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The End of the Well-Tempered Clavichord?

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Abstract

The scale used by all pianos today is based on the idea of the well-tempered clavichord, an instrument for which Bach wrote a famous collection of pieces showing its versatility. Our contention here is that, in an era of electronic instruments, the well-tempered scale has the potential to be completely outmoded, and replaced by an adaptive scale which uses natural harmonics whenever possible.

Introduction

Johann Sebastian Bach, in 1744, wrote a collection of 24 keyboard pieces entitled "The Well-Tempered Clavichord." Today these are usually played on the piano, but Bach wrote them for the clavichord, an instrument which, at the time, represented a technological advance. For the first time, a single musical instrument could be played, without being obviously out of tune, in each of the twelve major keys (C, Db, D, Eb, E, F, F#, G, Ab, A, Bb, and B) and each of the 12 minor keys (C, C#, D, Eb, E, F, F#, G, G#, A, Bb, and B). Bach celebrated this by writing 24 pieces, one for each of the 12 major and 12 minor keys. An instrument that is out of tune can be said to be "ill-tempered," so Bach's clavichord, by contrast, was "well-tempered."

Our concern here is with the fact that the design of the well-tempered clavichord involved a compromise. Despite appearances, there is a sense in which the well-tempered clavichord and its successor, the piano, are still out of tune. They are merely so little out of tune that the ordinary ear does not hear the difference. This compromise was necessary so long as the tuning of musical instruments was a painstaking process, performed once at the start of each concert or occasionally at the start of each piece within a concert. Today, however, computer-driven musical instruments can be tuned and retuned in less than a millisecond; and it is this that suggests to us a new tuning system in which compromise, although it is still there, is a fraction of what it was before.

Natural Production of Sound

To understand the compromise, it is necessary to review the mathematics of sound production in a tunable, non-computerized instrument. There is always something that is vibrating, whether it be a string, as with a violin or piano, or a column of air, as with a flute or trumpet. This string, or column of air, has a certain length L, and can be made to vibrate along its entire length or along parts of its length of equal size. If we play the E string of violin without touching it, we get the note E, a little more than an octave above middle C. If we play this same string while touching it very lightly at the midpoint of its length, we get a note one octave higher, also called E. This arises from the fact that the string is now vibrating in two equal parts.

Suppose now that we touch the string very lightly, not at the midpoint of its length, but at a point representing one-third of its length. The string will now vibrate in three equal parts, and the note will be B, almost three octaves above middle C. The string can also be touched at a point representing one-fourth, or one-fifth, or one-sixth of its length, causing it to vibrate in four, five, or six equal parts. The notes produced in this way are called harmonics of the original note. In the same way, the column of air inside a trumpet can be made to vibrate in anywhere from two through six equal parts, and sometimes more, producing harmonics of the *fundamental note*. This is the note that would be produced if the air column vibrated along its entire length (although actually doing that is rare in a trumpet).

This is natural music, without compromise. (It is also a bit of an oversimplification; see the section on Resonance below.) We can hear it when we hear the notes of a bugle before a horse race. Try to play a bugle call on the piano, and there is something mysteriously missing. What is missing is that the bugle is in tune, in a way that the piano is not. The bugle plays natural harmonics, while the piano plays only an approximation of harmonics, a result of the compromise which is necessary for the piano to play in 24 different keys. (Note that the bugle can play in only one key.) This effect is in addition to the fact that the bugle has a different timbre than the piano.

Natural Frequencies

To understand the compromise further, we will have to look at frequencies. The musical notation A-440, for example, means than anything which vibrates 440 times per second will produce the note A (just above middle C). There is an A string on the violin which vibrates that often; that is its frequency. If we touch the A string at the midpoint of its length, it will vibrate in two equal parts, and twice as frequently, or 880 times per second. If we make the string vibrate in N equal parts, its frequency will be 440 times N. If we make the air column in a trumpet vibrate in N equal parts, its frequency, similarly, will be N times the frequency of the fundamental note.

