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Polyhedral Sculptures with Hyperbolic Paraboloids

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Abstract

This paper describes the results of our experiments with gluing together partial hyperbolic paraboloids, or *hypars*. We make a paper model of each hypar by folding a polygonal piece of paper along concentric polygons in an alternating fashion. Gluing several hypars together along edges, we obtain a beautiful collection of closed, curved surfaces which we call *hyparhedra*. Our main examples are given in Figure 6.

We present an algorithm that constructs a hyparhedron given any polyhedron. The surface represents each face of the polyhedron by a "hat" of hypars. For a Platonic solid, the corners of the hypars include the vertices of the polyhedron and its dual, and one can easily reconstruct the input polyhedron from the hyparhedron. More generally, the hyparhedron captures the combinatorial topology of any polyhedron. We also present several possibilities for generalization.

1. Introduction

A *hyperbolic paraboloid* (shown in Figure 1) is a beautiful infinite surface discovered in the 17th century. This paper presents an original approach to combining hyperbolic paraboloids with polyhedra. In addition, we create our models using origami (paper folding) techniques. This uniqueness of representing the hyperbolic paraboloid curves in paper adds to the textural beauty. The richness of these forms excites us visually, and presents us with interesting mathematical problems.

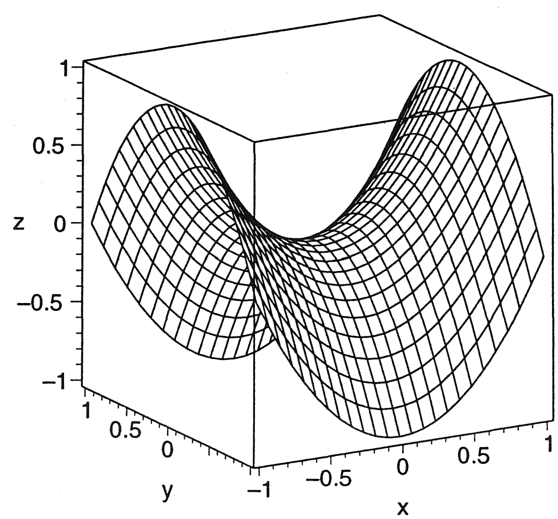

Figure 1: A hyperbolic paraboloid $z = x^2 - y^2$.

Hyperbolic paraboloids have been used sculpturally by many architects since the 1950's [14, p. 252]. Two examples are Philips pavilion at the 1958 Brussels exhibition, and the roof of the Girls' Grammar School in London, which are described briefly in Section 4.1.

Mathematically, a hyperbolic paraboloid is defined by the equation

$$z = x^2 - y^2.$$

The name "hyperbolic paraboloid" comes from the property that the xy cross-sections are hyperbolas, and the yz cross-sections are translated copies of a common parabola P. Note also that the zx cross-sections are translated upside-down copies of the same parabola P. Another way to look at the surface is as a saddle formed by a continuous family of yz parabolas hanging on a pair of zx parabolas.

We use the term *hypar* to mean a hyperbolic paraboloid shape, or more formally a partial hyperbolic paraboloid, cut from the full infinite surface. The term hypar was introduced by the architect Engel [6, p. 215].

This paper explores the joining of hypars by gluing them edge-to-edge, in order to form curved, closed surfaces which we call *hyparhedra*. Section 2 describes how we make paper models of hypars. In Section 3, we describe general models of joining hypars. Section 4 presents our main result: an algorithm to convert a polyhedron into a hyparhedron that "represents" its combinatorial topology, resulting in a wonderful variety of symmetrical, curved sculptures. Section 5 presents a more general algorithm that may also lead to interesting sculptures. Finally, we conclude in Section 6.

2. Folding Hypars

A major motivation for this work was the ease by which a pleated hypar form can be made by folding a square piece of paper. Remarkably, if one folds the diagonals of a square, and several concentric squares in alternating direction (a square of mountain folds, then a square of valley folds, and so on), then the piece of paper naturally forms a pleated hyperbolic paraboloid shape. See Figure 2. The piece of paper can also be collapsed into an "X" shape, which is helpful for reinforcing the creases.

The crease pattern for a pleated hypar is easy to fold when the number of concentric squares is a power of two. For example, the case of eight concentric squares is shown in Figure 2. Every step consists of folding one known edge (e.g., the boundary of the square) to another known edge (e.g., a previously made crease).

