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Mathematical Connections
in Art, Music, and Science

Mathematical "Abstracts"

Lawrence Holbrook
812 Lantern Hill Dr.
East Lansing, MI 48823, U.S.A.
E-mail: 105436.2505@compuserve.com

Abstract

"Abstract" art is derivable from mathematics in countless ways. Three techniques are discussed in this paper; two of which start with fractals generated from the iteration of almost any mathematical function. Fractals usually have observable and aesthetic properties such as symmetry and self-similarity. They also, by consequence of definition, are such that surface and perimeters appear convoluted and jagged. These patterns, of interest in themselves are the basis for more "painting-like" abstracts resulting from the incorporation of random numbers into the fractal equation or into some other controlling feature of the iteration process. Alternatively, fractals can be softened, modified and transformed combining graphics software techniques with color map design. Finally, non-fractal images are produced non-iteratively by mathematically averaging random numbers and processing the results with graphics software.

Background and Technique

Since ancient Greek times, mathematics and art have had a long association. We can find common mathematical properties in abstract art in the forms of geometric balance, replication, and variations on a theme. "Golden" rectangles, conic sections, and logarithmic spirals are standard examples[1]. For some unknown reason, aesthetic appeal often has a mathematical basis. With the advent of computers, the ancient and useful technique in which equations are repeatedly substituted back into themselves can produce spectacular patterns, some of which are called "fractals" following the pioneering work of Benoit Mandelbrot (1924--). Historically, the equation $z_{n+1} = z_n^2 + c$ is iterated with $z_0 = 0 + 0i$ where z and c are complex and c represents the complex coordinates of the point undergoing iteration. The artistic connection comes about when each point is mapped to a small square or "pixel" on the computer screen and colored according to some mathematical criterion. Whereas the equation is the image's "DNA", the pixel is the atomic element, and several millions (and kindred amounts of computer memory) are required to produce a high resolution 3-foot by 4-foot image. The iterative procedure could, theoretically, proceed forever; however, a decision criterion is introduced which specifies that the point's distance form the origin of the complex plane (modulus(z)) reaches or exceeds some limiting real number (such as 2.0 for the Mandelbrot set, and as proven for this set escapes to infinity) the process is terminated and the coloration criterion invoked. If this does not happen in an arbitrarily large number of iterations (say 150), it is assumed that the modulus of each subsequent iteration will be confined within the pre-set mathematical limits or "bounded". Bounded point sequences have an analogy in engineering, namely "stable" as opposed to "unstable" systems. However, bounded points have not received artistic attention since they have not provided an interesting basis for coloration even though an infinity of mathematical and statistical properties can be imputed to all sequences of iterated points. Bounded points of the Mandelbrot set images are usually colored black indicating regions of process stability. The points of usual artistic interest are those which do escape because their rate of escape (i.e., number of iterations required to exceed the modulus limitation criterion) can be used to determine the color assigned to the corresponding pixel. Colors preassigned to rate ranges reveal marvelous patterns and detail. One property often found is "self-similarity" where the same pattern is replicated at smaller and smaller scales [2], [3]. Indeed, unlike the physical world where reductions from mountains to quarks result in "qualitative" change across scale, in the infinitely divisible complex plane the repetition of pattern with no corresponding qualitative change may go on indefinitely. "Fleas upon their backs have little fleas to bite 'um and little fleas still lesser fleas, and so on ad infinitum."

The Mandelbrot set (or more precisely, the points just outside it) has been extensively examined [4]. It is known to produce striking images, and is the inspiration for the fractal portfolio. But the iterative procedure need not be confined to the equation of that set. In fact, any function can produce some kind of image depending on escape criteria and other computer and parameter settings. The number of functions, the number of escape criteria, and the number of coloration criteria each represent an infinity of possibilities. This infinity of joint infinities can engineer an infinity of images each of which is the embryonic basis for an artistically significant creation. How does one find a way through this forest of possibilities? By persevering, one can gain experience with a particular equation and identify parameter settings which control selected aesthetic properties. Then one can nudge these controls to produce an artistic effect. For example, different effects can be obtained by altering the escape criterion. Changing this criterion from **rate** of escape at the modulus threshold to the **region** where escape occurs produces a radically different image. The hands on trick is to select functions, settings, and color map numerical range assignments which produce images with the desirable aesthetic properties. These fundamental images can be further processed using graphics software to "tune" the final image. Ordinarily, one does not "set out" to design an image according to preconceived concepts. One starts out anywhere, observes artistic potential, and develops the piece accordingly. Throughout the process, one must both resist and exploit an infinitude of decision possibilities a few of which are listed in groupings below.

