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Mathematical Connections
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Symmetry and Ornament

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Abstract

After the symmetry analysis of the Paleolithic and Neolithic ornamental art, it is given the evidence of different symmetry and antisymmetry groups that originated from it, and are preserved in the entire history of ornamental art as a kind of "ornamental archetypes".

Introduction

Throughout history there were always links between geometry and the art of painting. These links become especially evident when to the study of ornamental art we apply the theory of symmetry. Therefore, ornamental art is called by H. Weyl [39] "the oldest aspect of higher mathematics given in an implicit form" and by A. Speiser the "prehistory of group theory".

The idea to study ornaments of different cultures from the point of view of the theory of symmetry, given by G. Pólya [27] and A. Speiser [34], was supported by the intensive development of the theory of symmetry in the 20th century. This caused the appearance of a whole series of works dedicated mostly to the ornamental art of ancient civilizations, to the cultures which contributed the most to the development of ornamental art (Egyptian, Arab, Moorish, etc.) [2,14,16,17,25,37], and to the ethnical ornamental art [9,10,11]. Only in some recent works, research has turned to the very roots, the origins of ornamental art– to the ornamental art of the Paleolithic and Neolithic [22]. The Figures in this paper are adapted from the more extensive collection appearing in [22], where the reader will also find more detailed reference to their specific Paleolithic and Neolithic sources. The extensions of the classical theory of symmetry– antisymmetry and colored symmetry, made possible the more profound analysis of the "black-white" [19,20,39] and colored ornamental motifs in the ornamental art of the Neolithic and ancient civilizations.

This work gives the results of the symmetry analysis of Paleolithic and Neolithic ornamental art. It is dedicated to the search for "ornamental archetypes"– the universal basis of the complete ornamental art. The development of ornamental art started together with the beginnings of mankind. It represents one of the oldest records of human attempts to note, understand and express regularity– the underlying basis of any scientific knowledge.

The final conclusion is that most of the ornamental motifs, which have been discussed from the standpoint of the theory of symmetry, are of a much earlier date than we can expect. This places the beginning of ornamental art, the oldest aspect of geometric cognition, back to several thousands years before the ancient civilizations, i.e. in the Paleolithic and Neolithic.

Since ornamental art is mostly limited to the two-dimensional plane presentation of ornamental motifs, the subject of this art, regarded from the point of view of the theory of symmetry, are the plane symmetry groups: symmetry groups of rosettes, friezes and ornaments. The discrete symmetry groups of rosettes consist of two infinite classes: cyclic groups and dihedral groups. A cyclic group $\mathbf{C_n}$ is generated by the rotation of order *n.* A dihedral group $\mathbf{D_n}$ is generated by two reflections in lines crossing in the invariant point– the center of rotation of order *n*. Seven discrete frieze symmetry groups can be denoted by symbols: **11, 1g, 12, m1, 1m, mg** and **mm.** In these concise symbols, the translation symbol **p** is omitted, **g** denotes glide reflection, **m** reflection, and **n** (n=1,2) a rotation of order *n*. All the symbols are treated in the coordinate sense. The elements of symmetry at the first position are perpendicular to the translation axis, and the elements of symmetry at the second position are parallel or perpendicular (exclusively for **2**-rotations) to the direction of the translation.

Analogously, in the symbols of symmetry groups of ornaments, the symbol **p** denotes a two-dimensional translation subgroup, while the symbols **m**, **g**, **n** (n=2,3,4,6) have respectively the same meaning as in the case of symmetry groups of friezes. When we talk about the continuous groups of symmetry of friezes, the presence of a continuous translation is denoted by a subscript 0, while in the antisymmetry groups, antigenerators are denoted by '. Antisymmetry groups are presented also by group/subgroup symbols *G*/*H* [30].

By the term "prescientific period" we understand the Paleolithic and Neolithic epochs, covering the period from 25000-10000 B.C., till the end of the IV millenium B.C., when we have the signs of first alphabets.

In the absence of written sources, the study of geometry of the prehistoric period is based on the analyses of artifacts, which offer information on geometric knowledge in an implicit form. Among the artifacts mentioned we distinguish few kinds of them. The oldest ones are ornamental motifs realized in the form of bone engravings, carvings and drawings on stone from the Paleolithic and Neolithic. Later we have ornamental motifs in ceramics from the Neolithic phase, obtained by engraving, pressing, drawing or coloring, as well as architectural objects and constructions from the Neolithic period, so called megalithic monuments.

Rosettes

The simplest ornamental motifs are rosettes, symmetrical figures with an invariant point, that correspond to the symmetry groups $\mathbf{C_n}$ and $\mathbf{D_n}$. They are denoted in Shubnikov's notation by **n** and **nm**, respectively [30].