This folding of a pleated hypar is fairly well-known in the origami (paper folding) community. Perhaps the earliest two publications are a book by Paul Jackson [8, pp. 138–141] and a diagram by Don Sigal [15]. A more accessible reference is Helena Verrill's origami web pages [16].

The pleated hypar folding was originally designed by John Emmet in England, according to Jackson [8, p. 138]. The more concentric squares one folds, the closer the pleated hypar is to a true hypar surface.

2.1. Non-Squares. We have generalized the folding of a hypar described in the previous section to other polygonal pieces of paper. The basic idea is to imagine *shrinking* the boundary polygon so that the shrunken edges remain parallel to the original edges. Fold several equally spaced shrunken copies of the polygon, as well as the line segments along which the vertices of the polygon travel during the shrinking process.

This folding can have a variety of effects, usually resulting in a hypar-like surface. If the polygon is a triangle, as in Figure 3 (left), the folding leaves the paper flat, not forming any curved

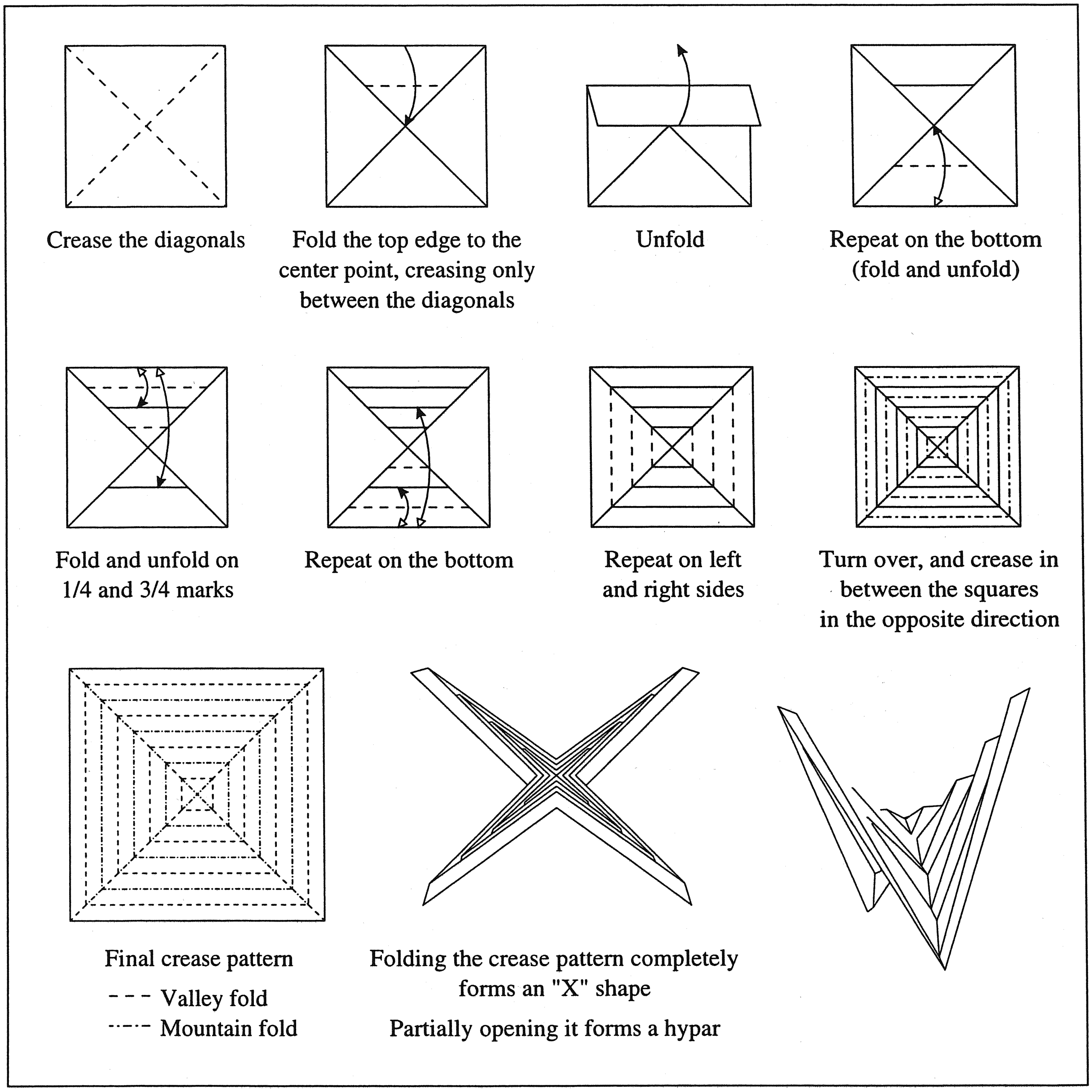

Figure 2: Folding a hypar from a square.

shape.[1] Another example is when the polygon is a pentagon, in which case it behaves similar to a quadrilateral. It is even possible to consider using non-convex polygons (polygons with "dents").

