BRIDGES
Mathematical Connections
in Art, Music, and Science

Some Briggsian Bridges between Sense Perception, Postage Stamps, and Stylized Proportions
or
A Musician-Mathematician Looks at Life

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Abstract

The properties of exponential and logarithmic functions and their historical development are reviewed, with emphasis on application to non-Euclidean metrics in Klein's Erlangen Program. Comparing the roles of angles and logarithms as arguments of circular and hyperbolic tangent functions, linear fractional transformations (l.f.t.'s) are introduced which show successive stages of hyperbolic rotations to be projectively equivalent to terms of a geometric sequence. The special role of powers of 2 in human perception of both color and pitch intervals is explored, using music staff paper to yield an aural model of economic growth. A possible explanation for the octave phenomenon is offered in the way it combines properties of both harmonic and geometric spacing, as illustrated by the ancient Egyptian artists' method of rendering human body proportions in murals.

Prologue in the Kitchen

Since the Northridge earthquake of 1994, my main office has been perforce my kitchen. The word processor dominates its table, crowding out the gatherings of friends that had been intended to meet around it, discussing M.F.K. Fisher's books over carefully prepared meals. The math books on the table greatly outnumber the cook books, but sometimes there are cross-overs:

If we array successive powers of some base such as 2 in equi-spaced manner, and then align successive powers of some other base such as 10 beneath them, we find that the latter are also automatically equi-spaced (as would be powers of any other base b > 1).

m	0	1	2	3	$3\frac{1}{3}$	4	5	6		7	8	9	10
2^m	**1**	**2**	**4**	**8**		**16**	**32**	**64**		**128**	**256**	**512**	**1024**
10^n	**1**				**10**				**100**				**1000**
n	0	0.3			1				2				3

Taking advantage of the coincidence that $1000 = 10^3 \approx 2^{10} = 1024$, we may take cube roots of both sides and find $10 \approx \sqrt[3]{2^{10}} = 2^{10/3} = 2^{3\frac{1}{3}}$, and similarly taking tenth roots of both sides we find $2 \approx 10^{3/10} = 10^{0.3}$. In the equi-spaced alignments shown above, it can be seen accordingly that 10 lies (very nearly) 1/3 of the way between 2^3 and 2^4 at the place for 2^m where $m \approx 3\frac{1}{3}$, and similarly that 2 lies (very nearly) 3/10 of the way between 10^0 and 10^1 at the place for 10^n where $n \approx 0.3$ or $\frac{3}{10}$. (The actual values are $\log_2 10 = 3.32193$ and $\log_{10} 2 = 0.30103$, to 5 decimal places.)

This coincidence is one reason why storage capacity on floppy disks is expressed in units of k = 1 kilobyte ≈ 1000 bytes (*kilo* being Greek prefix for 1000) when such storage is actually in powers of 2: it is a convenient linguistic fudge (there being no handy pre-existing word for 1024). Another might lie in the fact that it takes something on the order of 24 bytes per kilobyte to communicate between the floppy and hard drives (to produce displays on monitor screen and relay commands), hence leaving about 1000 bytes of net user-controlled memory per kilobyte, making the "k" designation more or less honest after all. Mrs. Beeton [1], that paragon of late Victorian kitchen-keeping, advised ladies whose husbands were stationed in colonial India not to take it necessarily amiss if native servants silently pocketed some of the marketing funds entrusted to them; if they had found good bargains, they were entitled to a share of the savings. PC's are among today's servants, and their use of those 24 bytes is for the equivalent of savvy housekeeping.

The Notion of Base

By definition, $\mathbf{a = b^c \Leftrightarrow c = \log_b a}$, read "a is the *exponential* of c base b if and only if c is the *logarithm* of a base b." (This is in contrast to $a = b^c \Leftrightarrow b = \sqrt[c]{a}$, read "a is the c^{th} *power* of b if and only if b is the c^{th} *root* of a," for since $b^c \neq c^b$ there must be distinct left and right inverses of exponential and power functions, namely logarithms and roots.) We take this for granted today, but must remember that logarithms and exponentials were not recognized as functions (distinct from power processes) until the early 17th century, were regarded as something miraculous, and took the work of several men of genius (Bürgi, Napier, Briggs, Oughtred, and Euler) to be gradually shaped into their modern forms of definitions, notations, and graphic displays.

The Babylonians practiced lending with compound interest and were able to pose and approximately solve such problems as finding how long it would take an investment returning 20% annually (written as <II = 12', or 12 parts out of 60 in their notation) to double in value, or $1.20^t = 2$ in modern terms, as evidenced by a clay tablet from about 1700 B.C.E. now in the Louvre [2]. Finding $1.20^3 = 1.7280 < 2 < 1.20^4 = 2.0736$ by repeated multiplication (i.e. by power processes), they knew that $3 < t < 4$ and were able to refine their estimate to $t \approx 3.7870$ by linear interpolation: in decimals, 2 is about 78.70% of the way between 1.7280 and 2.0736, so t must be about that far between 3 and 4 (they carried it out hexagesimally as 3 47' 13", but the idea is the same). The exact value is found today by taking logs of both sides, yielding $t = \log 2/\log 1.20 = 3.8018$ to 4 decimal places, confirming the Babylonian guess to be off by only about 4/10 of 1%.