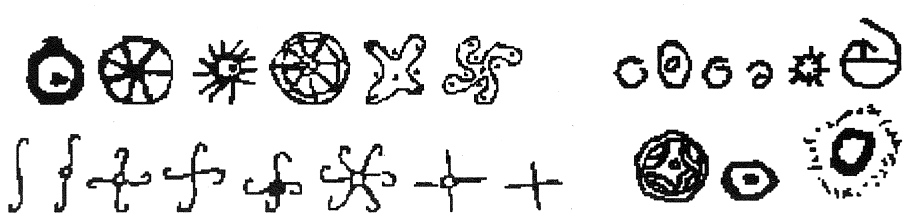


Figure 1. *Variations of the Sun symbol in the ornamental art of the Paleolithic and Neolithic.*

The continuous symmetry group of rosettes $\mathbf{D_\infty}$ (**∞m**) corresponds to the maximal symmetric rosette– a circle. Due to the maximal visual and constructional simplicity and maximal symmetry, a circle represents the primary geometric shape– geometric archetype. Within ornamental art it appears in the Paleolithic, as an independent rosette or in combination with some concentric rosette of a lower degree of symmetry, usually circumscribed or inscribed in a circle. Since the group $\mathbf{D_\infty}$ (**∞m**) contains all the other groups of symmetry of rosettes as subgroups, rosettes of a lower degree of symmetry are often derived by a desymmetrization of a circle. Owing to its visual-geometric properties: completeness, compactness, boundedness and uniformity of its structural segments, the circle may serve as a universal symbol of completeness and perfection. At the very beginning of ornamental art, the circle becomes the symbol of the Sun, remaining that throughout history (Figure 1).

The continuous symmetry group of rosettes $\mathbf{C}_\infty$ (∞) is the group of all rotations around a fixed point. A physical interpretation of it could be a circle uniformly rotating around the center, so in the static form this symmetry group is visually interpretable only by use of textures [31]: by using an asymmetric figure, statistically distributed in accordance with the desired symmetry $\mathbf{C}_\infty$ (∞).

The spiral is one of the oldest dynamic visual symbols. In the visual sense it suggests the rotational motion around the invariant point, and could be accepted as an adequate symbolic interpretation of the continuous symmetry group $\mathbf{C}_\infty$ (∞). In ornamental art, the spiral appeared already in the Paleolithic, as an independent ornamental motif, or in the form of a double spiral– a motif with symmetry group $\mathbf{C}_2$ **(2)** generated by two-fold rotation.

Among the elementary geometric forms we have a line segment, usually placed in accordance with the basic natural directions– vertical and horizontal line. To a line segment corresponds the symmetry group $\mathbf{D}_2$ **(2m)**, generated by two reflections: one in the mirror line perpendicular, and the other in the reflection line collinear to the line segment. However, from the point of view of visual perception, due to the action of the visual and gravitational dominant, the vertical line, we visually experience the symmetry of a line segment as $\mathbf{D}_1$ **(m)**. In this case, the horizontal reflection is neglected. The combination of the vertical and horizontal line segment results in the cross form with symmetry group $\mathbf{D}_1$ **(m)**, $\mathbf{D}_2$ **(2m)** or $\mathbf{D}_4$ **(4n)**. Rosettes with symmetry $\mathbf{D}_2$ **(2m)** and $\mathbf{D}_4$ **(4m)** possess another fundamental property: the existence of mutually perpendicular, vertical and horizontal reflection lines. The form of a cross with symmetry group $\mathbf{D}_4$ **(4m)** is often subjectively, visually perceived as the symmetry $\mathbf{D}_2$ **(2m)**, neglecting the presence of four-fold rotation.

Static rosettes with symmetry group $\mathbf{D}_1$ **(m)** or $\mathbf{D}_2$ **(2m)** are linked to the plane symmetry of a man, its vertical attitude and perpendicularity to the base. Besides the rational mirror symmetry, which originated from motifs in nature, we have in the ornamental art different aspects of symbolic symmetry $\mathbf{D}_1$ **(m)**: the duplicated figures, two-headed animals, etc. These examples result mostly from the common use of vertical mirror symmetry as a visual dominant.

In Paleolithic ornamental art we have also rosettes with the symmetry group $\mathbf{D}_n$ **(nm)**: $\mathbf{D}_3$ **(3m)**, $\mathbf{D}_4$ **(4m)** and $\mathbf{D}_6$ **(6n)**, as well as the corresponding regular polygons: equilateral triangle, square and regular hexagon (Figure 2). For rosettes, the principle of crystallographic restriction (n=1,2,3,4,6) is not respected. Anyway, prevailing are rosettes with the symmetry group $\mathbf{D}_n$ **(nm)** for the mentioned values of n. In the later stage, in Neolithic we have also rosettes with the symmetry group $\mathbf{D}_5$ **(5m)** with the use of regular pentagon and pentagram. The first appearance of pentagram is dated by H.S.M. Coxeter [7, pp. 8] in the VII century B.C. The visual characteristics of rosettes with the symmetry group $\mathbf{D}_n$ **(nm)** are stability, stationariness and absence of enantiomorphism. Enantiomorphism, the existence of a "right" and "left" modification of the same figure, appears with all figures possessing a symmetry group that does not contain indirect symmetry transformations.

