Bridges
Mathematical Connections
in Art, Music, and Science

The Generation of the Cube and the Cube as Generator

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Abstract

In this study we submit for consideration how, in parallel with the different ways of conceiving the generation of a geometric shape, often accompanied by certain arguments, we can encounter similar artistic developments which lead to original results and show the fecundity that scientific advances have for art. For now we will exclusively consider the cube, which is, perhaps after the sphere, one of the most productive shapes. Without intending to exhaust the subject, we shall limit ourselves to highlighting one of the most representative episodes.

Plato in the *Timaeus [1]*

In the Timaeus, Plato refers to the moment "when an essay was being made to order the universe," when "everything... were then first by the creator fashioned forth with forms and numbers." On the basis that "God formed them to the most fair and perfect," he goes about rationalizing "the disposition of each and their generation." As is well known, Plato ends by assigning to each of the elements: fire, air, water and earth (which, by observation, must be distinct from one another) the shapes of regular polyhedrons, since these must be the most beautiful and perfect forms. This is thus a preconceived idea.

Moreover Plato tries in his plausible reasoning to assume the notion that some elements are "capable of being generated out of each other by their dissolution." And this could in fact occur with the three polyhedrons whose faces are equilateral triangles, as long as these forms are considered from the point of view of the surface. But the two remaining polyhedrons have squares or pentagons as their faces. Because of its similarity to the sphere, the dodecahedron ends up being the most apt for the Universe, and the cube can better respond to the stability of the Earth. The choice therefore seems clear, and the cube will be selected to configure this element.

Thus we find that Plato, in order to explain the birth of the fourth element in such a way that it might have something in common with the other three, finds himself forced to subdivide the faces of all the polyhedrons chosen into right triangles. And so upon these he raises his argument in the following manner: "In the first place, that fire and earth and water and air are material bodies is evident to all. Every form of body has depth; and depth must be bounded by plane surfaces. Now every rectilinear plane is composed of triangles."

A triangle that contains a right angle is, in Plato's mind, more perfect than a triangle composed of any other sort of angles. And among these there are two classes: those of equal sides (isosceles) and those of unequal sides, which are infinite. However, the most perfect of the latter group should be those "which two conjoined form an equilateral triangle." And thereby the first type of triangle, the isosceles, shall originate the cube; and from this triangle the faces of the rest of the chosen polyhedrons shall come forth. As we can see, the force of the argument, which seeks to find a common process of formation, just as the idea that what is most regular is always most beautiful, is that which brings Plato to imagine a process of the cube's formation, which, seen from the today's perspective, seems surprising. Consequently, the complete description of the cube will be the following: "The isosceles triangle generated the fourth, combined in sets of four, with the right angles meeting at the centre, thus forming a single square. Six of these squares joined together formed eight solid angles, each produced by three plane right angles: and the shape of the body thus formed was cubical; having six squares planes for its surfaces." The reason for the number four in the triangles that form each face should perhaps be found in that this number harmonizes best with the definition of the square. For according to Plato, the issue that interests him is "what is the form in which each has been created, and by the combination of what numbers." In the cube, according to its description, we will find the numbers four, six, eight and three: four isosceles triangles, six equilateral rectangular figures, and eight solid angles of which each is made up of three flat angles.

In summary, the conception of the cube and the other polyhedrons is for Plato not volumetric but superficial, in this case the isosceles triangles being the generators of the squares that constitute its faces. With respect to the meaning that it has acquired in this process, the cube ends up being formed by the most beautiful of all possible triangles (right isosceles triangles). Furthermore it is the one which enjoys the greatest stability: "To earth let us give the cubical form; for earth is least mobile of the four and most plastic of bodies."[1] The cube has that stability, Plato comments, as much because the quadrilateral surfaces that form it are more stable than triangles, as because of its origin, since isosceles triangles of equal sides are naturally more stable than triangles of unequal sides. This is again a superficial conception and not a volumetric one, although the stability that this volume has with respect to gravity seems to us more evident when it rests on a flat surface. In effect, this is what seemingly takes place: "When earth meets with fire and is dissolved by keenness of it, it would drift about, whether it were dissolved in fire itself, or in being some mass of air or water, until the parts of it meeting and again being united became earth once more; for it never could pass into any other kind." [1] It could not, because there is no other element composed of right isosceles triangles.