We have found convex quadrilaterals to be the best for this folding, for example the rectangle in Figure 3 (middle). An interesting special case of a quadrilateral is when one of its diagonals is a line of symmetry (a "kite"), as in Figure 3 (right). In this case, the quadrilateral shrinks in a way

[1] Although we note that this crease pattern has been used by Nolan to make a paper airplane called a Delta Glider [11, p. 210].

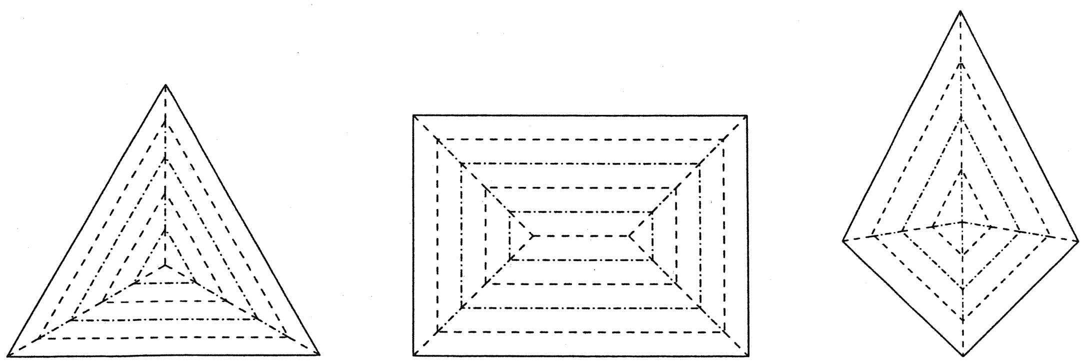

Figure 3: The generalized hypar crease pattern applied to a triangle, rectangle, and kite. The triangle crease pattern remains flat, but the others form a hypar shape.

that its vertices just travel along the angular bisectors and meet at a common point.

The advantage of quadrilaterals over squares is that it allows us to change the edge lengths of the hypars (i.e., the lengths along which the hypars are joined), permitting more irregularly shaped hyparhedra. For the majority of this paper, we will not use this level of flexibility, and instead just consider the case of a square piece of paper. Section 4.2 will consider the case of kites.

2.2. Straight Skeleton. The generalized "diagonals" (i.e., the line segments along which the vertices travel during the shrinking process) form what is known as the *straight skeleton.* This concept was independently discovered by several people, dating back to at least 1984 [10]. The term "straight skeleton" was coined by Aichholzer et al. [1, 2], who published the first formal definition. An interesting computational property of the straight skeleton is that it is surprisingly difficult to compute for non-convex polygons [7].

One of the original applications of the straight skeleton is that of designing roofs that fit on top of any given polygon, and do not collect any water [2]. The straight skeleton also has a variety of other applications in origami mathematics that we have explored [3, 4, 5], because it effectively captures the various symmetries in a polygon.

2.3. Other Possibilities. We note that the straight skeleton is not necessary to make a hypar shape out of a piece of paper. For example, it seems that folding smaller rectangles with vertices on the diagonals of a rectangular piece of paper, as in Figure 4, also leads to a hypar form. We leave such experimentation to future work.

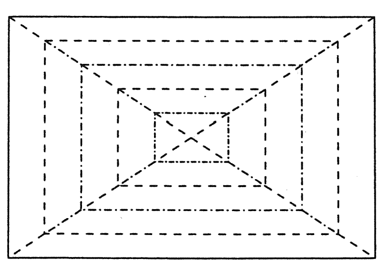

Figure 4: A different hypar folding from a rectangle.

3. General Models of Joined Hypars

Let us now look at the general case of joining of hypars. We consider edge-to-edge gluings, so two hypars can be glued together at one, two, or four edges: it turns out that if one tries to glue just three edges, the fourth one will be forced to be glued as well. If two hypars are joined at all four edges, they become effectively identical to just a single hypar. Hence, the interesting cases are when pairs of hypars are joined at one or two edges.

