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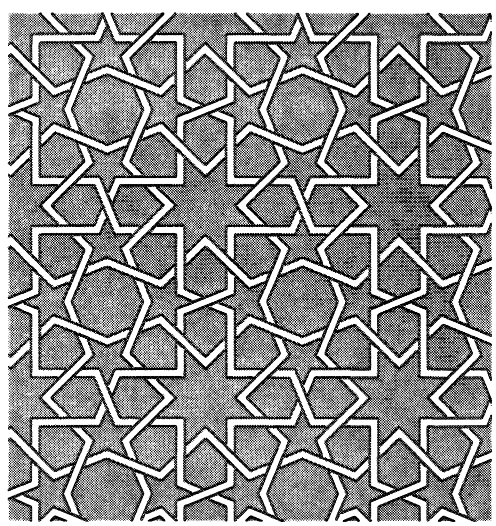
Computer Generated Islamic Star Patterns

Craig S. Kaplan
Department of Computer Science and Engineering
University of Washington
Box 352350, Seattle, WA 98195-2350 USA
csk@cs.washington.edu

Abstract

Islamic star patterns are a beautiful and highly geometric art form whose original design techniques are lost in history. We describe one procedure for constructing them based on placing radially-symmetric motifs in a formation dictated by a tiling of the plane, and show some styles in which they can be rendered. We also show some results generated with a software implementation of the technique.

1 Introduction



More than a thousand years ago, Islamic artisans began to adorn architectural surfaces with geometric patterns. As the centuries passed, this practice developed into a rich system of intricate ornamentation that followed the spread of Islamic culture into Africa, Europe, and Asia. The ornaments often took the form of a division of the plane into star-shaped regions, which we will simply call "Islamic star patterns"; a typical example appears on the right. To this day, architectural landmarks in places like Granada, Spain and Isfahan, Iran demonstrate the artistic mastery achieved by these ancient artisans.

Lurking in these geometric wonders is a long-standing historical puzzle. The original designers of these figures kept their techniques a closely guarded secret. Other than the finished works themselves, little information survives about the thought process behind their star patterns.

Many attempts have been made to reinvent the design process for star patterns, resulting in a variety of successful analyses and constructions. Grünbaum and Shephard [9] decompose periodic Islamic patterns by their symmetry groups, obtaining a fundamental region they use to derive properties of the original pattern. Abas and Salman apply this decomposition process to a large collection of patterns [2]. Elsewhere, they argue for a simple approach tied to the tools available to designers of the time [1]. Dewdney proposes a method of reflecting lines off of periodically-placed circles [5]. Castera presents a technique based on the construction of networks of eightfold stars and "safts" [7].

This paper presents a technique described by Hankin [10], based on his experiences seeing partially-finished installations of Islamic art. It also incorporates the work of Lee [11], who provides simple constructions for the common features of Islamic patterns. Given a tiling of the plane containing regular polygons and irregular regions, we fill the polygons with Lee's stars and rosettes, and infer geometry for the remaining regions. We have implemented this technique as a Java applet, which was used to produce the examples in this paper. The applet is available for experimentation at `http://www.cs.washington.edu/homes/csk/taprats/`.

The rest of the paper is organized as follows. Section 2 presents constructions for the common features of Islamic patterns: stars and rosettes. Section 3 shows how complete designs may be built using repeated copies of those features. Techniques for creating visually appealing renderings of the designs are given in Section 4. Some results appear in Section 5. The paper concludes in Section 6 by exploring some opportunities for future work.

2 Stars and Rosettes

In our method, a regular n-gon is filled with a figure of symmetry type d_n (which has all the symmetries of the n-gon). In practice, these figures belong to a small number of families which we describe below.