Regarding the squares that form its faces and their right angles, we have a significant reference from Proclus: "The Pythagoreans thought that the square more than any other four-sided figure carries the image of the divine nature. It is their favourite figure, indicating immaculate worth, for the rightness of the angles imitates integrity, and the quality of the sides abiding power."[2-3]

After the Timaeus of Plato, the theme of those forms since labeled the "platonic solids" –which include the cube– continued to be treated by philosophers, geometricians, astronomers, artists, etc. throughout the centuries. Among them Archimedes, Piero della Francesca, Luca Paccioli, Durero and Daniel Barbaro are indispensable. By way of again presenting a signification by cosmic attribution, we will only cite the contribution of Kepler (1571-1630). He, in moving from Plato's microscopic scale to the astronomic, attributes the regular polyhedrons to the planets, also providing himself with a rationalization.

Juan de Herrera in his *Discourse on the Cubic Figure [4]*

Based on his familiarity with Euclid, Juan de Herrera writes about the cube with the intent to illustrate the teachings of Ramon Llull. He considers that the cubic figure opens the door to "great and high mysteries and secrets which are difficult to comprehend," being the "root and foundation of so-called Llullian art." What interests the architect about the cube is in this case a process, an operation which is carried out in it, thereby having a dynamic conception of the figure, which, being natural, can also accompany rational discourse. As for the geometric figure, Herrera defines the cube, again superficially, as formed by six square surfaces, which he draws on a plane in the form of a Latin cross. He describes how these surfaces go about raising themselves perpendicularly or at right angles until the final surface comes into place, closing the cube(fig.1).

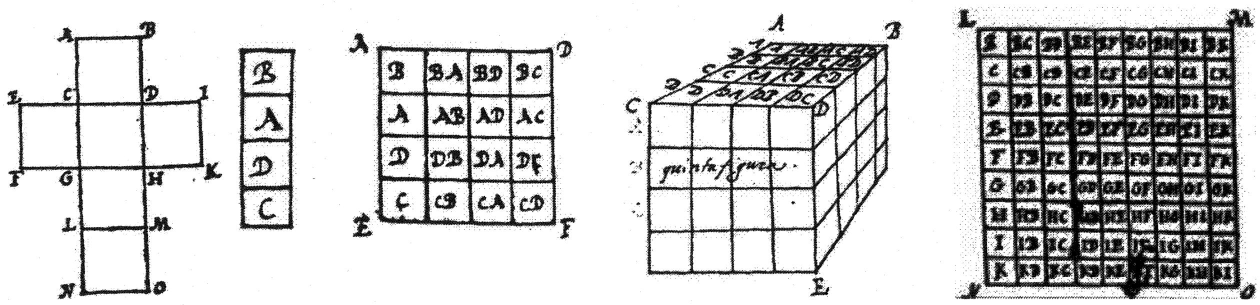

Figure 1: *Herreras' drawings*

But this is not the process which will serve him in his discourse, rather another, derived from the Greek conception of numbers: lineal, superficial or planar, and solid. The lineals have no other origin than themselves. Superficial numbers are those which proceed from two multiplied numbers, while the solid numbers come from three numbers multiplied together. If the two numbers that produce a superficial number are equal, a squared number is the result. If the three numbers that produce a solid number are equal, a cubed number results. Thus, Herrera, being attentive to Euclid's definitions, affirms: "The cubic number is that one which is formed by three equal numbers."

The unit number is imagined as a tiny corpuscle, in this case cubic. So the number 4, lineal, may be formed by four tiny cubes aligned. The superficial –square– 16, originating from two 4's multiplied together, should be formed by 16 tiny cubes arranged within a square area of 4x4 cubes. And the solid number –cubic– 64, originating from three 4's multiplied together, must be formed by 64 cubes arranged according to a macrocube of 4 cubes per edge (fig.1). This is a way of understanding the operation physically, as if the line were displaced as it is multiplied, forming the surface, and as though both the line and the surface did this in the other dimension, forming the volumetric body.

Herrera combines this operation with the Llullian concepts, and he explains it to us this way: "because the cube results from three dimensions and two operations: the first operation is of the *tivum* or the line itself from whose operation comes the *bile* or surface; the second operation, which is the *Are*, is jointly of the *tivum* or line and of the *bile* or surface. And the *are* together with the *tivum* and the *bile* are the cube, or the total plenitude of being and operating..." The cube is the plenitude of being and operating, because it contains the three essential dimensions: the *agente*, the *agible* and the *agere* (for example: the visual potential, the visible and vision), as Herrera says; and this is intrinsically as well as extrinsically. The line is active, the surface passive when engendered by the line, and the two together are again active when they produce the cube. Just as none of these three dimensions can be absent in the cube, neither can they be absent from being and operating.