In contradistinction to the static rosettes with the symmetry group $\mathbf{D}_n$ **(nm)**, rosettes with the symmetry group $\mathbf{C}_n$ **(n)** (e.g. triquetra with the symmetry group $\mathbf{C}_3$ **(3)**, swastika with the symmetry group $\mathbf{C}_4$ **(4)**) are visually dynamic rosettes. There exists the possibility for enantiomorphic modifications that suggest the impression of rotational motion (Figure 2).

In the next stage of the development of ornamental art, in Neolithic, after understanding the symmetry regularities on which the symmetry of rosettes is based and solving their elementary geometric constructions, the diversity of rosettes increases. This is followed by the application of plant and zoomorphic motifs and by varying the form of the fundamental region. Also, the superpositions of concentric rosettes resulting in a desymmetrization– a reduction to a lower degree of symmetry, are very common.

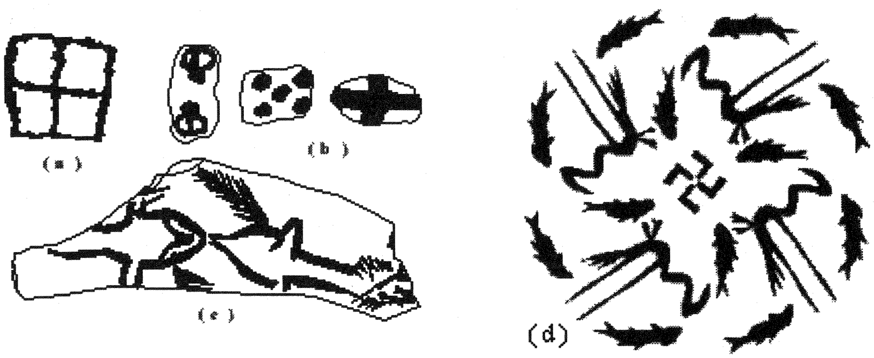


Figure 2. *Examples of rosettes with symmetry group* C_n ***(n)*** *and* D_n ***(nm)*** *in the ornamental art of the Paleolithic and Neolithic: (a) paleolithic of France,* D_4 ***(4m)****; (b) Maz d' azil,* D_2 ***(2m)*** *and* D_4 ***(4m)****; (c) Laugerie Basse,* C_2 *(2); (d) Neolithic ceramics of Middle Asia,* C_4 *(4), around 6000 B.C.*

In the Neolithic, with two-colored ceramics, we have the antisymmetric "black-white" rosettes (Figure 3). In this case, antisymmetry can be treated either as the mode of desymmetrization for obtaining the subgroups of index 2 of a given symmetry group, or as an independent form of symmetry. In the table of antisymmetry groups, every group is denoted by the group/subgroup symbol *G*/*H* [30] and followed by a system of (anti)generators. The factor-group *G*/*H* is isomorphic to a cyclic group of order 2– the group of color change "black"-"white". Hence, we have the following antisymmetry groups of rosettes: $\mathbf{D_{2n}/D_n}$ **(2nm/nm) = (2n)'m;** $\mathbf{D_n/C_n}$ **(nm/n) = nm';** $\mathbf{C_{2n}/C_n}$ **(2n/n) = (2n)'**.

In the case of antisymmetry groups, there is a possibility for interpreting the color change "black"-"white" as the alternating change of some physical or geometric bivalent property. In ornamental art color change mentioned introduces a space component, a suggestion of relations "in front"-"behind', "'up"-"down", "above"-"below". From the artistic point of view, it introduces the contrast between repeating congruent figures and specific equivalence of the "figure" and "background", thus expressing in a symbolical sense a dynamic conflict and duality.

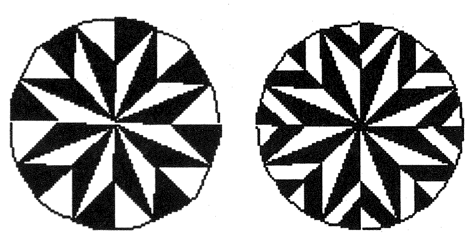


Figure 3. *Neolithic antisymmetry rosette* D_8/C_8*, Hajji Mohammed, around 5000 B.C.*

In ornamental art the use of color in the sense of regular coloring, i.e. antisymmetry and colored symmetry, opened and a large unexplored field. Hence, in the history of ornamental art, we can consider the Neolithic as its peak, a period in which after solving the basic technical and constructional problems, new possibilities for artistic research, imagination, variety of motifs and decorativeness were opened.

Friezes (one-dimensional patterns)

In the late Paleolithic (Magdalenian, about 25000-10000 B.C.) we find the oldest examples of the symmetry groups of friezes, plane symmetry groups without invariant points and with invariant line. We have the examples of all seven symmetry groups of friezes: **11, 1g, 12, ml, lm, mg, mm**, as well as two visually presentable continuous symmetry groups of friezes $_0$**ml** and $_0$**mm**.