For $n \geq 3$, let the unit circle be parameterized via $\gamma(t) = (\cos(2\pi t/n), \sin(2\pi t/n))$. We construct the n-pointed star polygon (n/d) by drawing, for $0 \leq i < n$, the line segment σ_i connecting $\gamma(i)$ and $\gamma(i+d)$. Note that $d < n/2$ and that $(n/1)$ is the regular n-gon. For some values of $k \neq i$, σ_i will intersect σ_k, dividing σ_i into a number of subsegments. We often choose to draw only the first s subsegments at either end of σ_i, which we indicate with the extended notation $(n/d)s$. Figure 1 shows the different stars that are possible when $n = 8$.

Our implementation generalizes this construction, allowing d to take on any real value in $[1, n/2)$. When d is not an integer, point P is computed as the intersection of line segments $\overline{\gamma(i)\gamma(i+d)}$ and $\overline{\gamma(i+\lfloor d \rfloor - d)\gamma(i+\lfloor d \rfloor)}$, and σ_i is replaced by the two line segments $\overline{\gamma(i)P}$ and $\overline{P\gamma(i+\lfloor d \rfloor)}$. Two examples of this generalization are given in Figure 2.

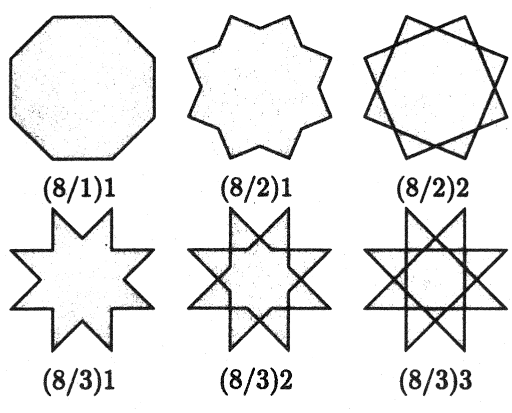


Figure 1 The six possible eight-pointed stars when d is an integer.

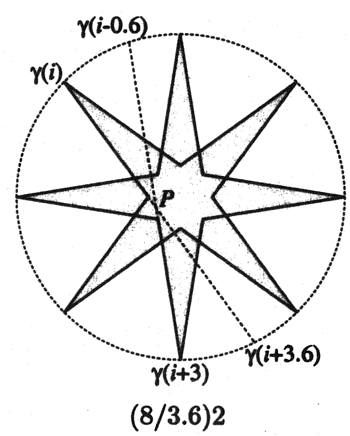


Figure 2 An $(n/d)s$ star for non-integral d.

When sixfold stars are arranged as on the left side of Figure 3, a higher-level structure emerges: every star is surrounded by a ring of regular hexagons. The pattern can be regarded as being composed of these surrounded stars, or **rosettes**. Placing copies of the rosette in the plane will leave behind gaps, which in this case happen to be more sixfold stars.

The rosette, a central star surrounded by hexagons, appears frequently in Islamic art. They do not only appear in the sixfold variety, meaning that we must generalize the construction of the rosette to handle

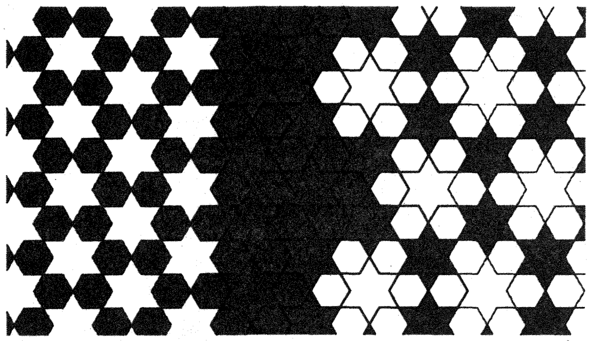


Figure 3 An arrangement of sixfold stars can be reinterpreted as rosettes. The pattern is one of the oldest in the Islamic tradition.