All of this applies to the four elements (earth, fire, air and water), considering a cube of four units per side, to form all of the possible combinations among them, which make all bodies possible. And in the same way, considering a cube of nine per side, it applies to the nine absolute principles and to Ramon Llull's nine relative principles, which will make way for all of the relationships of these principles in everything that exists. Later he will add Aristotle's accidents, which also number nine if considered in their three dimensions. Definitively, the cube comes to be "a type which is the rule and measure of the rest, treating all things with perfection." Going even further, Juan de Herrera comes to state that the ternary relation, which engenders the cube from the union of the line with the surface, is the greatest vestige of the Most Holy Trinity within creatures. Along with this, the signification assigned to the cube by the operations which it necessarily carries in all created beings becomes transcendent.

Following these discourses, Herrera will depict the four elements as a line of four tiny cubes, over each of which figures a letter (A,B,C,D). Next, this line will engender a surface of 4x4 tiny cubes, and in each one of the 12 remaining he will figure in all of the combinations of four letters taken two by two, and their variations, which are produced upon successively adding the other three letters to each letter of those that form the line. The precedence of one of them would imply its being predominant or active, as opposed to the other, passive element, which makes one note that there are no repetitions. When he explains that this combination of line and surface will engender at the same time the cubic figure, completing the relationship of all with all, he draws the corresponding cube of 4 small cubes per edge (fig.1).

Herrera follows the same process to combine the nine principles, with tiny cubes designated with the 9 letters of the Llullian alphabet. Thus he draws the surface of 9x9 cubes placing the 9 isolated principles on one line and over the remaining 72 he places the combinations of letters taken two by two with their variations (fig.1). And he again indicates the next process which results in the cube of 9 units per side, making a general reference to the combinations of all with all. It seems that in this way the 9 isolated relationships would have been obtained, the 72 combinations of these 9 taken two by two, and 648 possible combinations would remain (in the case of the accidents –which also number nine– Herrera mentions a total of 729 solid cubes). However, this number does not seem to coincide with any type of combinations of nine nor with their sum, which, including variations, would be 986,409.

We can observe through present understanding that if in fact the n elements taken in isolation (the combinations of n over 1) plus the combinations of n over 2 with their variations of sequence, always add up to n squared [n+n(n-1)]. However, continuing the sum of n over 3, n over 4, etc. until arriving at n over n with its variations, it does not result that the sum is n cubed, as Herrera, who never manages to prove it, would have wished. The sum would be instead equal –at its limit– to Euler's number multiplied by n! (n! being the product: n(n-1) (n-2)...3 x 2 x 1). When n is the number four we have the only exception that turns out to be truthful: so this total is equal to four cubed; and this exception is the one which permits our author to establish in a general sense the cubic simile as "plenitude and the absolute end of differences in plenitude, with neither lack nor excess, and the totality of mixtures and qualities."

In this way one arrives at the composition of all creatures that have in themselves the four elements and all of the substantial and accidental principles. And since they are composed "thus as many cubes that work together with one another, what will result from this multiplication of cubes with other cubes will necessarily be a cube," and the result of this is the elemental cube or individual.

What results from all of this is, therefore, a dynamic vision of the cube, fruit of the linear engendering element of the surface and of both together as generators of the volume. And at the same time, a sense of plenitude has been attached to this cubic body, as the bearer of the three dimensions necessary for being and operating; the cube also provides a combinatory mechanism that equally achieves the formative plenitude of the complex totalities, which pertain to every limited being created. In this way, the cube is at the base of every individual, of everything in existence, as an explanation of its being and operating, and beneath all reality as an explanation of its diversity in the specific and its plenitude in the combination. According to these explanations, it is no wonder that in the paintings of Luca Cambiaso on the ceiling of the Escorial, the great architectonic work of Herrera, should appear the figure of Our Lord Jesus Christ, supreme referent of all Creation, seated upon the cubic figure in the context of the Trinity.