Friezes are usually obtained by applying the rosettal method of construction, translational multiplication of an initial motif– a rosette, the symmetry of which directly conditions the symmetry of the frieze obtained. The other origin of friezes are models found in nature which, by themselves possess the symmetry of a frieze (Figure 4).

The way friezes are derived from models found in nature can be illustrated by examples: a herd of deer reduced to the frieze with the symmetry group **11**, the motif of cult-dance rendering the frieze with the symmetry group **m1**. Friezes with symmetry group **12** and **mg** can be considered as stylized waves. Models in nature with the symmetry groups **1g** and **1m** are found in the distribution of leaves of certain plants; they have served as the pretext for the construction of corresponding friezes in ornamental art. The importance of the plane symmetry in nature and the numerosity of rosettes with the symmetry group $\mathbf{D_1}$ **(m)** and $\mathbf{D_2}$ **(2m)** caused the appearance and frequent occurrence of friezes with symmetry group **mm**. These friezes can be derived by a translational multiplication of a rosette with the symmetry group $\mathbf{D_2}$ **(2m)**, where the translation axis is parallel with one reflection line of the rosette. The symmetry group of friezes **mm** is the maximal discrete group of symmetry of the friezes, generated by reflections. All the other symmetry groups of friezes are subgroups of the group **mm**. Hence the group **mm** can serve for derivation of all other symmetry groups of friezes by desymmetrization. Examples of all discrete frieze symmetry groups are found in Paleolithic ornamental art.

Besides friezes with a concrete meaning, which are based on material models found in nature, the appearance of certain friezes is caused also by the periodic change of many natural phenomena (the change of day and night, seasons, the tides, phases of the Moon, etc.). The corresponding friezes represent, at the same time, the oldest attempt to register the periodical change of natural phenomena, i.e. the first form of calendars. These friezes can also be understood as a way to register quantities, serving as tally boards, thus indicating the beginning of counting and recording the results of counting, i.e. the appearance of the set of natural numbers.

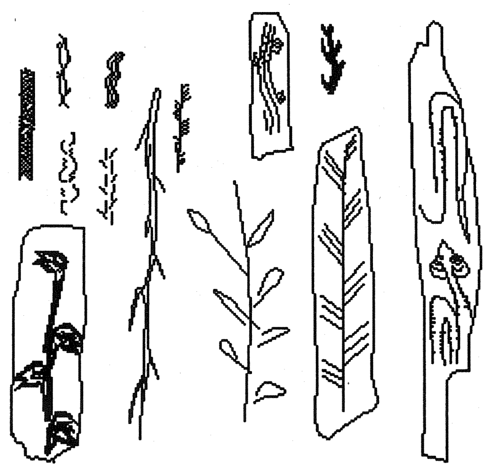


Figure 4. *Examples of frieze symmetry group **1g** in the ornamental art of the Paleolithic.*

Thanks to their symbolic meaning, certain "geometric" friezes became the means of visual communication. This is proved by the preserved names of friezes in the ethnical ornamental art [32]. This communication role of friezes, established in the Paleolithic, was partly preserved in the Neolithic. With the development of other communication forms, friezes lost their primary symbolic function, which was partly or completely replaced by their decorative function. The beginning of this process can be registered already in the Neolithic ornamental art.

The polarity, non-polarity and bipolarity of the translation axis of the friezes, the presence or absence of enantiomorphism implied by the presence or absence of indirect symmetries within the frieze symmetry group, etc. [22], represent some of the relevant geometric properties deserving more detailed geometrical consideration. At the same time, they define the visual characteristics of the friezes, thus conditioning also the spectrum of symbolic meanings which friezes with certain symmetry groups may possess.