By the turn of the 16th to 17th century of our era, investigators of more frequent compoundings had noticed that a unit of money lent at 100% interest led to $(1+1)^1 = 2$ units if compounded once a year, $(1+\frac{1}{2})^2 = 2.25$ if semiannually, $(1+ {}^1/_{12})^{12} = 2.613\cdots$ if monthly, $(1+ {}^1/_{365})^{365} = 2.714567\cdots$ if daily, and $2.718126\cdots$ if hourly, thus pressing towards Euler's constant $e = 2.71828\cdots$ "in the limit," as we would say, a half century before the limit process was used by Newton and Leibniz to found the calculus [3]. Bürgi first computed powers of a constant $(1+10^{-4})$ slightly greater than unity but didn't publish them [4]; Napier computed powers of a constant $(1-10^{-7})$ slightly less than unity and *did* publish them in 1614 [5]. Both men's formulations, however, were complicated by their desire to avoid decimals, Bürgi writing in effect the quasi-antilog $N = (1+10^{-4})^L \times 10^8$ and Napier $N = (1-10^{-7})^L \times 10^7$, whence for example if $N = 2 \times 10^8$ then Bürgi's quasi-logN would be $L \approx 6{,}931$, while if $N = \frac{1}{2} \times 10^7$ then Napier's quasi-logN would be $L \approx 6{,}931{,}471$, both recognizably related to today's ln2 (*logarithmus naturalis* of 2) $= \log_e 2 \approx 0.69314718$, sharing 4 and 7 digits with it, respectively, in displaced positions [6].

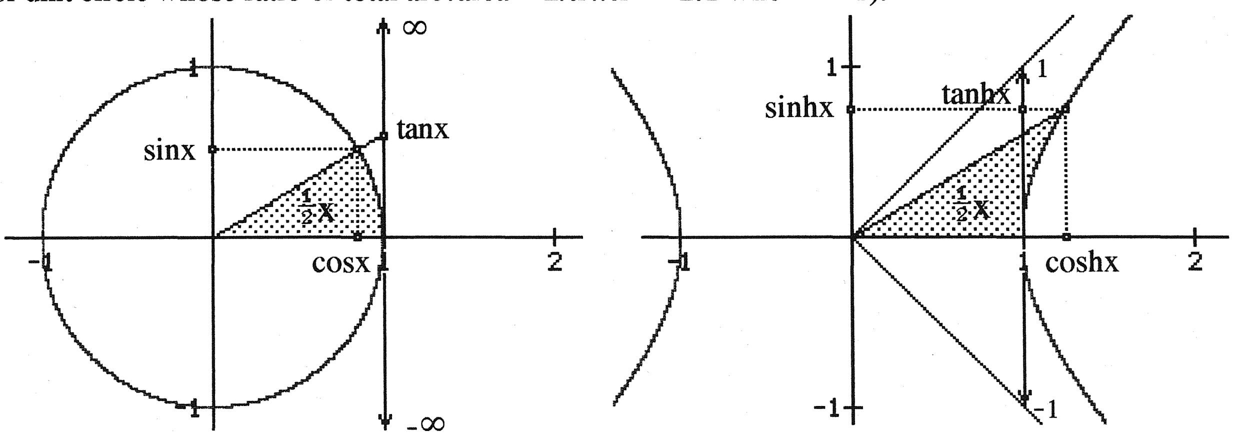
The man who first suggested the modern definition $a = b^c \Leftrightarrow c = \log_b a$ with no re-scaling was Henry Briggs; his were the first true modern logarithms [7]. Using base $b = 10$, they were known variously as Briggsian, decadic, or simply common logs. While Napier invented sliding sticks ("bones") of a different nature with which to perform calculations graphically, the idea of a slide rule as we know it is due to Oughtred [8]. The conversion of Bürgi's and Napier's place-shifted logs to modern form as logs base e or natural logs is due to Euler, who established the analytic properties of e^x as infinite series and e^{ix} as combination $\cos x + i \sin x$ trigonometrically [9], thereby returning to Napier's original inspiration for operation conversion in formulas relating products of circular trigonometric functions of two angles to combinations of functions of sums and differences of those angles.