After Descartes

The passage into modernity introduced a higher degree of abstraction, which while it distances us from the sensory permits us to overcome the material limitations which the sensory imposes. As Dan Pedoe states: "An enormous simplification came into being with the introduction of sensed magnitudes, which could be either positive or negative, and the introduction of co-ordinates by René Descartes in 1637 changed the face of geometry for all time."[3] The conception of geometry advances even beyond everything imaginable for man, giving rise to recognizably differentiated methods. Each way of operating opens its own possibilities, and mutual reference is always possible: generally, science expands its horizons and leaves the sum of prior understanding as a particular case of the latest conception developed.

Dan Pedoe make us take note again: "At about the same time that Descartes was producing his epoch-creating ideas, Girard Desargues (1593-1662), an architect from Lyons, was engaged in an extension of geometry which can be regarded as being more fundamental than that created by Descartes. We call this projective geometry...in which there is no reference to measurement." And along with many other scholars he came to demonstrate that "Euclidean geometry is merely projective geometry with reference to a special pair of points" (the complex points I and J on a line in infinity). The set theories would end up again orienting these studies in another direction that is even more universal: By beginning with a set of points within which the lines would be determined subsets, and by establishing certain axioms, one would find a geometry the deductions of which could be extended to other sets of objects that fulfill those determined conditions.[3]

The scientific development outlined by Pedoe arrives at the possibility of considering more than three dimensions for geometry, and even doing so without denying a representation of three dimensions in space of what is defined in four. This representation would be an intersection of such a four-dimensional space in a space of three dimensions, in the same way that we can consider the intersection of a plane in an object defined three-dimensionally. He notes that this fourth dimension might be time, and he offers the example of how the inhabitants of a two-dimensional world might interpret as a variation in time the figure that would be traced in a plane by the intersection of a three-dimensional sphere which, moving through three-dimensional space, would pass through that plane; the first point would be produced, then a series of circles, increasing until one reaches the diameter of the sphere, and then diminishing until once again ending in a point just before disappearing. The concept of the hypercube is assumed in terms of mathematics. Although we may not be able to draw them nor even imagine them, it is possible to speak of bodies equivalent to a cube in three dimensions, defined in four, five or more dimensions. The hypercube may be a body in a space of n dimensions that has equal sides. In this case, in the definition of the hypercube, what holds a singular importance is the identical dimension of its sides.

In speaking about fractal geometry, Mandelbrot characterizes Euclidean geometry as that of the "sets for which all the useful dimensions coincide," that is, "dimensionally concordant sets," while fractal geometry observes that "the different dimensions of the sets fail to coincide; these sets are dimensionally discordant."[5] He refers to the topological dimension (which is always a whole number) and to the Hausdorff-Besicovitch or fractal dimension; in fractals the latter dimension is greater than the former, while in Euclidean geometry the two remain equal. In the new fractal geometry, also, the latter dimension is typically not a whole number, in contrast to Euclidean geometry in which it is always a whole number. Indeed, a special form of the fractal dimension is the self-similarity dimension. If we divide a cube into an equal number of small cubes (in a way similar to the constructions of Herrera), we find that the number of resultant cubes is equal to 1 divided by the factor of reduction (the relationship between the edge of the small cube with respect to that of the initial cube) raised to the dimension of self-similarity. This relationship fulfills any case considered. Whatever the factor of reduction that we consider, the dimension remains constant, and in the cube it is equal to its topological dimension, that is, 3.[6]

The cube as generator of art forms

This immense panorama opened by science can do no less than to influence the imagination of those artists who work with plastic elements: painters, sculptors and above all, architects. It is commonly said that in their art, consciously or unconsciously, they metaphorically render many of these argumentations, they symbolize them or even come to predict them, in the most unusual occasions. Although the complete development of these ideas would require more space, we would like to outline at least some examples.