A gluing can be modeled by a *graph* in which up to two edges are allowed to join the same pair of vertices, and in which every vertex has precisely four adjacent edges. Succinctly, such a graph is called a 4-regular loopless multigraph with all multiplicities at most two.

Of course, not every such graph will correspond to a physical joining of hypars. This may happen for a variety of reasons: two hypars that should be joined may not be oriented properly or may be too far away from each other, or hypars may have to intersect each other to form the join. An interesting open problem would be to classify the physically realizable joinings, but this seems difficult.

This leads us to look at specialized joinings of hypars. We will examine two methods. The first, described in Section 4, uses "hats" to make beautiful symmetric representations of polyhedra. The second, described in Section 5, considers more general ways to join hypars based on "matchings" in polyhedra.

4. Joining Hats to Form a Polyhedron

Perhaps the simplest way to join hypars is called a *hat* (see Figure 5). In the graph representation, a hat corresponds to a cycle. In other words, the hypars are arranged in a cycle, and each hypar is joined to its two neighbors along adjacent edges.

Each hypar can have one of two orientations, which we call "normal" and "inverted." First, we need some terminology. Call the vertex shared by all the hypars in a hat the "top vertex." The "tip" of one of the hypars is the vertex opposite from the top vertex. Now consider the plane that passes through all the vertices adjacent to the top vertex. If the tip of a hypar is on the same side of the plane as the top vertex, we say that it is in *normal position*. Otherwise (the tip is on the opposite side of the plane), we say that it is in *inverted position*. Note that a hypar can be inverted between the two orientations simply by applying pressure to the tip.

We will focus on *normal hats*, where every hypar is in normal position. Specifically, a k-hat is a normal hat made out of k hypars. Two degenerate cases are a 1-hat (which is just a hypar by itself), and a 2-hat (which is two hypars joined along two adjacent edges).

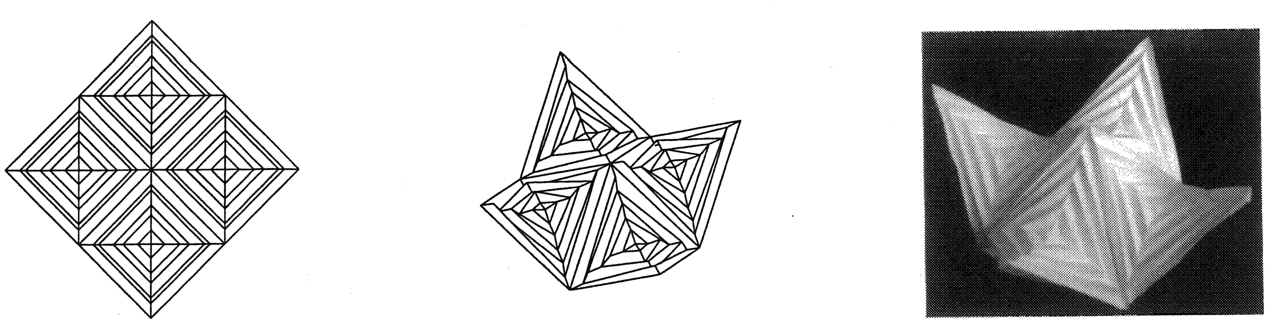

Figure 5: A 4-hat from overhead and side views.

A k-hat quite naturally represents a k-sided polygon (or k-*gon*). This suggests the following algorithm for converting any polyhedron into a gluing of hats, or *hyparhedron*.

- Replace each k-sided face with a k-hat, each hypar of which corresponds to an edge of the face.
- When two faces share an edge, glue the corresponding two hypars of the hats at their remaining two free edges.

Figure 6 shows the wonderfully symmetric examples of the Platonic solids. For these solids, every vertex of the polyhedron (and its dual) is a corner of a hypar. In this case, we say that the hyparhedron captures the *geometry* of the polyhedron. For other polyhedra, the generated hyparhedra only represent the *combinatorial topology*. For example, all tetrahedra (which have the same combinatorial topology but have different geometries) produce the same polyhedron.

We can also apply the algorithm to unusual polyhedra, such as those involving regular 1-gons (segments) or 2-gons (double segments). Recall that a 1-hat consists of a single hypar, and a 2-hat consists of two hypars joined together. For example, if we take a doubly covered triangle, and fit three 1-gons in between the two triangles, we obtain the hyparhedron shown in Figure 7 (left).