Now let us look, specifically, at the harmonics corresponding to four, five, six, and eight times the fundamental note. This is the so-called major chord; in the key of C, it is C, E, G, and C, while in the key of A, it would be A, C#, E, and A. If the fundamental note has frequency 110, then these four notes have frequencies 440, 550, 660, and 880. We can also start with the lowest note of the major chord, in this case A, and calculate the other frequencies from that. If we start at 440, then to get to 550, we multiply by $^5/_4$; to get to 660, we multiply by $^6/_4$, or $^3/_2$. To get to 880, which is an octave higher, we multiply by 2; and indeed multiplying the frequency by 2 always produces the next octave up. The same thing happens in the key of C; if we start at C, with frequency N, then E will have frequency $^5/_4$N, G will have frequency $^3/_2$N, and C, an octave up, will have frequency 2N.

These are the frequencies of the natural harmonics. They are not the frequencies of the well-tempered clavichord. To understand why not, let us start at C, with frequency N; go up to E, with frequency $^5/_4$N; and then *start* at E and go up to the next note in the E major chord, which is G#. Now the frequency is $^5/_4$ times $^5/_4$N, or $^{25}/_{16}$N. But on the piano, G# is the same as Ab, and we can now go up to the next note on the Ab major chord, which is C again. This time the frequency is $^5/_4$ times $^{25}/_{16}$N, or $^{125}/_{64}$N. The trouble is that we are now an octave higher than we were before, and therefore the frequency should be 2N. Instead, it is $^{125}/_{64}$N, which is a little smaller than 2N (which would be $^{128}/_{64}$N, since 64 times 2 is 128).

This is the bane of natural harmonics: *they don't come out even*. We can't play the same tunable instrument in even the three keys of C major, E major, and Ab major without some form of compromise. Once you think of it, the compromise is obvious. We multiplied N by $^5/_4$, three times, and we wanted to end up with 2N. This didn't work, so we need another number, not $^5/_4$ but very close to $^5/_4$. Instead of 1.25 (or $^5/_4$), the number is approximately 1.25992, the cube root of 2. If we multiply

N by this three times, we clearly get 2N, because the definition of the cube root of 2 (call it X) is that $X^3 = 2$.

The Well-Tempered Scale

The basic principle of the well-tempered clavichord is an extension of this idea from three multiplications to twelve. Instead of a number X with $X^3 = 2$, we now want a number R with $R^{12} = 2$, also known as the 12th root of 2, or approximately 1.05946. If the frequency for C is N, then the frequency for C# is N times R; the frequency for D is N times R times R, and so on up the chromatic scale. In particular, the frequency for G, which in natural harmonics would be $^3/_2$N, or N times 1.5, is N times about 1.49831, on the well-tempered scale. (Other bases than 12 may be used; with base 24, for example, we get the so-called quarter-tone scale. All such scales are sometimes referred to as "just scales," where "just" means "fair," since all 24 keys are treated fairly and equally.)

More generally, let us look at the notes of the major scale: C, D, E, F, G, A, B, C. This is made up of the major chord (C, E, G, C), sometimes called the *tonic chord*, together with the first three notes of the *subdominant chord* (F, A, C) and the *dominant chord* (G, B, D, but with the D an octave down). The natural frequencies for these are:

C	D	E	F	G	A	B	C
N	$^9/_8$N	$^5/_4$N	$^4/_3$N	$^3/_2$N	$^5/_3$N	$^{15}/_8$N	2N

It is easy to see that, going up the major chord that starts with F ($^4/_3$N), you get A ($^5/_4$ times $^4/_3$N, or $^5/_3$N) and C ($^3/_2$ times $^4/_3$N, or $^4/_2$N, or 2N). Going up the major chord that starts with G ($^3/_2$N), you get B ($^5/_4$ times $^3/_2$N, or $^{15}/_8$N) and D ($^3/_2$ times $^3/_2$N, or $^9/_4$N), and one octave down from there would be half of that, or $^9/_8$N. Now let us see the extent of the compromise, when we use the well-tempered scale. Our factors, to be multiplied by N in each case, are:

Note	C	D	E	F	G	A	B	C
Natural harmonics	1	1.125	1.25	1.333	1.5	1.667	1.875	2
Well-tempered scale	1	1.12246	1.25992	1.33484	1.49831	1.68179	1.88775	2
Percentage off by	0%	−0.3%	+0.8%	+0.1%	−0.1%	+0.9%	+0.7%	0%

The reason that the well-tempered scale works as well as it does is that none of these notes is off by as much as one percent. Nevertheless, they are off, and today, with computers, we can do better.