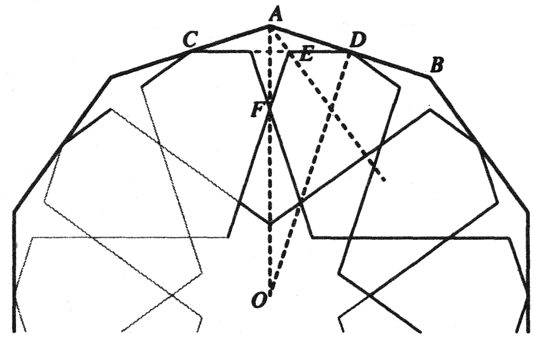


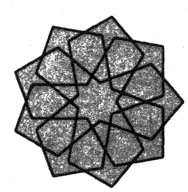
Figure 4 The construction of a ten-pointed rosette.

arbitrary n. The construction given by Lee [11] yields an n-fold rosette for any $n \geq 5$ while preserving most of the symmetry of the hexagons. Each hexagon has four edges not adjacent to the central star; all four edges are congruent. Moreover, the outermost edges lie on the regular n-gon joining the rosette's tips, and the two "radial" edges are parallel.

A diagram of Lee's construction process is shown in Figure 4. To begin, inscribe a regular n-gon in the unit circle and draw the n-gon whose vertices bisect its edges. Let A and B be adjacent vertices of the outer n-gon and C and D be adjacent vertices of the inner n-gon with D bisecting $\overline{AB}$. The key is to then identify point E, computed as the intersection of $\overline{CD}$ with the bisector of $\angle OAB$. Then F is the intersection of $\overline{OA}$ with the line through E parallel to $\overline{OD}$. The rest of the rosette follows through application of symmetry group d_n: edges $\overline{DE}$ and $\overline{EF}$ lead to the outer edges of the hexagons, while copies of F become the points of the inner star, which can be completed with the construction given earlier.

By sliding E along the bisector of $\angle OAB$, we can continuously vary the shape of the rosette while preserving the congruence of the four outer hexagonal edges.

Some Islamic designs feature a motif slightly more complicated than a basic rosette, where opposing limiting edges from adjacent tips of the rosette are joined up. The resulting object has the same symmetries and number of outer points as the rosette, but with an additional layer of geometry on its outside. We refer to these as "extended rosettes". A ninefold extended rosette appears on the right.



3 Filling the Plane

Equipped with a taxonomy of typically Islamic motifs that can be inscribed in regular polygons, we are now ready to create complete periodic designs. We start with a periodic tiling containing regular polygons, with irregular polygons thrown in as needed to fill gaps. For each regular n-gon with $n > 4$, we choose an n-fold star, rosette or extended rosette to place in it and replicate that motif everywhere the n-gon appears in the tiling. The motif is placed so that its points bisect the edges of the n-gon.

The result is a design like that of Figure 5(b). There are still large gaps where motifs were not placed, corresponding here to to the squares in the original tiling. Each square edge is adjacent to an edge of an octagon, and so a vertex of the chosen motif is incident to it. The presence of these vertices suggests a technique for filling the gaps in a natural way, by extending the line segments that terminate on the boundary of the region until they meet other extended segments in the region's interior. Except for degenerate cases,

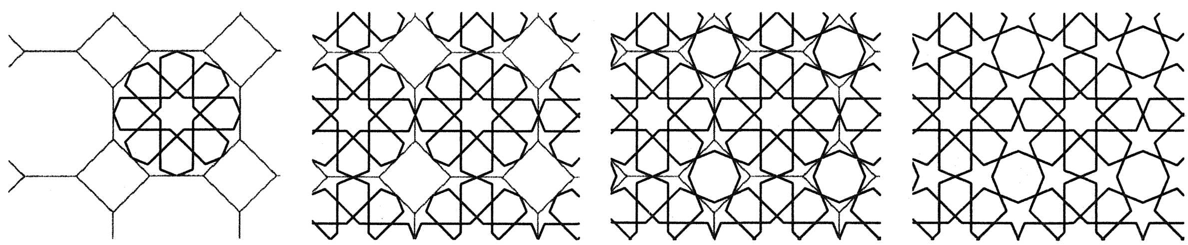


Figure 5 Given the octagon and square tiling shown in (a), we decide to place 8-fold rosettes in the octagons and let the system infer geometry for the squares. The rosette is copied to all octagons in (b), and lines from unattached tips are extended into the interstitial spaces until they meet in (c). The construction lines are removed, resulting in the final design shown in (d).