It follows from the algorithm that if the input polyhedron is closed, the generated surface will also be closed. A surprising result is that even if the input polyhedron is flat, the hyparhedron may not be. This principle was just illustrated by the previous example: the triangles and three line segments lay flat on each other, but the hyparhedron in Figure 7 (left) has a large volume.

Another interesting example is a collection of 1-gons joined together in a cycle. This produces a kind of star shape, as shown in Figure 7 (right), and we call the hyparhedron a k-star where k is the number of hypars. A pleated 4-star was independently designed by Johnson [9, p. 72]; we have used a pleated 5-star for the top of our Christmas tree.

Finally, we should mention that the algorithm is not guaranteed to create a realizable hyparhedron, although we have not encountered a counterexample yet.

4.1. Hats in Architecture. We note that k-hats for small k, as well as a few other special ways to join hypars, have been exploited in architecture. For example, Siegel [14, p. 256] illustrates the roof of the Girls' Grammar School in London (designed by Chamberlin, Powell, and Bonn) which is a 5-hat with the five hypars spread apart slightly. Later [14, p. 264] the idea of joining two 5-hats is suggested, although the two hats are cut to have a curved boundary, making them easy to join. Siegel [14, p. 260] also shows a photo of the Philips pavilion at the 1958 Brussels exhibition (designed by Le Corbusier) which is a beautiful surface made of eight or so hypars that rests on the ground. A few more wonderful examples with straight boundaries are illustrated by Engel [6, pp. 228–229], each involving between five and twelve hypars. Finally, a grid of connecting 4-hats is illustrated and analyzed by Otto [12, p. 64].

4.2. Pulling on Edge Points. For each edge of the polyhedron, the algorithm joins a pair of hypars along two adjacent edges. These two edges share a vertex, which we call an *edge point*, because there is precisely one edge point per edge of the polyhedron. The "height" of this edge point is arbitrary.

A slight modification to the algorithm is to allow "pulling" on these points. Each edge point can be pulled by the desired amount, allowing one to change the shape sculpturally. Pulling can be performed by modifying the two adjacent hypars to use kites instead of squares. By adjusting the two edge lengths adjacent to the point being pulled, we can "pull" the point out or "push" it in.

This is a good example of how non-square pieces of paper can be used to result in more general hyparhedra.