Euclidean and Non-Euclidean Metrics

When Ricatti introduced the analogous definitions of hyperbolic functions in the 18th century giving e^x as combination coshx + sinhx [10], the stage was set to recognize logarithms as the hyperbolic equivalent of angles, enabling Klein in 19th century to unify the descriptions of the various non-Euclidean geometries due to Lobachevsky and Riemann [11]: By his Erlangen Program, studying transformations by what they leave invariant, he was able to define the *distance* between any two points or *angle* between any two lines as 1/2 the natural log of the projectively invariant cross-ratio of those two elements with the two ideal elements on same line or through same point:

$$d(P_1,P_2) = \tfrac{1}{2}\ln(P_1P_2,M_1M_2) \quad \text{and} \quad \angle(\ell_1,\ell_2) = \tfrac{1}{2}i\ln(\ell_1\ell_2,m_1m_2).$$

The first cross-ratio is real $\neq 0$ or 1 for points on a line in hyperbolic or Lobachevsky metric and corresponds to a step in a *geometric* sequence, as we shall see, but 1 in case of parabolic or Euclidean metric necessitating arbitrary choice of origin and units (in. or cm.) forming steps in an *arithmetic* sequence. The second, used for distances in the elliptic (circular) metric of Riemannian geometry and for angles in all three of these Euclidean and non-Euclidean geometries alike, is unimodular complex-valued; hence its logarithm is pure imaginary since $\ln(\cos 2x + i\sin 2x) = \ln(e^{2ix}) = 2ix$, and must be scaled by another i to yield the conventional real value as *rotation*. The double angle that must be halved, finally, arises trigonometrically out of such products as $2\cos x \sin x = \sin 2x$ that first caught Napier's eye, and analytically from the fact that the ratio of angle subtended from origin to area bounded by curve is 2:1 for both unit circle and hyperbola, whence areas can serve as common argument for both kinds of functions (seen most readily in case of unit circle whose ratio of total arc:area = $2\pi r:\pi r^2$ = 2:1 when r = 1).



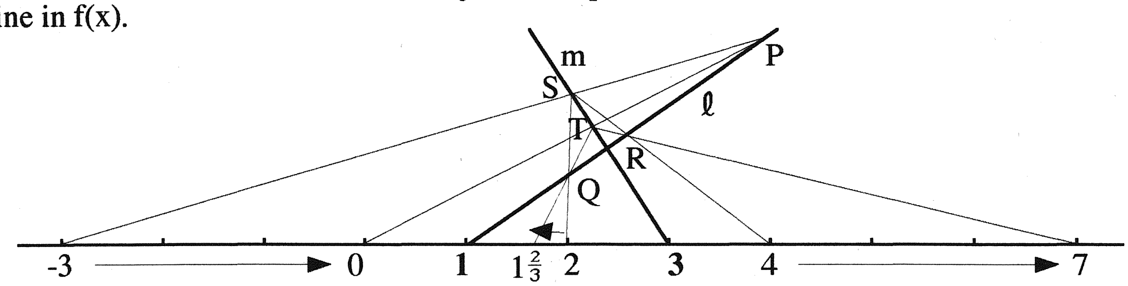
The Bridge between Special Metric and General Projective Sequences

Suppose a line is to be transformed, inwardly stirred in some fashion, but in a manner befitting a line: by linear methods. If one or two points are moved arbitrarily to other locations, there is as yet no entailment; other points don't "know" where to go. But as soon as a third point is moved, there is an implicit orderly rule to which all others must *conform* (the mathematical equivalent of fairy tales in which three wishes are granted but whose consequences may surprise the wisher). Suppose, for example, that -3 moves up to 0, 2 is nudged down to $1\frac{2}{3}$, and 4 is sent zooming out to 7. Then there is a *linear fractional transformation* f(x) = (ax+b)/(cx+d), also known as a *conformal map*, which will do the job, as shown by Möbius in mid-19th century [12].

By their nature as ratios, they are unique up to proportion; therefore, one of the coefficients (say d) may be presumed without loss of generality to be a unit. Solving the resulting system of 3 linear equations in 3 unknowns yields a = $\frac{1}{5}$, b = $\frac{3}{5}$, and c = -$\frac{1}{5}$, from which (multiplying by 5/5 to clear fractions) we find f(x) = (x+3)/(5-x). Checking, we confirm that f(-3) = 0, f(2) = $1\frac{2}{3}$, and f(4) = 7, as required.

But notice that the first and second points move toward one another, suggesting some intermediate point will be fixed in position by being "pushed" equally from either side. Similarly, we notice the second and third points move away from one another, suggesting another intermediate point "pulled" equally from either side. These are the two *invariant* points of the transformation, found by setting f(x) = x and solving the resulting quadratic for its two roots x = **1** and **3** (boldface).

Geometrically, if ℓ and m are any other lines through the fixed points 1 and 3, respectively, and P is any point on ℓ, joining P to -3 and to f(-3) = 0 yields lines cutting m in a pair of points S and T such that other well-defined points Q and R will be found on ℓ whose joins through S and T lead to 2 and f(2) = $1\frac{2}{3}$, 4 and f(4) = 7, respectively. To find f(x) for any other point x of the line, one joins x to S, finds where that cuts ℓ, joins that point to T, and finds where that line cuts original line in f(x).