The Computer-Driven Grand Piano

One might object at this point that computers are used only in synthesizers and electric pianos, and not in serious musical instruments. It is true that violins and trumpets have nothing to do with computers, although trumpets already use natural harmonics, while violinists can use an infinite gradation of frequencies by pressing their fingers firmly on the strings to change the length of that part of the string which vibrates. Meanwhile, there is the electronic grand piano, which produces a quality of sound equivalent to that of most conventional grand pianos.

We will need some basic information on computer production of sound. The earliest sounds produced by computer involved producing clicks at a given frequency. If the frequency is 440, then, after each click, the computer would wait for 1/440 of a second before producing the next click. This is feasible because 1/440 of a second is 2,273 microseconds — a long time, for a computer. However, the sound produced in such a way is monotonous.

Today's computers calculate the sound wave amplitude y associated with a particular sound. This time, if the frequency is 440, the wave equation is $y = sin\ (440x/2\pi)$, where x represents time in seconds. The value of this can now be calculated (for example) once every microsecond, and transmitted to the sound-producing device. The point is that this device is now capable of accepting, not just a yes-or-no signal, but an amplitude. The result is that waveforms of any kind may be generated. Furthermore, the wave equation is the result of a numerical calculation in which frequencies appear; and changing the frequencies in the equation can be done in less than one microsecond.

Fixed Harmonic Scales

We start by considering an idea which is inferior, although well adapted to computers. This would require the player, before playing a piece, to set the major key; and then all the notes in all the major scales of that key would be set to the natural harmonics of that major key. Whatever the frequency N of the tonic note, the frequencies of the next six notes would be found by multiplying N, respectively, by $^9/_8$, $^5/_4$, $^4/_3$, $^3/_2$, $^5/_3$, and $^{15}/_8$, as indicated earlier. These are then the frequencies that would be played.

The problem with this is that a piece in C major (for example) is not *entirely* in C major. A typical piece in C major will go into F, G, and D major at times; and this is only the beginning. If the harmonic scale is fixed, then, as soon as you get out of the major key, some of the notes will be even worse off than they were before.

What needs to be done is that the computer should sense, at all times, what key is being played in, and adjust the frequencies to the needs of that particular key. What we need is a way of translating a chord C into the key K in which that chord naturally occurs. The question, however, is how this is to be done, given the wide variety of possible chords C that can be played.

The Universal Adaptive Scale

The solution to the above problem arises from our observation that, in considering a chord, one only has to consider twelve notes, not 88. That is to say, for the purposes of determining K from C, it is not significant what octave a note is in. Thus, for example, C-E-G is in C major, regardless of whether the G is above the C, on the piano, or below the C, in a different octave. In fact, each of the 12 notes is either in the chord, or it is not; and thus the total number of possible chords is 2^{12}, or 4096. A table that size is well within the capabilities of today's memory chips, each of which is at least 1,000,000 bytes in size.

We thus envision a table of 4,096 positions, which, for each note combination, gives the key, major or minor. Whatever chord is being played, the frequencies of all notes will be adjusted according to the chord, and it is those frequencies that will be played. Some cases will be complete dissonances, such as C-C#-D-Eb; and some cases will be ambiguous, such as C-E-G-A, which could be either C major 6th or A minor 7th. It is important to note, however, that *most cases will not be ambiguous*.This is what we call the Universal Adaptive Scale.

