations of notes or pitch classes within a discrete system of 12 objects, even though many of the available combinations were not syntactically acceptable in the compositional practice of their time. Certainly these authors were motivated by the increasing chromaticism of later nineteenth- and early twentieth-century music, which introduced sonorities that were unexplainable in familiar theoretical terms, but their mathematical methods transcended music-stylistic boundaries and theoretical conventions. While Loquin and Bacon are unusual and intriguing for their independence from mainstream currents in music theory, their role in this study is to document the tenacity in music theory of abstract or metaphorical space articulated through combinatorial mathematics.

The atonal revolution in harmonic language in the early twentieth century and the evolution of the twelve-tone or serial method of composition, spearheaded by Arnold Schoenberg and extended by others, brings a new chapter to the study of the increasingly more significant role played by combinatorics in music theory. While the limited scope of this paper makes it impossible to address this new chapter in any depth, it can be said that the compulsion to classify all possible combinations of pitches within the discrete, modular system of twelve pitch classes proved to be much more than a curiosity. A taxonomy of equivalence classes in the universe of 4096 pitch-class sets is entrenched as a foundational component of contemporary atonal music theory, first codified by Allen Forte, and relations and transformations within equivalence classes of pitch-class sets are shown to be firmly grounded in combinatorial mathematics as well as in set theory and group theory [9]. The spatial metaphor maintains a strong presence in music theory in analogies of geometric transformations and transformations of pitch-class sets, in the writings of Robert Morris on compositional and other musical spaces [10], and in the theories of generalized musical intervals and transformational networks of David Lewin [11].

In accord with the Weyl quotation that introduced this essay, space constitutes an important point of interdisciplinary intersection, exemplified vividly in the powerful resource of combinatorics at the crossroads of music theory and mathematics.

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[8] The four harmonies represented by these interval successions can be further grouped into two larger equivalence classes (<1-1-1-4-5> and <1-1-1-5-4>; <1-1-4-1-5> and <1-1-5-1-4>) based on inversional as well as transpositional equivalence.

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