Now that the function has been found which does the moving, we may investigate the nature of moving in successive steps to form sequences. Starting at 2 = $f^0(2)$, we can go on beyond $1\frac{2}{3}$ = $f^1(2)$ to find $1\frac{2}{5}$ = $f^2(2)$, $1\frac{2}{9}$ = $f^3(2)$, etc., moving in ever smaller steps toward the infinitely approachable but never reachable 1 as *sink* to which these values are *attracted*. And starting at 4 = $f^0(4)$, we can go not only beyond 7 = $f^1(4)$ but beyond $\pm\infty$ to return Magellan-like from the other side and find –5 = $f^2(4)$, –$\frac{1}{5}$ = $f^3(4)$, etc., slowing down again with approach to the same value 1 from other side. The other fixed point 3 is the *source* from which these two sequences may be thought to arise and from which their member points are *repelled* and flee in successive steps.

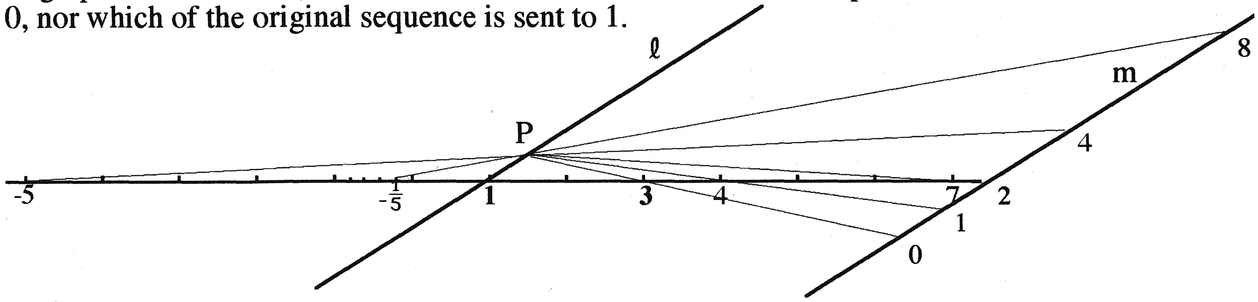
Klein's brilliant insight was that all of these goings-on can be characterized neatly by what *doesn't* change — not just the fixed points, but a single number epitomizing the entire process. This is the *cross-ratio* (AB,MN), defined as ($\overline{AM}/\overline{BM}$)/($\overline{AN}/\overline{BN}$) = ((M-A)/(M-B))/((N-A)/(N-B)). Taking M = 1 and N = 3 from the above example, then whether we begin with A = 2 moving to B = $1\frac{2}{3}$ or with A = 4 moving to B = 7, the cross-ratio remains the same. In the first case we have $((1\text{-}2)/(1\text{-}1\frac{2}{3}))/((3\text{-}2)/(3\text{-}1\frac{2}{3})) = ((\text{-}1)/(\text{-}\frac{2}{3}))/((1)/(1\frac{1}{3})) = (\frac{3}{2})/(\frac{3}{4}) = 2$, and in the second case we have $((1\text{-}4)/(1\text{-}7))/((3\text{-}4)/(3\text{-}7)) = ((\text{-}3)/(\text{-}6))/((\text{-}1)/(\text{-}4)) = (\frac{1}{2})/(\frac{1}{4}) = 2$ again. Taking any other A = x and B = f(x) would yield the same result.

The Hyperbolic Nature of Geometric Sequences

Recalling that hyperbolic tangent functions can only take on values between ±1 (as shown at top right on previous page), we may project the fixed points found above to these values and see where the intermediate sequence members land as a result. To send 3 to -1, 2 to 0, and 1 to 1, fixed point to fixed point and mid-point to mid-point, we can take linear g(x) = 2-x; the sequence $2,1\frac{2}{3},1\frac{2}{5},1\frac{2}{9},\cdots$ is then sent to $0,\frac{1}{3},\frac{3}{5},\frac{7}{9},\cdots$. To send 3 to -1, 4 to 0, and 1 to 1, we need linear fractional h(x) = (x–4)/(2x–5), but it sends the sequence $4,7,\text{-}5,\text{-}\frac{1}{5},\cdots$ to same $0,\frac{1}{3},\frac{3}{5},\frac{7}{9},\cdots$. Any such conformal map sending fixed points to ±1 and another point x to 0 will send the sequence of transforms $f^n(x)$ to this same sequence, which must therefore also typify the motion called f(x).