Notes Other Than The Major Scale

In our treatment of natural harmonics for the notes of the scale (C, D, E, F, G, A, B, C) we left out the five additional notes (the black notes, in the scale of C). The natural harmonics here can be chosen in various ways, of which the following are the simplest:

C	Db	D	Eb	E	F	F#	G	Ab	A	Bb	B	C
N	$^{16}/_{15}$N	$^{9}/_{8}$N	$^{6}/_{5}$N	$^{5}/_{4}$N	$^{4}/_{3}$N	$^{25}/_{18}$N	$^{3}/_{2}$N	$^{8}/_{5}$N	$^{5}/_{3}$N	$^{9}/_{5}$N	$^{15}/_{8}$N	2N

These are justified as follows:

- Eb is $^{6}/_{5}$N because the Eb major scale (Eb, G, Bb) includes G, at $^{5}/_{4}$ times $^{6}/_{5}$N or $^{6}/_{4}$N (or $^{3}/_{2}$N), as specified.
- Ab is $^{8}/_{5}$N because the Ab major scale (Ab, C, Eb) includes C, at $^{8}/_{5}$ times $^{5}/_{4}$N or 2N.
- Bb is $^{9}/_{5}$N because the Eb major scale includes Bb, at $^{3}/_{2}$ times $^{6}/_{5}$N or $^{18}/_{10}$N (or $^{9}/_{5}$N). This is simpler than tying Bb to F through the Bb major scale, which would result in $^{16}/_{9}$N (since $^{16}/_{9}$ times $^{3}/_{2}$ is $^{8}/_{3}$, and then $^{4}/_{3}$ is an octave below that).
- Db is $^{16}/_{15}$N because the Db major scale (Db, F, Ab) includes F, at $^{5}/_{4}$ times $^{16}/_{15}$N or $^{4}/_{3}$N, as specified. The fraction $^{16}/_{15}$ may seem complex, but it is $1^{1}/_{15}$; any other close fraction would be just as complex.
- F# is $^{25}/_{18}$N because going up from F# to A is like going up from C to Eb: you multiply by $^{6}/_{5}$, producing $^{6}/_{5}$ times $^{25}/_{18}$ or $^{5}/_{3}$, as specified. This gives a simpler fraction than any of the obvious alternatives.

Monophonic Passages

A passage, or small part of a piece of music, is monophonic if only one note is being played at a time. Obviously the key in which to play such a passage cannot be inferred from the passage itself. It can be inferred from the preceding chord in most cases (as, for example, when a chromatic scale is being played), but not in all. To improve the way that such a passage sounds, a computerized keyboard might have a mode in which the left hand plays one or two notes (which do not sound), indicating the key, while the right hand plays a monophonic passage.

Relations Among Scales

We mentioned that, when a piece in C major goes into G major, some of the frequencies change. Actually, not as many of them change as one might think. It is still necessary, using the universal adaptive scale, for the computer to know what the tonic is (in this case C major), because the G major scale, for example, will now be based on G as it appears in the C scale, not in any other scale (including the well-tempered scale). Let us look only at the two major scales involved, one based on N and the other on M = $^{3}/_{2}$N:

C Scale:	C	D	E	F	G	A	B	C
	N	$^{9}/_{8}$N	$^{5}/_{4}$N	$^{4}/_{3}$N	$^{3}/_{2}$N	$^{5}/_{3}$N	$^{15}/_{8}$N	2N
G Scale:	G	A	B	C	D	E	F#	G
(in terms of M)	M	$^{9}/_{8}$M	$^{5}/_{4}$M	$^{4}/_{3}$M	$^{3}/_{2}$M	$^{5}/_{3}$M	$^{15}/_{8}$M	2M
(in terms of N)	$^{3}/_{2}$N	$^{27}/_{16}$N	$^{15}/_{8}$N	2N	$^{9}/_{4}$N	$^{5}/_{2}$N	$^{45}/_{16}$N	3N

Lowering some notes by an octave (that is, dividing the frequency by two), and rearranging individual notes so that they correspond, we obtain the following comparison:

Note:	C	D	E	F	F#	G	A	B	C
C Scale:	N	$^{9}/_{8}$N	$^{5}/_{4}$N	$^{4}/_{3}$N		$^{3}/_{2}$N	$^{5}/_{3}$N	$^{15}/_{8}$N	2N
G Scale:	N	$^{9}/_{8}$N	$^{5}/_{4}$N		$^{45}/_{32}$N	$^{3}/_{2}$N	$^{27}/_{16}$N	$^{15}/_{8}$N	2N

Clearly F# is missing in the scale of C, while F is missing in the scale of G. Aside from these two, however, the only note with a different frequency in the two scales is A, which is 1.3% higher in the G scale than in the C scale.