Area of the Complex Plane to Explore. With the Mandelbrot set, various regions have been extensively explored and mapped so something is known about the kind of images that various locations generate. Other functions should afford the same kind of information provided exploratory experience is systematically acquired and intuitively stockpiled. One can also expect various forms of symmetry when working near the complex plane origin. A useful strategy is to first explore an area around the origin in order to exploit the aesthetics of symmetry and then move off axis and zoom in.

Alterations in the Basic Equation. By slightly altering parameters in the basic equation, one can alter and mold shapes in accordance with artistic insight. This can be tricky when dealing with exponents, and "division by zero" is an occupational hazard when exploring overly complex functions. These problems can usually be avoided by building a polynomial function with simple additive rather than concatenated exponential or logarithmic terms. Unfortunately, exponential functions are quite rich in artistic potential. If one proceeds incrementally with parameter adjustments, small image changes are more likely than radical conversions. With experience one develops a sense of parameter sensitivity.

Use of Randomly Generated Numbers (Random Numbers). Random numbers, inserted into the equation, the escape threshold, or the initialization coordinates provide an excellent way to convert the basic image into an impressionist abstract. One can easily control the degree of randomness introduced and subsequently process the result with graphics software to create the paint stroke effect.

Graphic Software Distortions and Transformations. Graphics software is very effective in remaking the basic image. One can add to, stretch, clone, distort, diffuse, blend, solarize or pixelate to achieve transformations which may bear little or no similarity to the original mathematically determined image. However, in the most successful images, some original image fatherhood can generally be found because even abstracts have pattern and at least loosely affiliated mathematical relationships.

Alternative Colorations. Graphics software can be used to edit colors in an image. This can take the form of color replacement or one can "bleed" color into selected image regions creating new pattern. These tend to be final stage adjustments. More sweeping and dramatic results are obtained with the use of color maps. Color maps relate colors to number ranges and the flexibility of this relationship creates considerable opportunity for substantial image transformation. This is because areas hitherto uniform in a color triggered by a broad range of numbers can become color differentiated when a spectrum of hues is assigned to subranges of numbers within that range. The reverse can also happen. The result can be a radically new or more highly detailed image bearing only distant resemblance to its mathematical twin.

Examples

The author was engaged to design the cover for a book entitled *Corrections: A Humanistic Approach* by professor of psychology Hans Toch currently with the School of Criminal Justice at SUNY. The book, published in 1998, addresses incarceration from the perspective of the prisoner and seeks to advance incarceration understanding so that beneficial reforms can be designed and instituted. The problem was to design an image which reflects the abstract concept of incarceration in a controlled but humanistic environment. The iterative abstract (tetration) shown in Figure 1 was chosen by the publisher because the supple, enveloping, curvilinear nature suggests something other than the traditional spare, rigid, "boxed-in" connotation of prison confinement. In contrast, the white bars (added with graphics software) against the black background suggest the isolation and severely limited freedoms imposed by incarceration in a dungeon-like setting. The equation used to generate this abstract is in logarithmic form: $\log(z_{n+1}) = z_n \log(c)$ where c represents the complex coordinates of the point undergoing iteration with, in this case, an escape criterion of modulus(z) >= 4.0. Or, in other words, the value of the n+1-th iteration is the result of raising the complex coordinates of the pixel in question to the power of the complex coordinates of the n-th iteration. A small section of the complex plane was selected for magnification in grey-scale coloration.