The nature of that motion's steps can be revealed by removing one mask at a time to find $x = 0, \frac{1}{3}, \frac{3}{5}, \frac{7}{9}, \cdots \rightarrow \tanh^{-1}x = 0$, 0.346573···, 0.693147···, 1.039720···, etc., which we may call y. The original rational x terms progress by sending each fraction's numerator N to 2N+1 and each denominator D to 2D–1. The irrational y values to which these are then sent by the inverse hyperbolic tangent function include 0.693147··· which we again recognize as ln2; accordingly we may remove one more mask by sending that sequence to e^y, obtaining $1, \sqrt{2}, 2, 2\sqrt{2}, \cdots$, whose squares pair $f^n(x)$ ultimately with 2^n. The l.f.t . 3(x-3)/(x-1) sending $3,4,1 \rightarrow 0,1,\infty$ sends $4,7,-5,-\frac{1}{5},\cdots$ to $1,2,4,8,\cdots$, i.e. $f^n(x) \rightarrow 2^n$, and can be realized geometrically by a suitable parallel construction:

Draw any line ℓ through one fixed point (say 1), and any other line m parallel to it. When joined through any point P on first line, other points of the sequence project onto the second line as successive powers of the cross-ratio as base, the other fixed point (3) projecting to 0. The choice of placement of parallel lines ℓ and m as well as projection point P is completely arbitrary (a matter of graphic convenience), nor does it matter which of the two fixed points is sent to ∞ and which to 0, nor which of the original sequence is sent to 1.

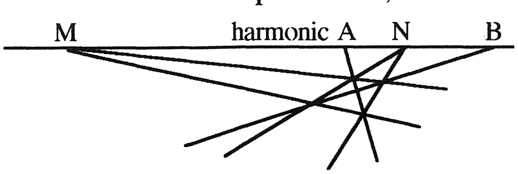


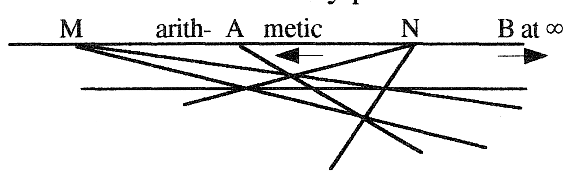
The $f^n(x)$ values themselves are thus projections of tanh(nx) for n = 0,1,2,3,···, representing stages of a *hyperbolic rotation.* After unmasking, we find them corresponding to terms of a *geometric sequence* of exponentials of n to same *base* 2 that was predicted by the invariant *cross-ratio*. They appeared $\frac{1}{2}$-powered (square-rooted) for same reason that Klein's distance and angle metrics were halved: the half-power of cross-ratio is serving as angle-like argument of the hyperbolic tangent function, and logs of half-powers are half-multiples of logs [13].

The Place of Geometric Sequences between Arithmetic and Harmonic in Human Sense Perception: Hearing between Touch and Sight

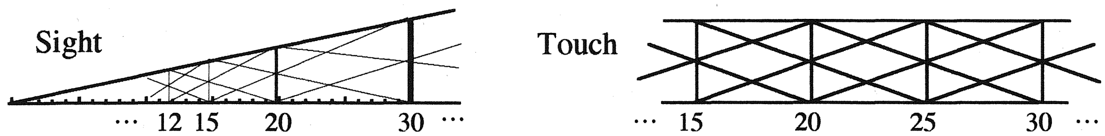
As noted earlier, the base b of the projectively equivalent geometric sequence is predicted by the invariant cross-ratio (AB,MN) of any point A moving to f(A) = B with respect to M & N as ideal fixed points on same line. After some experience computing a few of them, we can observe further that, when M & N are specially placed at ±1 and A is begun at 0 to bring out the connection with hyperbolic rotation stages, then B is at (b-1)/(b+1) for base b and rotation angle $x = \frac{1}{2}\ln b$.

While b = 0 would seem to be allowed by such an expression, we rule it out as falsely predicting B at -1 = tanh(-∞) as next step; it corresponds to cross-ratio 0 of the completely constant transformation, so is not of interest in any case. b = -1, on the other hand, would seem to be disallowed from such an expression, predicting B ambiguously at ±∞ as next step, but that is in fact just what happens! b = -1 is algebraically characteristic of an *involution* satisfying $f^2(x) = x$ just like (and because of) $(-1)^2 = 1$, and geometrically characteristic of *harmonic mean* between two extremes M & N (i.e. $\overline{AB}$ between $\overline{MB}$ and $\overline{NB}$, as lengths); when harmonically-placed A tends toward central arithmetic placement, its mate B tends toward ±∞ remoteness by parallelism.