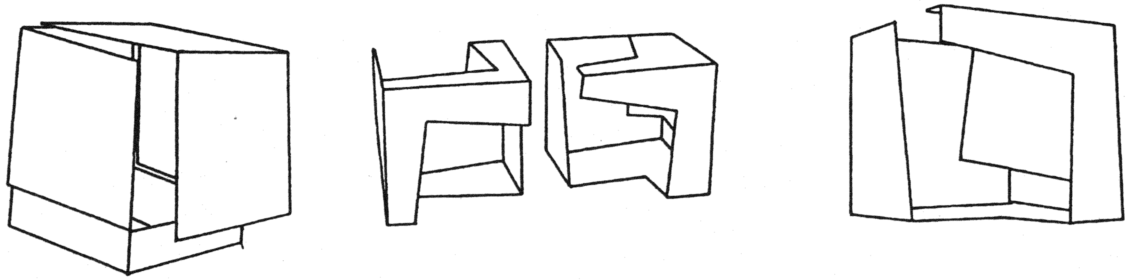

Figure 2: *Jorge Oteiza's sculptures*

The generation of the cube in a superficial manner, which we see mostly in the case of the Platonic argument, could be tied to the experiments with the cube carried out in some of Jorge Oteiza's sculptures. Oteiza is interested in the empty space that these planes define in their interior, which is shown to the viewer by opening up the surface that encloses it, and in this way said space is "deoccupied," according to the expression used by the artist. The sculpture is defined by materializing those surfaces in steel sheets of very little thickness, and by showing this extreme to the viewer (fig.2). In *The Metaphysical Box of Fra Angelico. Conjunction of Two Trihedrons[7]*, from 1958, we are permitted to access the interior void by slightly separating the two trihedrons that form the cube, as its name indicates; along with this, six of the edges are duplicated and the total virtual volume increases. In *Empty Box*, from 1958, each surface appears cut back by the sculptor's removing a trapezoidal shape from each face; although without a uniform law, the edges appear affected, partially disappearing, while the volume of the virtual cube remains identical.[8]

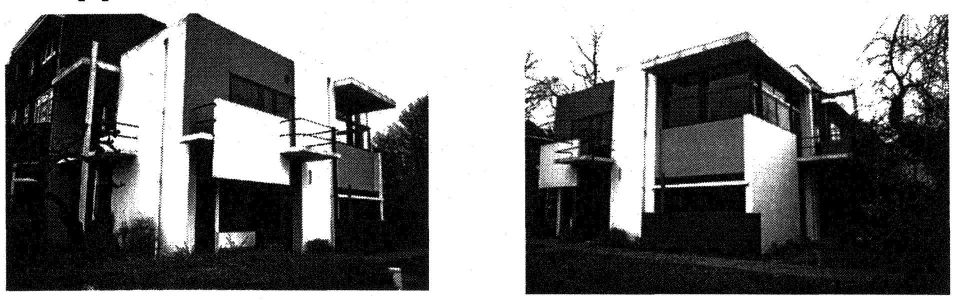

Figure 3: *Schröder's house*

The generation of the cube as a line that moves, engendering a flat surface, so that both (line and plane) are displaced, engendering the cube, could be tied to the neoplasticist constructions, especially Rietveld's. In these, the lines and planes, which remain as if frozen in their distinct positions, are emphasized, always avoiding the form of dihedrons –and that of trihedrons even more so– which would give the impression of a closed volume. Especially representative is Schröder's house (fig.3), where, as Theodore M. Brown already had already affirmed in 1958 [9] the architectural elements extend beyond the points of their intersection, showing lines and separated planes in their complete integrity. The void and the glass act as separators between the superficial and linear elements. Also, the different colors reinforce the singularity of the superficial and linear elements. And above all, the lines of the elements in I (being extended) act as axes of a coordinated three-dimensional system. The reference to infinite space where this architectonic body is situated and defined is therefore evident in this case. We should point out, however, some peculiarities. The generating planes are fragmentary and are established in the three directions simultaneously: these planes are vertical as well as horizontal. Among the horizontal ones, which extend between the eaves and the floor at the ground level, we should include the fragmented flooring of the first level, because of the distinct color that it takes on.