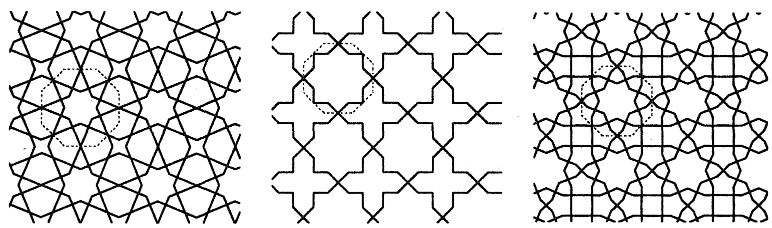


Figure 6 Some alternative patterns based on the octagon-square tiling that can be constructed by varying the motif placed in the octagons.

following this procedure guarantees that the resulting design will admit an interlacing.

Figure 5(c) shows the design with the free rosette tips extended into the gaps. Here, the natural extension creates regular octagons in the interstitial regions. To complete the construction, the original tiling is removed, resulting in the design in Figure 5(d), a well-known Islamic star pattern [3, plate 48].

Given a tiling containing regular polygons and gaps, we can now construct a wide range of different designs by choosing different motifs for the regular polygons. Even when restricted to the octagon-square tiling used above, many different designs can be created. Three alternative designs appear in Figure 6. Of course, we can expand the range of this technique in the other dimension by also encoding a large number of different tilings.

The implementation currently encodes fourteen tilings from which Islamic star patterns may be produced. Some are familiar regular or semi-regular tilings [8, Section 2.1]. Some are derived by examination of well-known Islamic patterns. The remaining tilings were discovered by experimentation and lead to novel Islamic designs shown in Section 5.

4 Rendering

The output of the construction process is a planar graph. To be sure, the graph has an intrinsic beauty that holds up when it is rendered as simple line art. Historically, however, these designs were never merely drawn as lines. Islamic star patterns are typically used as a decoration for walls and floors. The faces of the planar graph are realized as a mosaic of small terracotta tiles in a style known as "Zellij". Often, the edges are thickened and incorporated into the mosaic with narrow tiles, sometimes broken up to suggest an interlacing pattern. Islamic designs can also be found carved into wood or stone and built into trellises and latticework.

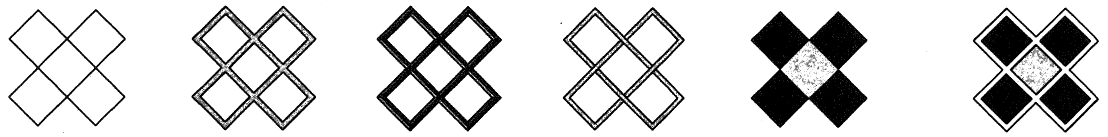


Figure 7 Rendering styles.

To increase the aesthetic appeal of our implementation, we provide the ability to render the planar graph in a manner reminiscent of some of these techniques (see Figure 7). The **outline** style thickens the edges of the planar graph, adding weight and character to the lines of the plain style. The **emboss** style adds a 3D effect to the outline style, simulating the appearance of a wooden trellis; the centre of each thickened edge is raised towards the viewer and the graph is rendered by specifying the direction of a fictitious light source. The **interlace** style adds line segments at each crossing to suggest an over-under relationship between the crossing edges. When every vertex in the graph has degree two or four, the crossings can always be chosen so that the graph is broken into strands that adhere to a strict alternation of over and under in their intersections with other strands. The final style, **checkerboard**, renders the faces of the graph and not the edges. When all vertices have even degree (as they must in an interlace design), it is always possible to colour the faces with only two colours in such a way that faces with the same colour never share an edge. The checkerboard style walks the graph, creating a consistent 2-colouring.

