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The Mathematics of the Just Intonation Used in the Music of Terry Riley

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Abstract

Several twentieth-century musicians compose for instruments that offer tunings first described by ancient Greek mathematicians. These tunings allow composers to add to the expressiveness of their music by taking advantage of the greater variety of intervals available and of the richness of the enhanced overtones. In particular, in *The Harp of New Albion*, contemporary American composer Terry Riley has used a five-limit system of just intonation. An analysis of the structure of Riley's system reveals a deviation from more symmetric systems, such as the system of just intonation of Friedrich Wilhelm Marpurg from 1776. Musical and mathematical reasons for the deviation will be considered.

Terry Riley and *The Harp of New Albion*

Terry Riley has composed and improvised music on a piano tuned in a manner different from that of the standard equal-tempered system employed in most pianos. For example, his 1984 composition/improvisation, *The Harp of New Albion*, employs a particular five-limit just intonation. What does this mean? Are there mathematical reasons that can support the selection of this particular tuning system? After a short look at the place of this composition in Riley's career, I'll discuss these questions.

Terry Riley is, with LaMonte Young, Steve Reich, and Philip Glass, one of the composers credited with founding Minimalism, which is characterized by the use of repetitive patterns. In fact, in 1964 Riley's *In C* was "the first publicly performed work to employ the repeated patterns that would come to characterize Minimalism." [7, p. 67]

Concurrently with this exploration of pattern, Riley explored the natural intervals arising from the overtone series via drones and his studies of Indian and Arabic music. The piece, *The Harp of New Albion*, created for piano in just intonation, is "the third work in a series of Native American Mythological Portraits"; the other two works, *The Medicine Wheel* and *Cadenzas on the Night Plain*, are for string quartets. The title *The Harp of New Albion* refers to a legendary instrument left in the New World, specifically in the San Francisco Bay area (New Albion), by a crewman of Sir Francis Drake on his voyage of 1577-1580. [4, liner notes by Riley] The composition has not been notated on paper, rather, it is an improvisation for a piano tuned in just intonation.

The term "just intonation" refers to any selection of frequencies for the pitches of a musical scale such that the ratios of the frequencies are rational numbers. Almost all pianos are at present tuned in "equal temperament," in which every half-step is the same size. That is, the ratio of the frequency of the pitches of any key to the next on the keyboard is the same frequency ratio, namely $\sqrt[12]{2}:1$, which is an irrational number. Thus, equal temperament is not a just intonation.

The first performance of *The Harp of New Albion* was in 1984 in Köln; the two-disc recording of the full piece [4] and the excerpts on *The Padova Concert* [6] were from two performances in 1986. Riley's insistence in this period on playing on a piano tuned in just intonation was demonstrated during a joint residency with the Kronos Quartet at Oklahoma State University in 1987, during which he declined to play any of his music on an equal-tempered piano. He recently relaxed this standard in a concert given in Lisbon in 1995; *Lisbon Concert* [5] is played entirely on a piano in equal temperament.

Mathematical Principles of Keyboard Tuning

The selection of pitches in most scales for keyboards in western music is based on a few mathematical principles:
(1) Intervals are determined by the ratios of the fundamental frequencies of the pitches. For example, the ratios of the frequencies between a middle C on the piano and the C below it is 2:1. Similarly, the ratio of the frequencies between the F above middle C and the F below middle C is also 2:1, so this interval between the F's is the same as that between the C's. Often the intervals will be referred to by their ratios.
(2) A consequence of this first principle is that adding intervals corresponds to multiplying ratios. That is, if we consider pitches C, G, and the octave C, such that the interval from C to G corresponds to the frequency ratio 3:2, and the interval from G to the octave C corresponds to the frequency ratio 4:3, then the interval from C to the octave C corresponds to the frequency ratio (3:2)*(4:3) = 4:2 = 2:1.
(3) The octave, namely the interval with the ratio 2:1, is to be sacrosanct, in that any two pitches with the same name must have frequencies that are in the ratio of a power of 2.
(4) In contrast, the term "fifth," without modification, will be used for any interval between two pitches that are produced by keys on the keyboard that are seven keys apart. For example, the interval from middle C to the G above is a fifth; beginning with C, we in turn count the keys C#, D, D#, E, F, F#, ending with G. The ratio corresponding to a fifth may be 3:2, in which case the interval in question is a perfect fifth, or may be a ratio close to this, such as $2^{7/12}$, an equal-tempered fifth. Thus the term "fifth" in fact refers to a family of intervals. Keys that are five keys apart form a family of intervals called a "fourth" (e.g., C to F); keys that are four keys apart produce a family of intervals called a "major third" (e.g., C to E).

The goal in producing a tuning system for a piano is to identify ratios for the frequency of notes for the eleven piano keys between one pitch and the octave above it. The frequencies for all other piano keys can be determined by multiplying by the appropriate positive or negative power of 2. Conversely, any frequency ratio larger than 2 or smaller than 1 can be multiplied by an appropriate power of 2 to give a unique ratio between 1 and 2. For example, if we decide to begin a tuning system with C as the base pitch, then we can find a frequency ratio for D by using two perfect fifths (namely C to G and G to D), giving us a ratio of (3:2)*(3:2) = 9:4, then multiplying by 1/2 to obtain (1/2)*(9:4) = 9:8, a ratio between 1 and 2.