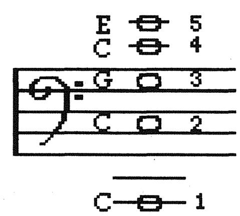




For points $f^n(A)$ to remain bounded between M & N and not "ping-pong" in and out between them, the base must be positive. Among such values, $b = 1$ corresponds to the "degenerate" case with $M = N$. Sending that coincident pair together to ∞ then lets $f^n(A) = x = \cdots, -1, 0, 1, 2, 3, \cdots$, with arbitrarily-chosen 0. These, in turn, are projectively equivalent to any l.f.t. of their values, including $1/x = \cdots, -1, \pm\infty, 1, \frac{1}{2}, \frac{1}{3}, \cdots$. Thus the archetypal cases of arithmetic and harmonic sequences (the integers and their reciprocals) arise together for base $b = 1$, one modelling the experience of our hands and feet sensing the arithmetic spacing of consecutive fence posts we build or walk past, the other modelling their appearance to our eyes in perspective:



The numerical definitions of these two kinds of means are that p, q, r are arithmetic iff $q = \frac{1}{2}(p+r)$ or average between p and r, while p, q, r are harmonic iff $1/q = \frac{1}{2}(1/p+1/r)$, so that e.g. $20 = \frac{1}{2}(15+25)$ at above right while $1/20 = \frac{1}{2}(1/15+1/30)$ at left. Said another way, $\cdots,15,20,25,30,\cdots$ are 5 times integers $\cdots,3,4,5,6,\cdots$ while $\cdots,12,15,20,30,\cdots$ are 60 times reciprocals $\cdots,\frac{1}{5},\frac{1}{4},\frac{1}{3},\frac{1}{2},\cdots$. The geometric mean $q = (pr)^{\frac{1}{2}} = \sqrt{pr}$ is the mean of these two means, for not only does e.g. $15\sqrt{2}$ agree with $\sqrt{15\cdot 30} = \sqrt{450}$ as geometric mean between 15 and 30, it also agrees with $\sqrt{20\cdot 22\frac{1}{2}} = \sqrt{450}$ as the geometric mean between their harmonic and arithmetic means 20 and $22\frac{1}{2}$.



Musically, the integers 1,2,3,4,5,··· model the *vibration rates* of successive harmonics above a fundamental such as C,C,G,C,E,···, scaled to find cycles per second in Bach's era by multiplying by 64 (concert A at $426\frac{2}{3}$ c.p.s.) and nowadays by 66 (for A at 440 c.p.s.), while their reciprocals $1,\frac{1}{2},\frac{1}{3},\frac{1}{4},\frac{1}{5},\cdots$ model the *lengths of the organ pipes* producing those sounds, scaled in Bach's era by multiplying by 8 feet (for low C).

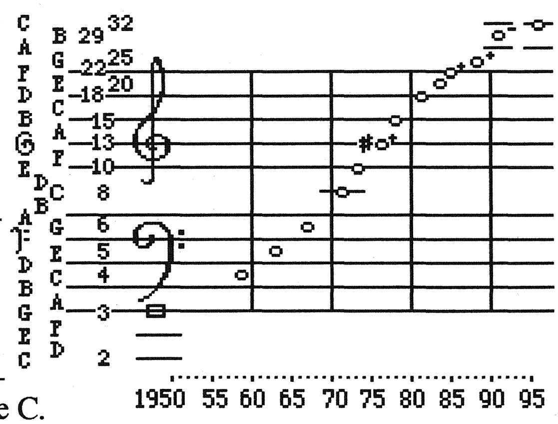
Both of these sequences are thus primarily concerned with the *physics* of music. The miraculous conversion to the *human* perception of musical tones as an æsthetically pleasurable *art* medium is achieved once the physical density waves of air compression have struck our ear drums and their physical poundings e.g. in proportions such as $2{:}4{:}8 = 2^1{:}2^2{:}2^3$ for the 1st, 2nd, 3rd octaves of C (or triple those for octaves of G etc.) are converted by the cochlea into simple 1,2,3 — *the logarithms of the physical stimuli*, as discovered in the 19th century by psycho-physiologists Weber and Fechner [14], solving differential equation $dR = k\,(1/S)\,dS$ to find $R = k\int(1/S)dS = k\ln S$ (response is proportional to logarithm of stimulus).

That the 2nd octave is perceived as lying midway between the 1st and 3rd octave at equal musical intervals below and above it is due the ear's using the geometric mean of vibration rates as its standard of judgment. But whereas hands and eyes had but one such standard apiece, the ear goes on to use the other two means as well as standards for judging the musical equivalent of *coloration* in major and minor modes: In a major triad such as CEG the "warm" placement of note E proportional to 5 c.p.s. is due to its being arithmetic mean between the C and G proportional to 4 and 6 below and above it, while in a minor triad such as EGB the "cool" placement of note G proportional to 12 c.p.s. is due to being harmonic mean between E and B proportional to 10 and 15.