A further enhancement can be achieved by layering one of the edge-based rendering styles on top of the checkerboard style. This combination comes closest to the appearance of Zellij.

5 Results

Figures 8 and 9 present a selection of finished computer-generated drawings. The first group, Figure 8, is made up of reproductions of well-known Islamic star patterns which can be found in Bourgoin [3] or Abas and Salman [2]. Figure 9 contains designs that do not appear in either of those sources. Three of them are based on polygonal tilings that do not seem to be used by any known designs. These last three are moderately successful, though they seem to lack the harmonious balance of the well-known designs. Still, in an artform with a thousand-year tradition, any sort of novel design is certainly of interest.

6 Future Work

Our software implementation and the technique on which it is based allow access to a wide variety of designs without offering so much flexibility that it becomes overly easy to wander out of the space of recognizably Islamic patterns. There are, however, opportunities for future work that do not compromise the focus of the system.

The set of available tilings from which to form patterns is open-ended. More tilings could be implemented. Some new ones can easily be derived by inspection of patterns in Bourgoin or Abas and Salman. We could move away from periodicity by implementing aperiodic tilings with regular polygons. Castera has constructed several ingenious aperiodic Islamic star pattern based on Penrose rhombs [4]. Finally, the hyperbolic plane offers tremendous freedom in the construction of tilings with regular polygons. We hope