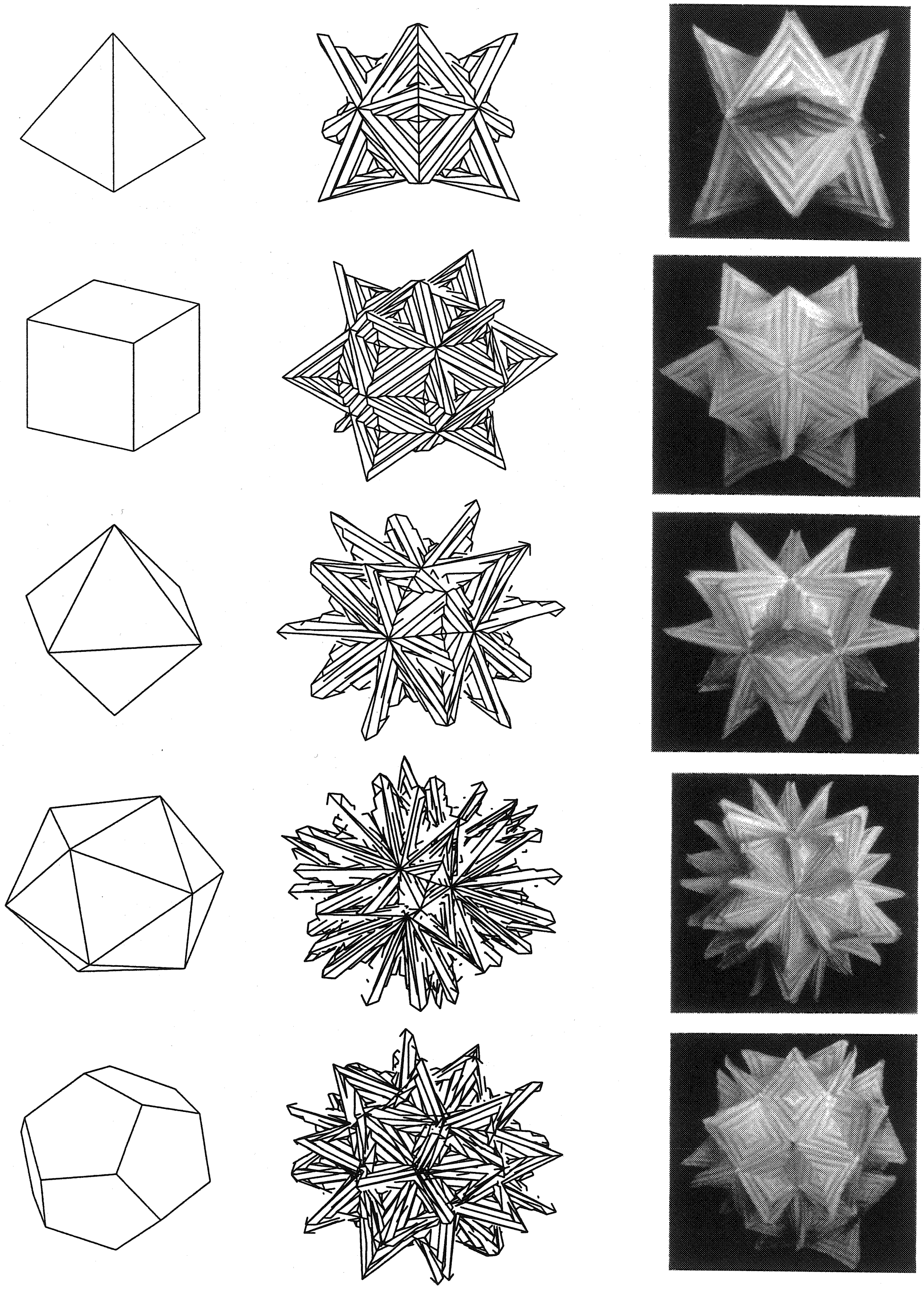

Figure 6: The Platonic solids and corresponding hyparhedra (computer and physical models). The view angles are the same for the left two columns, except for the tetrahedron. The physical models were constructed from six-inch squares of paper, and take 2–6 hours each to construct.

Figure 7: (Left) The hyparhedron from a doubly covered triangle with three line segments in between. (Right) The hyparhedron from a bundle of five line segments connected in a cycle, called a 5-star.

5. Matching Triangulated Polyhedra

Another way to look at the algorithm in the previous section is as follows. Subdivide each face of the polyhedron into a cycle of triangles, by cutting from each vertex to a point interior to the face. Consider each of these triangles to be *matched* with the (unique) adjacent triangle on the adjacent face. Now apply the following algorithm:

- Replace each triangle with a hypar.
- Join two adjacent hypars that are not matched together by a single boundary edge.
- Join two adjacent hypars that are matched together by two adjacent boundary edges.

This algorithm applies to any triangulated polyhedron with a perfect matching. A *perfect matching* is a pairing up of triangles so that every triangle has a unique partner. The matching specifies which pairs of hypars are joined along two adjacent boundary edges; all other pairs of adjacent hypars are joined by single boundary edges. Because each triangular face has three edges, two of which are joined by single edges and one of which is joined by two edges, this corresponds to the four sides of the hypar.

Petersen's theorem from 1891 [13] states that every triangulated polyhedron has a perfect matching, and hence we can apply this algorithm to any triangulated polyhedron. It is interesting to apply the algorithm to all the different perfect matchings in a polyhedron. Again, we are not guaranteed for the hyparhedra to be realizable, but we know of no counterexamples.

6. Conclusion

We have presented algorithms to compute a wide variety of mathematical sculptures made out of hyperbolic paraboloids (hypars), which we call *hyparhedra*. Several avenues remain to be explored in using the exciting building tool of a hypar. For example, we can take a k-hat and then "invert" one of the hypars (as described in Section 4), so that it points in the opposite direction of all the others. Indeed, if we invert all of the hypars, we obtain the star shape described in Section 4. What new hyparhedra can be created with this concept of inversion, and how can they be represented?

Another interesting problem is to *geometrically* represent more polyhedra than just the Platonic solids, that is, make the vertices of the polyhedron be corners of the hypars. Currently, we only know how to "topologically" represent general polyhedra, that is, represent their combinatorial structure. It is likely that not all polyhedra can be geometrically realized, although we have been able to make general triangles, and general tetrahedra are likely possible.

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