The color analogy is exact, since the octave phenomenon (a doubled stimulation perceived as analog of another stimulation, hence given same name) occurs in chromatic vision as well: The human eye sees only the range of electromagnetic radiation between dark red bordering on infrared and dark blue bordering on ultraviolet, calling the colors at both ends by same name "violet" whether approached via reds or blues, bending that segment in effect into the familiar color circle with darkest violet opposite the brightest-perceived color yellow, and other complementary colors opposite one another around the rest of the circle. Just as pitches proportional to 5 and 6 forming notes E and G contained between 4 and 8 as octaves of C satisfy $4{\cdot}2^n = 5$ and 6 for $n \approx 4/12 = 1/3$ and 7/12 (counting numbers of black and white keys on piano keyboard as 1/12th's of octave between the C's), so wavelengths of the simple lights we perceive to have primary colors blue and yellow also lie at $383\frac{1}{3}{\cdot}2^n$ mμ (millimicrons) for the same $n \approx 1/3$ and 7/12, namely ca. 483 and $574\frac{1}{3}$ mμ (or 4830 and 5743 ångström units) between $383\frac{1}{3}$ mμ and its double as wavelengths of ultraviolet and infrared perception thresholds where we see darkest red fade to black. Both take a linear scale of physical stimuli and bend a 1:2 portion of it around into a circle, divide that circle into 1/12th's, and partition those 12 parts into 4+3+5 (for major, 3+4+5 for minor) triads [15].

P[l]aying the Price of Postage as Expression of Recent U.S. Economic History

As a gesture of national unity in wake of the Civil War, the cost of sending a 1 oz. 1st class letter to any state or territory was fixed in 1863 at 3¢, for which special nickel and silver 3¢ coins were minted on the east and west coasts. This rate held for nearly a century, not being raised to 4¢ until 1958. Thereafter it suffered a 20-year period of inflation with approximate 10-year doubling, a 16¢ rate announced for 1978 being reduced to 15¢ and quickly raised to 18¢, before easing again.

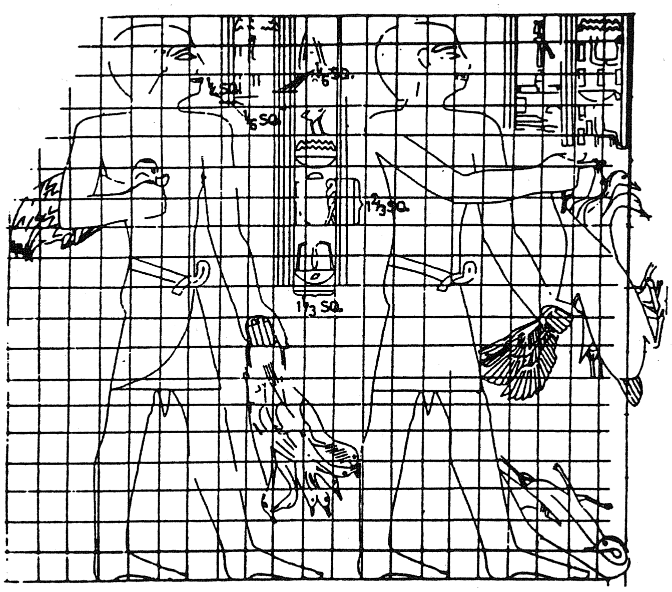


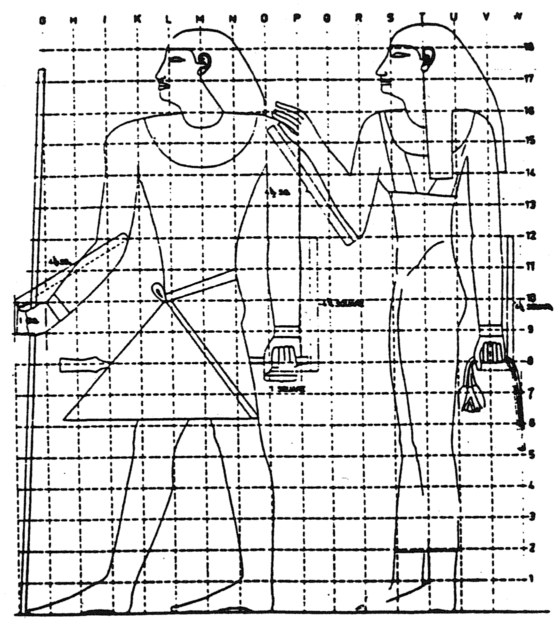
Taking powers of 2 (i.e. exponentials base 2) as C's (for Bach era simplicity) and rates of form $3{\cdot}2^n$ for G's and $5{\cdot}2^n$ for E's etc. and helping non-musicians get over fear of note-reading by offering simple decipherment "key" (*clef*) that the gizmos looking like 𝄢 and 𝄞 are just that: the indicators of lines F and G in so-called bass and treble clefs, counting from there by alphabet A to G to name all other notes (spaces on treble clef famously spelling "FACE" etc.), *we can use ordinary music paper just like semi-log graph paper* — the only catch is that the two clefs must be closed up properly, leaving only room for line for middle C. Then, as we saw at the beginning, once powers of 2 (the octaves) are equispaced graphically, so are powers of every other base, including 10, and *we can graph any function of form* $y = kb^x$ *on it to plot straight*. Above, we can *see and hear* the costs of those 1 oz. 1st class letters from early 50's to late 90's: After a long G (century of 3¢), a C major chord unfolds along straight line of steady 7% inflation. The OPEC crisis causes modulation to A minor (E_7 chord), returning (via G_7) to C after Desert Storm.