This brings up an important issue. Every so often, the universal adaptive scale is going to present the listener with two notes, one after the other, which "ought to be the same" and yet are noticeably different. The Blue Danube Waltz is a good example of this; if it is played in C, there are three measures in which the melody is

B / B D A / A

where the first two bars are in G major (actually G7) and the last bar is in C major. According to the correspondence above, the last two notes of this, even though they are both A, have different frequencies, and an astute musical ear will recognize this. Furthermore, this phenomenon does not occur on the piano or on any other instrument using the well-tempered scale.

Will the listeners, being aware of this, criticize it and prefer, on that account, the well-tempered scale to the universal adaptive scale? Our contention (admittedly debatable) is that they will not. The reason is that the universal adaptive scale, or any scale involving natural harmonics, exhibits the phenomenon of resonance in a way that the well-tempered scale never can. We will now explain this.

Resonance

To understand resonance, we have to go back to the study of vibrations. Whenever a string or an air column vibrates, even if it is vibrating on the fundamental note, it is also, to a very small extent, vibrating on one or more of the harmonics as well. These are called *resonant harmonics*; their amplitudes differ from one musical instrument to another, and it is the relative amplitudes of the various resonant harmonics which determine, to a great extent, the timbre of a particular musical instrument.

Now suppose that we are playing a chord which involves several natural harmonics. As a specific example, let us assume that three trumpets are playing Bb, D, and F, while trombones and tuba are playing low Bb at the same time. The natural harmonics being played by the trumpets will also, in a way imperceptible to the human ear, be coming from the trombones and tuba. If all these instruments have been properly tuned, then the lower instruments will be adding sound to the higher instruments, on the exact notes that the higher instruments are playing.

But they will be doing more than that. Resonance, as a phenomenon in physics, has to do with the effect of transmitting a certain frequency in an environment in which there are other transmitters for which that is a natural frequency. In particular, resonance can occur when one instrument naturally resonates with the frequency, or a multiple of the frequency, being played by another instrument nearby.

In electronics, resonance can be inappropriate. The annoying low-pitched hum that comes from speakers when they are not properly adjusted, for example, is due to resonance in the speakers caused by ordinary 60-cycle current (it is always on a B, two octaves below middle C, whose frequency is approximately 60). In sound production, however, resonance is desirable. In particular, a low Bb played by trombones and tuba will resonate in nearby trumpets, even if no one is playing the trumpets.

All this, however, depends on natural harmonics, and doesn't work when the well-tempered scale is used, except for octaves. If a chord involving a fundamental note and one or more of its harmonics is played on the piano, or on an old-fashioned clavichord, the only resonance will be between octaves. There the intervals are exact, even in the well-tempered scale. Indeed, if the damper pedal is down, a low F#, for example, will resonate through all the unplayed F# strings, up and down the piano, and similarly for any other low note.

Other harmonics don't work here on pianos or clavichords because the natural harmonic frequencies are not exact. A low Bb doesn't resonate in a higher D, even though D would be one of the harmonics, because the D is tuned to the Bb times 1.25992 times 4 (the 4 is for the two-octave difference). This is the Bb times 5.03968, which is not the same as the Bb times 5, for the fifth (or as it is sometimes called, the fourth) harmonic. Indeed, when two notes that are almost, but not quite, the same are played at the same time, the result can be harsh and grating, as can be heard when listening to a band which is not properly in tune with itself.

Our thesis here is that the universal adaptive scale will bring resonance back to the electronic grand piano and similar instruments. The compromise that was necessary in the design of the well-tempered scale will now be necessary only in a few situations, such as chromatic scales and atonal music. With the new possibilities brought about by computers, we can look forward to a new and resonating musical future.

Other Studies of Computer Music

Computer music has a long and rich history, which is admirably summarized in the 1,234-page reference work by Roads et al. [1]. This covers sound analysis and synthesis, music languages and editors, mixing and signal processing, psychoacoustics, performance software, and algorithmic composition systems, among other subjects. It does not, however (if its 24-page index is to be trusted), address such subjects as the clavichord, well-tempered scales, just scales, or quarter tones.

Reference

[1] C. Roads, et al., *The Computer Music Tutorial*, MIT Press. 1996.

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