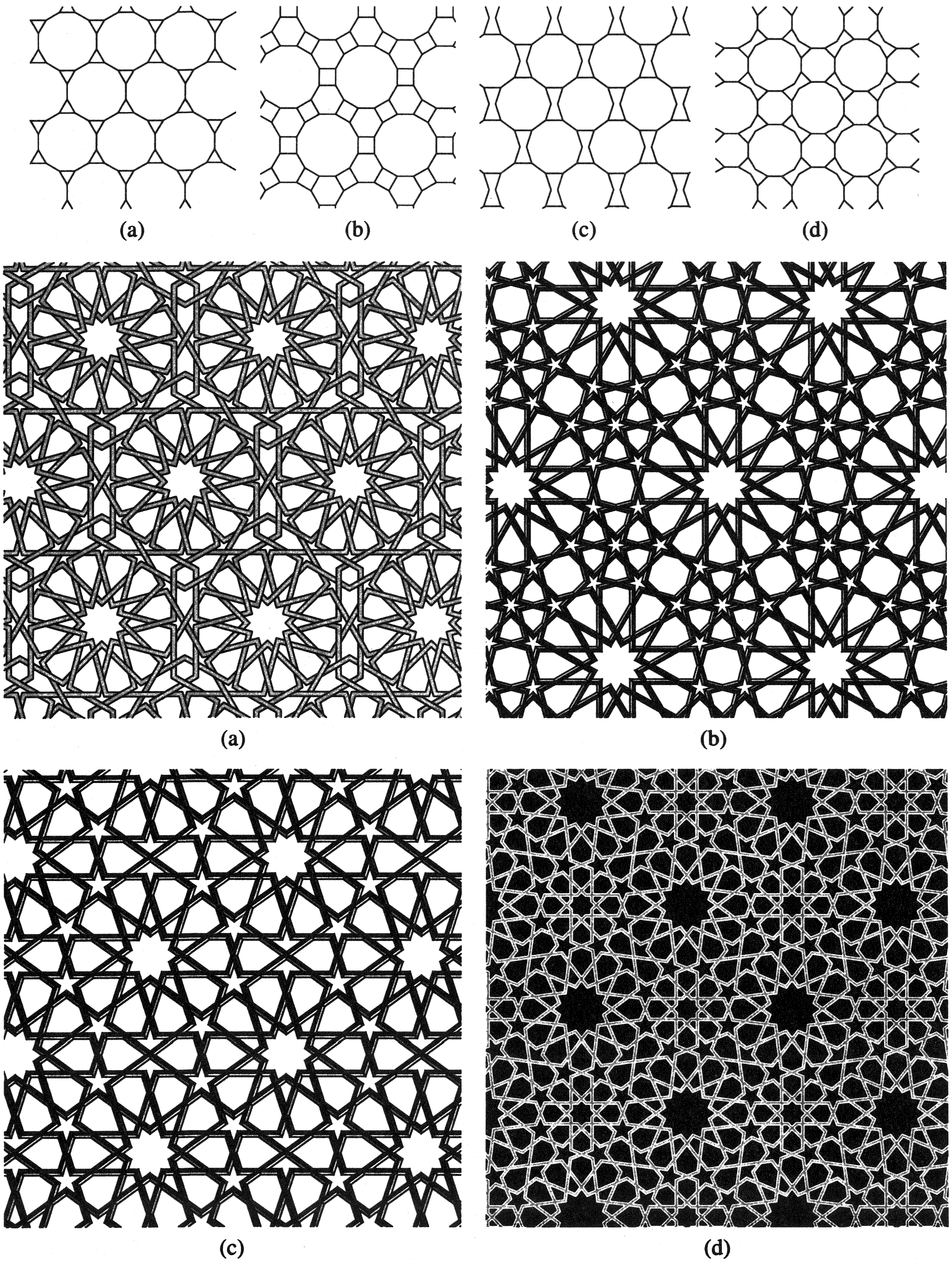


Figure 8 Some sample results based on well-known tilings from Islamic ornament. Each final design is based on the corresponding tiling in the top row.