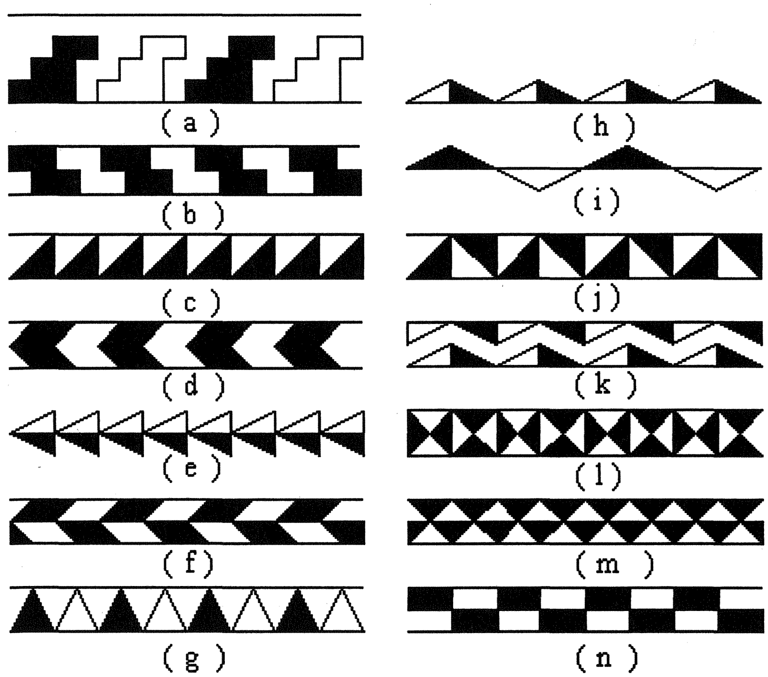


Figure 5. *Examples of 14 antisymmetry groups of friezes in Neolithic ornamental art: (a) Greece,* ***11/11****, about 3000 B.C.; (b) Greece,* ***12/12****; (c) Near East,* ***12/11****, about 5000 B.C.; (d) Near East,* ***1m/1m****, about 5000 B.C.; (e) Near East,* ***1m/11****; (f) Anatolia,* ***1m/1g****, around 5000 B.C.; (g) Near East,* ***m1/m1****; (h) Near East,* ***m1/11****; (i) Greece,* ***mg/m1****; (j) Near East,* ***mg/1g****, about 5000 B.C.; (k) Anatolia,* ***mg/12****; (l) Tell el Hallaf,* ***mm/mm****, about 4900-4500 B.C.; (m) Hacilar,* ***mm/m1****, about 5500-5200 B.C.; (n) Near East,* ***mm/mg****.*

With regard to the frequency of occurrence, besides friezes originating directly from models found in nature, in the ornamental art of the prescientific period, friezes which satisfy the criterion of visual entropy [22]: maximal visual and constructional simplicity and maximal symmetry, are dominant.

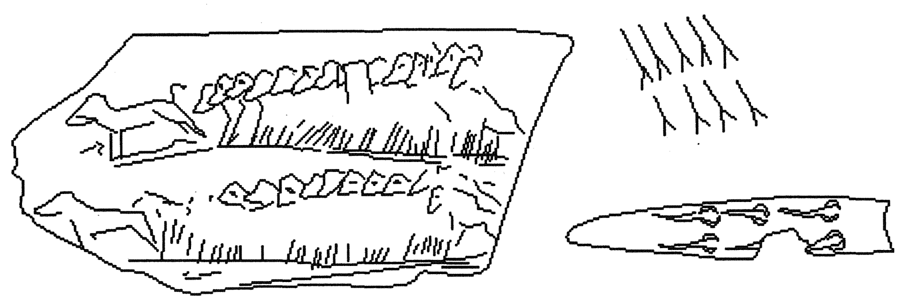
The oldest examples of antisymmetry friezes, so called "black-white" friezes, date back to the Neolithic epoch, in which we have the examples of the most of the 17 antisymmetry groups of friezes. Further investigations should show whether or not from that period originate examples of all the 17 antisymmetry groups of friezes. With regard to the frequency of occurrence, the most numerous are "black-white" friezes derived from the most frequent classical-symmetry friezes by the use of antisymmetry desymmetrization method (Figure 5).

The frequency of occurrence of antisymmetry friezes depends also on the antisymmetry properties. Therefore, more frequent are antisymmetry friezes with oppositely colored adjacent fundamental regions. A domination of "geometric" antisymmetry friezes over antisymmetry friezes inspired by models found in nature is also evident, due to the absence of antisymmetry in nature among models from plant and animal life. In contradistinction to this, many natural alternating phenomena followed by bivalent changes (e.g. the change of day and night, etc.), already in Neolithic ornamental art are represented by antisymmetry friezes.

There is a possibility to treat the antiidentity transformation of order 2 (color change "black"-"white") as the way to represent in a plane the space symmetry structures– bands (three-dimensional symmetry groups with invariant plane and line contained in it, and without invariant points). The 31 antisymmetry groups of the friezes (7 generating + 7 senior + 17 junior antisymmetry groups) correspond to the 31 symmetry groups of bands. From the artistic point of view, that gives a possibility to suggest space in a flat plane of drawing. Besides this possibility, there are also many different geometric or non-geometric interpretations of the antiidentity transformation. In prehistoric ornamental art a primary symbolic function of "black-white" friezes, is evident.

Ornaments (two-dimensional patterns)

In the theory of symmetry and ornamental art, the most interesting field of study are the 17 groups of symmetry of ornaments, two-dimensional symmetry groups without invariant lines and points. The common characteristic of ornaments is the presence of discrete two-dimensional translation subgroup, generated by two independent translations. The difficulty of discovering and constructing examples of all the 17 symmetry groups of ornaments is shownby the fact that many cultures with a very rich ornamental art do not possess within their early ornamental art the examples of all these groups [16,17]. The same is proved by the fact that in the mathematical studies of symmetry, the complete derivation of the symmetry groups of ornaments can be found only in 1890, in the works of E.S. Fedorov, although this problem attracted also many other important mathematicians (for example, C. Jordan, L. Sohncke).