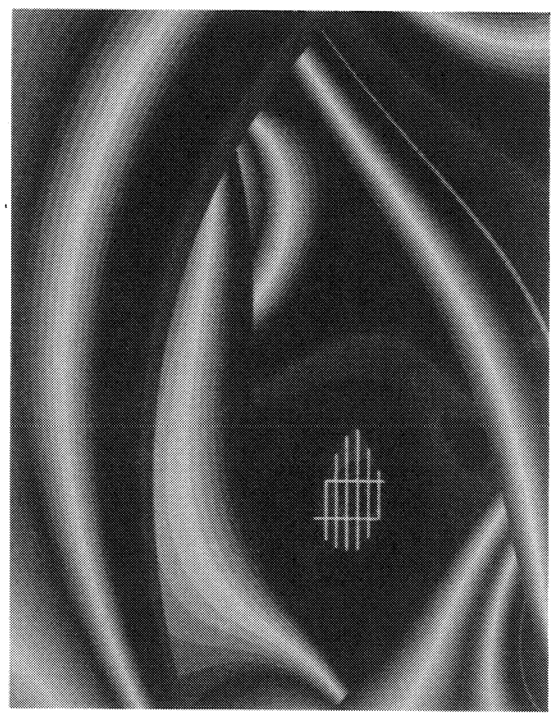

Figure 1: *Humanistic incarceration.*

An entirely different pattern, which exhibits regional self-similarity, results from the iteration of the trigonometric function: $z_{n+1} = z_n^2 + \tanh(z_n) + c + k(rand)$. The "rand" term designates a random number (uniformly distributed in the range 0,1) assigned for each iteration of each point. The multiplier k determines the extent to which randomness creeps into the image. With k = 0, we see the original pattern shown in Figure 2. With k>0, the equation produces a less well defined image, and because of color scattering, reminds one of an impressionist painting In color, the result is considerably more attractive than the black and white image of Figure 3. Initially, the result is grainy because of the random numbers, but can be made more "painting like" with graphics software which can smear the points. One then "blends" in 24-bit color (nearly seventeen million colors) and then transforms to the far less color differentiated palette of 256 colors. This process was used to produce the image in Figure 3 where "blending" and palette exchange reduce local mathematical variance and consequently promote uniform coloration at the paint stroke level. One can also introduce abstract impressionism through random numbers by either specifying the randomness function initially at the pixel level or finally at the escape criterion level.

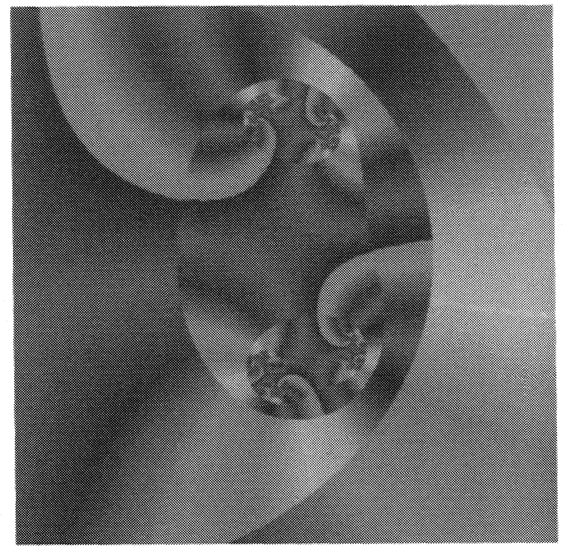

Figure 2: *Initial self-similar image.*

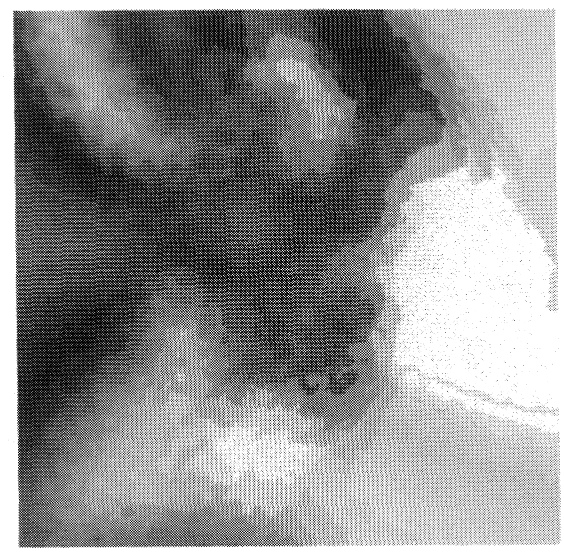

Figure 3: *Fig. 2 random number processed.*

Using the iteration technique, one can also produce images which, though abstract, have realistic interpretations. Figure 4 shows the image titled "River Sunrise" which in deep blues, yellows, oranges, and reds, clearly reminds one of the sun setting or rising over water. A series of exponential functions produces the basic image which was originally symmetric about the vertical (imaginary) axis. The image was back rotated ninety degrees to capitalize on this symmetry. This produced the reflection effect of a sunrise over water. The clouds were produced by stretching horizontally the rotated image.