The generation of the cube as orthogonal lines that extend in the three Cartesian directions with identical measures is present in the preference of Le Corbusier for the module of 2.26 x 2.26 x 2.26 meters (the measure of a man with his arm raised upward, the origin of his Modulor [10]), which gives way to the proposition of honeycomb structure in which to develop human dwellings. In this structure, according to its author, one discovers " 'un contenent d'hommes', affirmation d'un élément volumétrique capable de mettre de l'ordre, de transformer les règlements et d'aider l'architecture des temps modernes dans sa lourde tâche de créer les logis de la civilisation machiniste"; also adding at another moment: "l'exactitude est encore ici une source de confort physique et intellectuel." The author himself carries out the application of this structural network, according to his patent of December 15, 1950, in such studies as "ROQ" y "ROB" in Cap Martin [11] on the Azure Coast, such as the "Maison de l'Homme" (Le Corbusier Center) in Zurich, where he uses it in the structure enclosing the interior space[12]. In order to better understand the meaning that this cubic structure had for Le Corbusier, we can recall what he left us written on the Modulor. As he states, Mediterranean architectural art was *the spirit placed beneath the sign of the square,* while *the spirit placed beneath the sign of the triangle and of the convex pentagon or star, and its volumetric consequences: the icosahedron and the dodecahedron*, characterized the Germanic. Masculinity, referring to architecture, pertained to the former, while femininity, as a subjective and abstract symbolism, characterized the latter. *The man of the ruler* would be the first of these artifices, *the man of the compass* the second, the one which appears to have been predominant in those times; however, our author considers that today (in his time), the ruler is necessary and the compass dangerous. [13] And in accord with these ideas he draws and writes from 1947 to 1953 his "Poem of the Right Angle." [10]

The imaginary generation of a four-dimensional cube that leaves three-dimensional traces interpreted as temporal variations could be placed in relation with the experiments of the Eisenman's houses, in which the deep formal structures introduced, or the transformations suffered by the initial solid during the process of creation, leave their impressions on the final result. Those transformations or deep formal structures are independent of both the function and the construction, because of which they never stop being simple geometric play that enriches the shape and the final space.

We can say that the law of generation itself, more than its result, is the object that minimalist works attempt to represent; so it happens in many sculptures, especially the works of Sol LeWitt, which consider the cube as a point of departure. It is a law of generation that, in contrast with the cases considered up to this point, is presented physically, that is, with a presence simultaneous with all the intermediate steps (instead of being deducible by the trace that the process utilized leaves in the final body). Each process is a work and each work is a process. Examples of this may be "Serial project (ABCD)" of 1966 or "3 part set" of 1968. The self-similarity also seems inherent in these works, apparently conceited, that try to capture the attention of the viewer and focus it onto themselves, impeding other associations or analogies that might transcend that which is immediately present. In some works by Donald Judd we find this self-similarity in the way in which identical cubes are repeated; in other works by Sol LeWitt such as "Modular Cube" of 1962, we find it as cubes obtained from another cube by subdivision (the application of a factor of reduction) or if we consider the inverse, by aggregation.

A segment of the most current architecture seems once again to be inspired by this poetry that minimalist architecture emphasizes. In this case, its signification is reinforced by the complexity that its greatest scale introduces, in terms of construction, perception and use. But even a passing glance a the most emblematic cases would still require a more extensive study that merits an exclusive development.

In the pictorial representation of the cube in two dimensions, the possibilities increase, since one can draw even that which is impossible to construct in three dimensions, such as tricks of false perspective show us. Escher is an emblematic example of the utilization of these resources, as observed already by mathematicians, in the most figurative manner. The empty cube that changes the connection with the base of the anterior and posterior vertical edges is found represented in a drawing and in a model that is held in the hand of a character in Escher's work "Belvedere" of 1958, and at the same time the principal motif of the drawing produces this same effect in an architectural belvedere of two floors that is as impossible as the cube, occupied by characters who, like those who operate the ladder, contribute even more to make us note its impossibility. In his "Metamorphosis II," xylograph from 1939-40, Escher himself attends more to the superficial aspect that the full cube presents, by means of drawings such as the well-known floor of rhombuses (white, black and gray) that simulate relief in cubes with three faces visible. It is not without significance that in this work the cubes are subdivided producing architectonic conglomerations of little Mediterranean houses.

All of the productions of this artist acquire a disquieting effect, nearly surreal, that traps the viewer, making him/her participate in these impressions. Meditation on the fragility of the real meaning of perception itself, which captures only appearances, and the suspicion about the instability of the perceived world which could correspond to that perception, is supported paradoxically on properties arising from the commonly considered aseptic and firm world of science.

The examples mentioned here may give an idea of the stimulus that, for the artist, means emphasizing the different conceptions of the cube which have been given to it through the various geometries. We have seen moreover that artists and architects tend to find associated meanings for the cube that transcend the mere physical conception of the figures. The fecundity that this very basic and apparently very simple shape has had and continues to have in the art of our time leads us to consider in this sense the expression "infinity is a cube without vertices," a statement which we can make by paraphrasing the one who wrote that infinity is a square without corners. [14]

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