(a) (b)

Figure 6. *Examples of ornaments with the symmetry group **p1** in Paleolithic ornamental art: (a) Chaffaud; (b) bone engravings, Europe.*

This is the reason why it is rather surprising that already in the ornamental art of the Paleolithic we can find examples of the nine symmetry groups of ornaments: **p1**, **p2**, **pm**, **pmm**, **pmg**, **cm**, **cmm**, **p4m** and **p6m** [22]. In the Neolithic phase we have the appearance of five other symmetry groups of ornaments: **pg**, **pgg**, **p4**, **p4g** and **p6**. Examples of symmetry groups of ornaments **p3**, **p3m1** and **p31m** can be found in the early ornamental art of ancient civilizations, and probably also in the late Neolithic.

According to the stated presence of the corresponding symmetry groups **p4m** and **p6m** in the Paleolithic, all three regular tessellations: {4,4} with symmetry group **p4m**, {6,3} and {3,6} with symmetry group **p6m**, have been known. Besides the regular square, hexagonal and triangular lattices, in Paleolithic ornamental art we find the remaining two Bravais lattices: the lattice of parallelograms with symmetry group **p2** and the rhombic lattice with symmetry group **cmm**.

In the ornamental art of the Paleolithic and Neolithic, with regard to the construction methods used in obtaining the ornaments we distinguish four construction methods: multiplication of the friezes, multiplication of the rosettes, the method of Bravais lattices and the desymmetrization method. The first construction method is based on the translational repetition of a certain frieze by means of a discrete translation, non-parallel to a frieze axis. Because of the simplicity of this construction, and because of the existence of early examples of all seven discrete symmetry groups of friezes, this method was probably often used for the construction of ornaments. In the Paleolithic, it is probably used for the construction of ornaments with symmetry group **p1**, **p2**, **pm**, **(pg)**, **pmg** and **pmm**. The similar rosette method of construction is based on the multiplication of a rosette by two independent discrete translations. The symmetry of the ornament obtained is completely defined by the properties of these translations and by the symmetry group of the rosette. The appearance of the Bravais lattices in the Paleolithic and Neolithic ornamental art originates from the models in nature (e.g. honeycomb, different net structures). Another cause is a very high degree of visual and constructional simplicity of the Bravais lattices. The most frequent Bravais lattices, regular tessellations {4,4}, {6,3} and {3,6}, to which correspond the maximal symmetry groups of ornaments **p4m** and **p6m** generated by reflections, have often served as the basis for the application of the desymmetrization method. The importance of this construction method increases especially with the appearance of (two) colored ceramics in the Neolithic, i.e. with the beginning of antisymmetry and colored symmetry ornaments. All these construction methods probably were used in the ornamental art of the prescientific period.

Since they point to the very roots of ornamental art, ornaments from the Paleolithic, realized as bone engravings or stone carvings and drawings, deserve special attention.

Ornaments with the symmetry group **p1** are based on the multiplication of a frieze with the symmetry group **11** by a discrete translation, or on the multiplication of an asymmetric figure by two discrete translations. Because of a low degree of symmetry, they occur relatively seldom, and most often appear with stylized asymmetric models found in nature (Figure 6).

Ornaments with the symmetry group **p2** appear in the most elementary form: as a lattice of parallelograms. A highlight of Paleolithic ornamental art are ornaments with the application of the meander motif or double spiral, the rosette with symmetry group C_2 (**2**), which originates most probably from the territory of the Ukraine and Russia (Mezin, Mal'ta). That motifs will be, later on, often used in the ornamental art of almost all Neolithic cultures, mostly as a variation of the motif of waves. Because the forms with symmetry group **p2** are very rare in nature, ornaments with symmetry group **p2** are almost completely limited to geometric motifs or to symbolic stylized motifs (Figure 7).

Ornaments with the symmetry group **pm**, due to the presence of the reflections, belong to the class of static ornaments. Besides geometric motifs, there is a frequent use of models with mirror plane symmetry that are found in nature.

Although according to [22] no examples of ornaments with symmetry group **pg** have been found in Paleolithic, there are grounds to believe that they do appear in Paleolithic ornamental art, as there are examples of the frieze symmetry group **1g**.