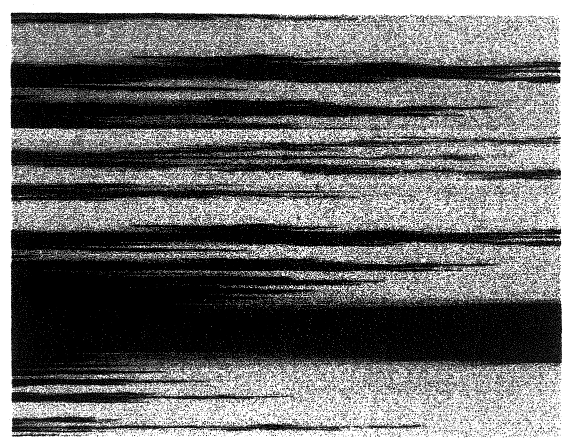

Figure 4: *Stretched symmetric fractal.*

Artists such as the mannerists have portrayed reality through a distorted lens. Distortion style can define artistic identity. The distorted reality effect shown in Figure 5 results from z raised to the power of 2 + 0.1i. The fractal titled "Zen Maple" software edited in yellow and orange coloration gives the distinct impression of a fall scene. Zen Maple was stretched and subdivided into five parts to produce a 6 x 12 foot set of panels for a public building currently under construction in Lansing.

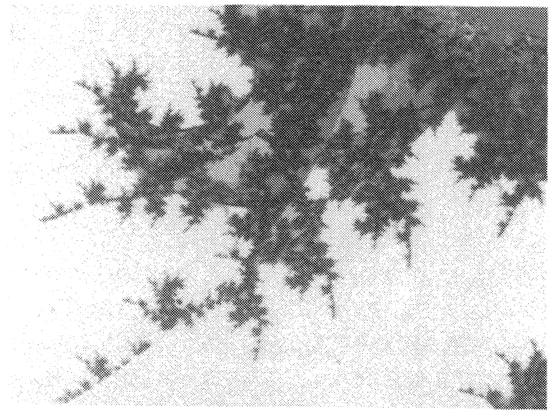

Figure 5: *Distorted reality: Zen Maple.*

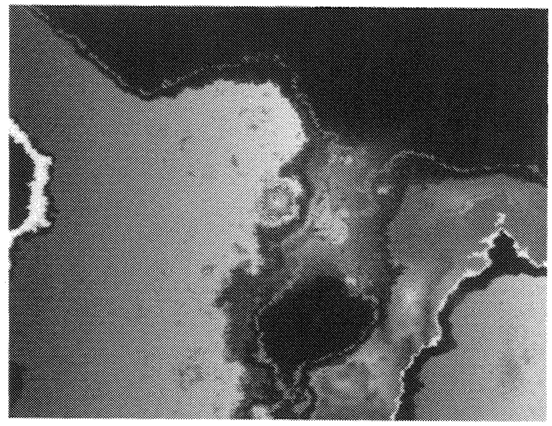

Figure 6: *Processed random numbers.*

Another way to mathematically create abstract images is to randomly assign numbers to pixels in the real plane and color them in accordance with a color map which assigns colors to number ranges. This procedure does not involve equation iteration. By itself, this produces a chaotic image of no artistic interest. "Order" can be introduced artificially by averaging a pixel's random number with those of nearby pixels. This way, information is shared and continuity introduced. This "breeding" of numerical material transforms disorder into order. One method assigns a number to a rectangular area's center and its four corners and recursively subdivides quartered rectangles adding new random numbers to the center of the subdivisions and assigning flanking neighbor averages to the two new corners. This transmits regional information. Thus, information is disseminated or shared as in moving averages. Alternatively, one can, borrowing from temporally based signal analysis (exponentially mapped past) and econometrics (distributed lag models), average spatially with nearby pixel numbers assigning weights proportionate to proximity. These procedures introduce a surprising degree of continuity suppressing abrupt numerical and consequently color variations. The effect is further enhanced with color graphics which allow one to capitalize on regional detail and control the degree of color "bleeding" into surrounding areas of different mathematical composition. This is an interaction of graphical coloring technique with numerical patterns based on random numbers and an averaging algorithm relating them in a kind of two-dimensional autocorrelation. "Biogenesis", shown in Figure 6, was generated by a field of random numbers and random number averages. Whatever the sequence of operations, none of these quasi-abstracts involved any "drawing", a matter of considerable importance to those of us not blessed with the hand-eye coordination of the traditional artist.

Conclusion

Described herein are three methods of mathematically producing abstract art. Traditional fractal technique makes use of mathematical function iteration. This often produces images which can be symmetric and self-similar. These images benefit from graphics software modifications such as color replacement which breaks the work away from the rigidity of the original formulation and affords the artist with additional creative opportunity. Working with random numbers, one can adjust them to one another using averaging techniques. Color graphics can then exploit patterns accordingly produced. Finally, one can introduce randomness into the fractal iteration process. This has the effect of muting rigid pattern and creating an impressionistic appearance. This transformation can be completed using

References

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