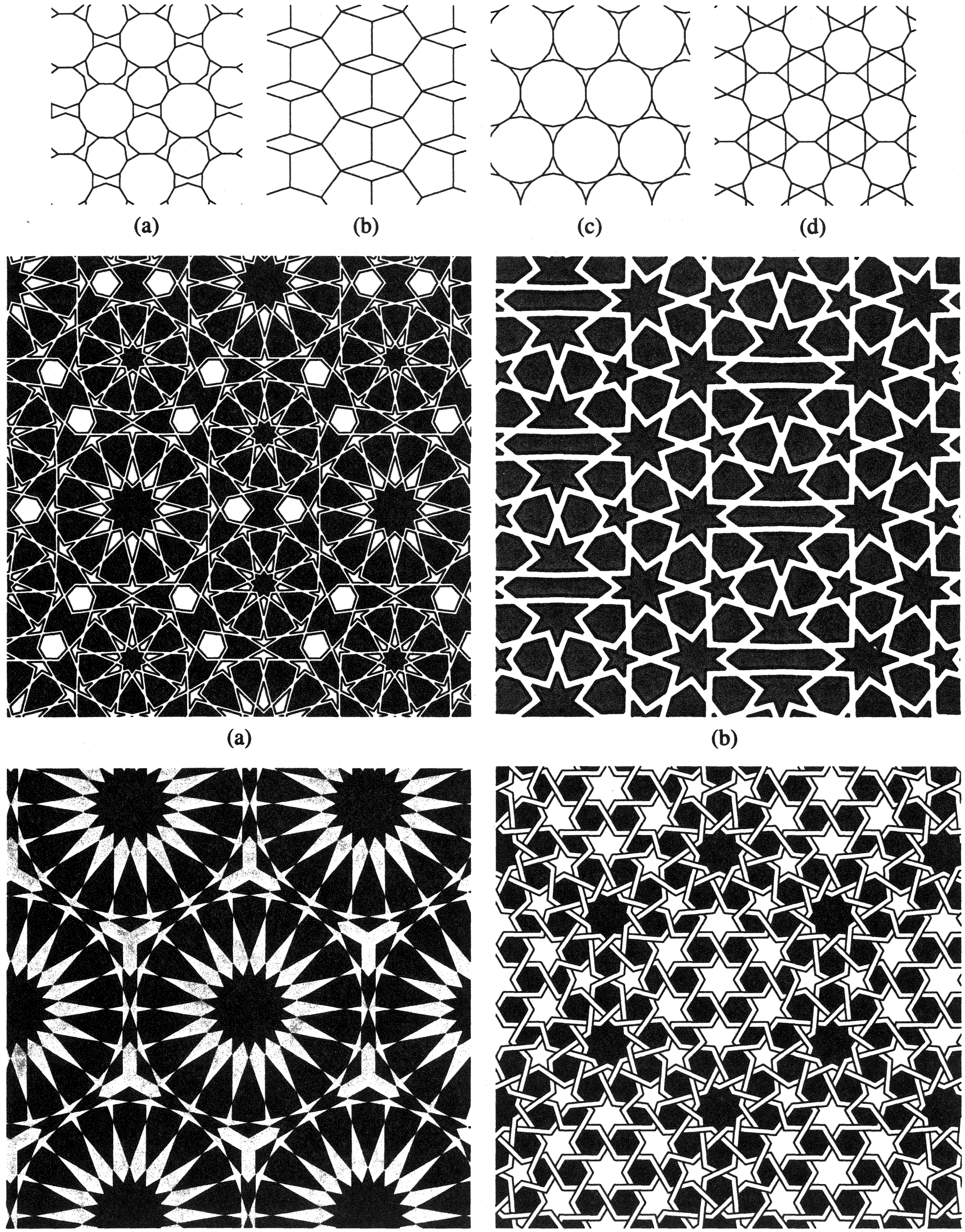


Figure 9 Sample results not found in the literature. The pattern in (a) is similar to one found in Abas and Salman [2, p. 93], using extended rosettes instead of ordinary rosettes. The other three patterns are based on previously unused tilings.

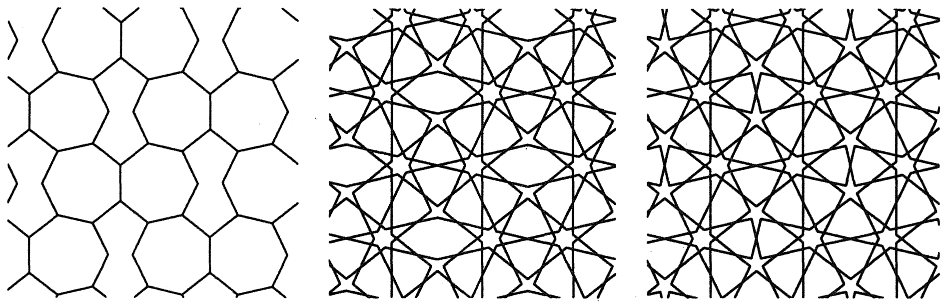


Figure 10 A novel pattern with 7-stars. The design in the centre, resulting from the natural extension of star edges, leaves behind disproportionate octagons. The design on the right, constructed manually, corrects this by redistributing the area to new 5-stars.

to adapt the technique described in this paper to the Poincaré model of the hyperbolic plane, much the same way Dunham has done with Escher patterns [6].

One last aspect of the system we hope to improve is the naïve extension of lines into interstitial regions. Our algorithm can easily fail to produce attractive results. In Figure 10, a novel grid based on regular heptagons is turned into an Islamic pattern by placing $(7/3)2$ stars in the heptagons. The natural extension of star edges into the gaps leaves large, unattractive octagonal areas. With the appropriate heuristics, our inference algorithm could detect cases such as this and add some complexity to the inferred geometry in order to improve the final design.

Acknowledgments

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