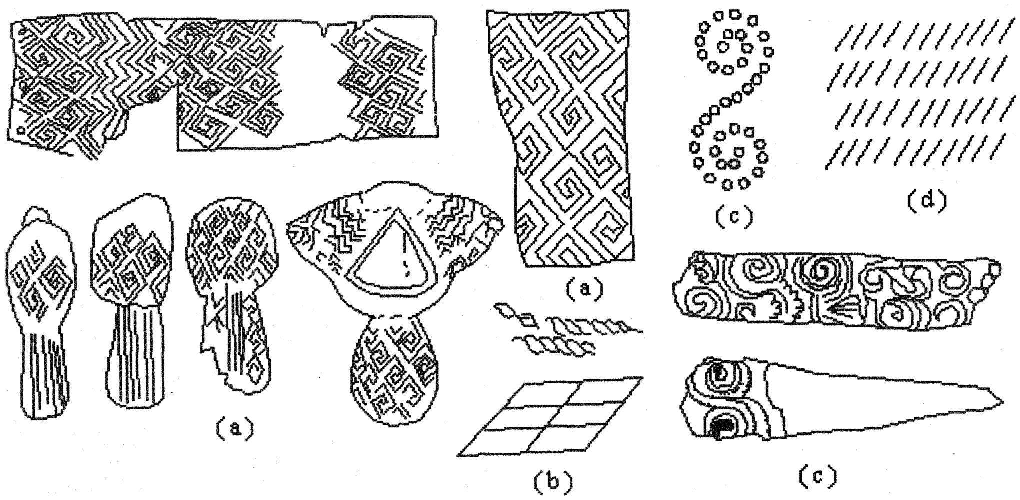


Figure 7. *Examples of ornaments with the symmetry group* **p2** *in Paleolithic ornamental art: Mezin, Ukraine, about 23000-18000 B.C.; (b) the Paleolithic of Western Europe; (c) the motif of double spiral, Mal'ta, Russia; (d) the application of the motif of double spiral, Arudy, Isturiz.*

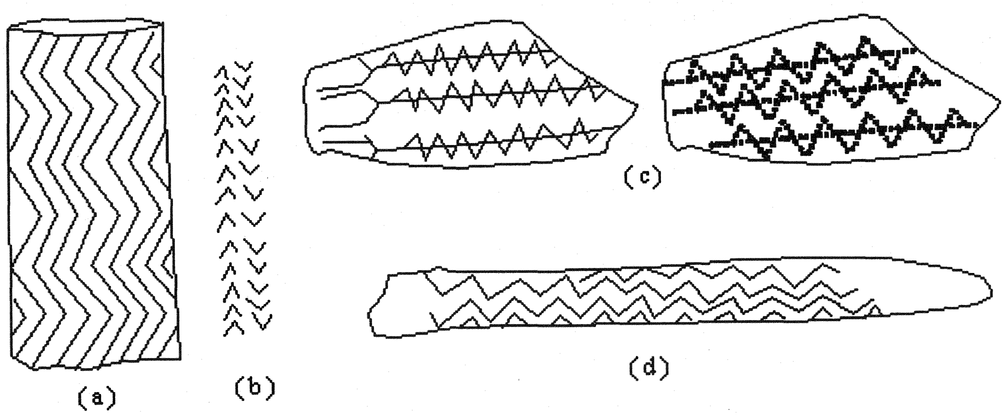


Figure 8. *Examples of ornaments with the symmetry group* **pmg** *in Paleolithic ornamental art: (a) Mezin, Ukraine; (b) Western Europe; (c) Pernak, Estonia; (d) Shtetin.*

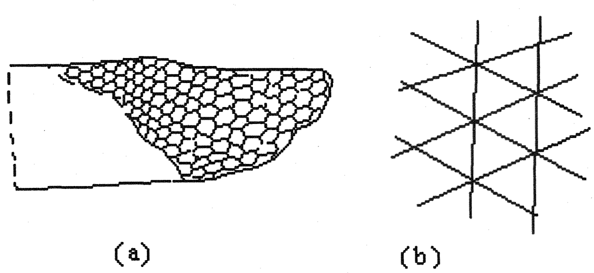


Figure 9. *Examples of the regular tessellations with the symmetry group* **p6m***: (a) {6,3}, Yeliseevichi, Russia, 10000 B.C.; (b) regular tessellation {3,6}.*

Regarding the frequency of occurrence and their variety in Paleolithic and Neolithic ornamental art, ornaments with symmetry groups **pmg** and **pmm** prevail. Both of these ornaments can be obtained by the frieze method of construction, by translational multiplication of a frieze **mg** and **mm**, respectively. The ornaments with the symmetry group **pmg** appear in their primary form almost always within the geometric ornaments, as a stylization of the wave motif. The symmetry group **pmg** offers the possibility for different variations, expressing in the visual sense a specific balance between the static visual component caused by the presence of reflections and dynamic component, resulting from the presence of the glide reflection which suggests the alternating motion (Figure 8).