Why Powers of 2? — A Possible Non-Euclidean Clue from Ancient Egyptian Art

Since at least the 4th dynasty of Old Kingdom Egypt until 26th dynasty of the New, fragments of artists' sketches for murals survive attesting use of a common grid system involving 18 units of height from ground to hairline (above which a variety of wigs or hats might lie) and 6 units of width at the shoulders (labelled I to P below), one vertical (K) through front armpit, the next (L) through front point of hairline, and next (M) through ear, according to Iversen [16], and experimental applications even to much earlier works like the pre-dynastic Narmer palette work well [17]. In illustration on next page, king (Sesostris I, 12th dynasty) and consort, as well as their servants bringing offerings (a brace of ducks for dinner), though to different scales to show social status (servants are 1/2 size of royalty in original panel), are drawn to same canon: Height 6 lies at knees, 9 at wrist, 12 at waist and elbows, $14\frac{2}{5}$ at armpit, 16 at shoulder, 18 at hairline.

John Anthony West [18] mistakenly considered it a harmonic sequence since *each term is the harmonic mean between the previous term and the ultimate or limit term* (here hairline height 18), *but this is in fact a peculiarity of hyperbolic sequences equivalent to powers of* 2: the l.f.t. (x-9)/9 sends $0, \cdots, 6, 9, 12, 14\frac{2}{5}, 16, \cdots, 18$ to $-1, \cdots, -\frac{1}{3}, 0, \frac{1}{3}, \frac{3}{5}, \frac{7}{9}, \cdots, 1$ [19]. By calling our attention to this canon, however, he may have hit upon a clue to what makes powers of 2 so special for human perception through several senses. Depending on the order of its 4 elements, every cross-ratio can be evaluated in $4! = 24$ ways, but not all of the resulting values are numerically distinct. Usually there are 4 sets of 6 values given by λ,μ,ν and their reciprocals, related to one another via $\lambda = 1/(1-\mu)$, $\mu = 1/(1-\nu)$, and $\nu = 1/(1-\lambda)$. It is the peculiar property of harmonic 4's (or "throws" in 19th century parlance) that $\lambda,\mu,\nu = -1, \frac{1}{2}, 2$, so that $\lambda = 1/\lambda$ and $\mu = 1/\nu$, reducing the 6 values to only 3, occurring 8 times apiece. On one hand, the involutory property that characterizes harmonicity geometrically (whereby A and B are mates with respect to M and N in the earlier example) is literally based on the numerical property $(-1)^2 = 1$. On the other hand (by letting, say, $A \rightarrow N$ with M and B fixed), a projective transformation is determined which, if iterated, is equivalent to repeated multiplication or division by 2. *The two processes arise together — harmony and octaves.*





Notes and Bibliography

[1] Mrs. Isabella Beeton, *International Cookery*, Platt & Peck, New York, 1st ed. (19th cent.) — no date.

[2] Carl B. Boyer, *A History of Mathematics*, rev. ed., John Wiley, New York, 1989, p. 36.

[3] Eli Maor, *e: The Story of a Number*, Princeton University Press, 1994, p. xiii.

[4] ibid., pp. 13-14. [5] ibid., pp. 7-8 and 11 ff.

[6] Raymond Ayoub, "What is a Napierian Logarithm?", *Amer. Math. Monthly*, 100-4, April 1993, p. 362.

[7] Eli Maor, op. cit., pp. 11-13. [8] ibid., pp. 15-16. [9] ibid., Chapters 13 and 14. [10] ibid., p. 145.

[11] Dirk Struik, *Lectures on Analytic and Projective Geometry*, Addison-Wesley, Cambridge, 1953, pp. 163 ff.

[12] Ferdinand Möbius, *Theorie der Kreisverwandtschaften*, Gesammelte Werke II, 1855.

[13] The complete "unmasking" that converts repeated projection steps into a geometric sequence is $(h(g(x)))^2 = e^{2\tanh^{-1}(x)}$ which is reminiscent of Lobachevsky's formula for the angle of parallelism $\Pi(x) = 2\tan^{-1}(e^{-x})$. [The preliminary projection that rescales $f^n(x)$ to lie between ± 1 is suppressed here for purpose of analogy.]

[14] Fritz Winckel, *Music, Sound and Sensation*, Dover Publications, New York, 1967.

[15] Stephen Eberhart, *Math. Through the Liberal Arts*, C.S.U. Northridge, 1991, pp. 64-65, and [19] p. 75.

[16] Erik Iversen and Yoshiaki Shibata, *Canon and Proportions in Egyptian Art*, Aris and Phillips Ltd., Warminster, 1975, p. 47 and Plate 8. [17] ibid., pp. 60-61 and Plate 15. *(After 26th dyn. a canon of 21 was used.)

[18] John Anthony West, *Serpent in the Sky*, Harper & Row, N.Y. et al., 1979, p. 188.