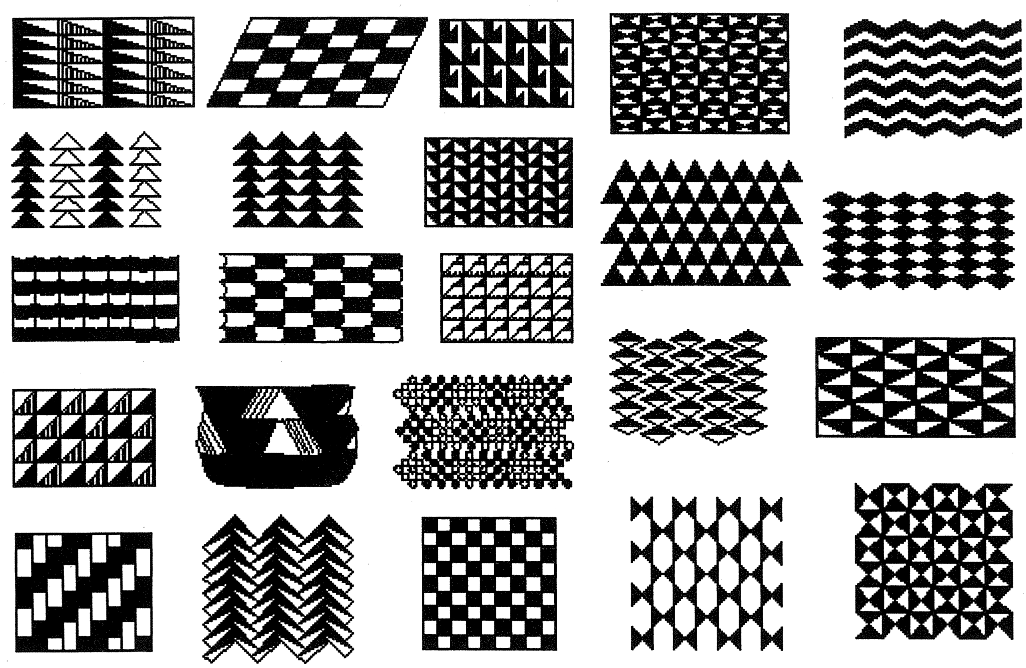


Figure 10. *The examples of 23 antisymmetry groups of ornaments in Neolithic ornamental art.*

The static ornaments **pmm** generated by reflections are realized in their earliest form as a rectangular lattice. The other forms are obtained by the multiplication of a frieze with the symmetry group **mm** by means of a translation perpendicular to the frieze axis, or by the rosette method of construction. There, a rosette with the symmetry group $\mathbf{D_2}$ **(2m)** is multiplied by means of two translations perpendicular to the corresponding reflection lines of the rosette.

Ornaments with the symmetry group **cmm** appear in the Paleolithic in the form of the rhombic lattice. These ornaments can be constructed from an ornament with the symmetry group **pmm** by centering it, i.e. by the procedure in which the gaps between the rosettes $\mathbf{D_2}$ **(2m)** forming the original ornament, are filled with the same rosettes.

The ornaments with the symmetry group **cm** are obtained from the ornaments with the symmetry group **pm** by the same procedure– by centering.

The symmetry groups of ornaments **p4m** and **p6m** correspond to the regular tessellations {4,4}, {6,3} and {3,6}. The regular tessellation consisting of regular hexagons, three of which are incident with each vertex of tessellation, most probably originates from its model in nature: the honeycomb (Figure 9). The regular tessellations {3,6} and {4,4} are from the same period, the Paleolithic.

The principle of visual entropy: maximal visual and constructional simplicity and maximal symmetry is a common, universal characteristic of all Paleolithic ornaments. Hence, among Paleolithic ornaments five of the nine existing symmetry groups of ornaments correspond to the Bravais lattices, seven of the nine groups contain reflections and belong to a class of static ornaments. In them, the almost complete absence of the dynamic elements of symmetry– polar translations, polar rotations and glide reflections, is evident.

In the Neolithic period we have the appearance of almost all the remaining symmetry groups of ornaments. The "black-white" ornaments, i.e. those having antisymmetry, have a special place in Neolithic ornamental art. Very many of the 46 antisymmetry groups of ornaments appear in the Neolithic ornamental art, in particular in the ornamental art of the Near and Middle East (Tel el Hallaf, Hacilar, Catal Hüjük). If we treat antisymmetry ornaments with the antisymmetry group **p6m/p3m1** as the classical-symmetry ornaments obtained by the method of antisymmetry desymmetrization, we can add to the list of symmetry groups of ornaments appearing in the Neolithic, also the symmetry group **p3m1**.

Neolithic ornamental art is one of the richest sources of different ornaments in all the history of ornamental art. The examples of the 14 symmetry groups of ornaments and 23 antisymmetry groups of ornaments (Figure 10) found in Neolithic ornamental art are the most complete testimony about the artistic creativity of Neolithic peoples.

Ornaments with the symmetry group **p3**, **p3m1** and **p31m** represent quite a problem with regard to their construction. In classical-symmetry sense, they first appear in the ornamental art of ancient civilizations, or maybe earlier, in late Neolithic ornamental art.

Very interesting and insufficiently explored fields related to the geometry of the prescientific period are still the following: dating of the appearance of all the plane symmetry structures and corresponding classical-symmetry, antisymmetry and color-symmetry groups, the registering of the most significant archaeological excavation sites from the point of view of the theory of symmetry and ornamental art, the links between the ornamental art of different cultures, the links between the friezes, natural numbers and calendars, etc. All these and many other similar questions relevant to the history of mathematics of the prescientific period should become a common field of research for mathematicians, archaeologists and specialists of different sciences.

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