Five-Limit Just Intonation Systems

Just intonation systems are those in which the frequency ratios of all the intervals are rational numbers. One justification for such a tuning system is that it is in accordance with the series of harmonics of a plucked string, which is producing not only the fundamental pitch corresponding to the vibration of the full length of the string, but also pitches with frequencies that are integer multiples of the fundamental frequency from vibrations of halves, thirds, fourths, and so on, of the string. Another justification is that the upper harmonics of strings vibrating at frequencies that are rational multiples of each other in fact reinforce each other.

The fundamental theorem of arithmetic guarantees that every whole number can be expressed uniquely as a product of prime numbers. Just intonation tuning systems in which the ratio of any two frequencies is a rational number that can be expressed in terms of powers of only the primes 2, 3, and 5, are called "five-limit" systems. "Five-limit" refers both to the largest prime employed as well as to the highest member of the harmonic series that is considered as relevant in the tuning. Such a system admits as basic intervals the octave, with frequency ratio 2:1, the perfect fifth, with frequency ratio 3:2, and the "perfect" major third, with frequency ratio 5:4.

Pythagorean Tunings. Even with this restriction to a five-limit system, there are many different tunings possible for the standard octave of twelve keys found on a piano keyboard. For example, the Pythagorean tuning is based on the cycle of fifths, playing on the fact that twelve perfect fifths is nearly the same as seven octaves. Specifically, beginning with any note on the keyboard, counting up either seven octaves or twelve perfect fifths would lead to the same key. The pitch played by this key should have a frequency ratio $(2:1)^7 = 128:1$ above the original key, taking into account the seven octaves, but also a frequency ratio $(3:2)^{12} = 531441:4096$, based on the twelve perfect fifths. These ratios are not the same; in fact, the fundamental theorem of arithmetic guarantees that they must be different. They are nearly equal, in that the ratio of the larger to the smaller has decimal approximation 1.0136, an interval called the Pythagorean comma. To use the Pythagorean tuning to obtain ratios of frequencies that could be used for the pitches in a piano keyboard octave, simply multiply by appropriate powers of 2 to pull the frequencies into the interval between 1 and 2. The corresponding pitch frequencies would then be those found in the first column of ratios in Table 1, where the lowest pitch C is found in the bottom row and the octave C in the top. Notice that the interval F to C, while a fifth, has the ratio (2:1) / (177147:131072) = 262144:177147 rather than 3:2, so that this is not a perfect fifth. Also, the major third C to E has the ratio 81:64 rather than the "perfect" 5:4. In fact, in this tuning, eleven of the twelve fifths are perfectly tuned, while none of the major thirds are.

Pitch	Pythagorean ratios	Pythagorean ratios, recentered
C	2:1	2:1
B	243:128	243:128
A#	59049:32768	16: 9
A	27:16	27:16
G#	6561:4096	128: 81
G	3:2	3:2
F#	729:512	729:512
F	177147:131072	4:3
E	81:64	81:64
D#	19683:16384	32:27
D	9:8	9:8
C#	2187:2048	256:243
C	1:1	1:1

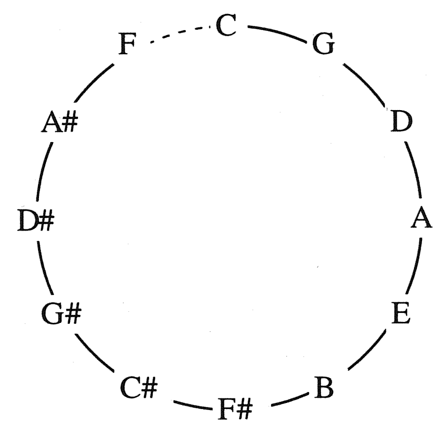
Table 1

Diagrammatically, this Pythagorean tuning is represented in Figure 1 by showing the cycle of fifths with solid horizontal lines between the pitches tuned so that the intervals formed have ratio 3:2 and the dotted horizontal lines indicate fifths that are not perfectly tuned. This line can be regarded as continuing without end in each direction. Because of the periodic nature of the letters indicating the pitches, it is convenient to identify them, changing the line to a circle, shown in Figure 2.

...–D# – A# – F···C – G – D – A – E – B – F# – C# – G# – D# – A# – F···C – G –...

Solid lines represent perfect fifths and dotted lines non-perfect fifths

Figure 1: Pythagorean Tuning



Solid arcs represent perfect fifths and dotted arcs non-perfect fifths

Figure 2: Circle of Fifths representing Pythagorean Tuning

In practice, it may be more convenient to use the ratios in the second column of Table 1, which is constructed by going down five perfect fifths from C and up six perfect fifths from C. Now the mistuned fifth is between C# and F#. The diagram below indicates this recentered version of the Pythagorean tuning.

...–E – B – F#···C# – G# – D# – A# – F – C – G – D – A – E – B – F#···C# – G# –...

Solid lines represent perfect fifths and dotted lines non-perfect fifths

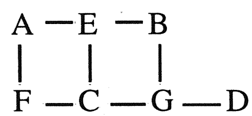
Figure 3: Pythagorean Tuning, recentered

Syntonon Diatonic Scale. The Pythagorean tuning ignores any consideration of perfectly tuning major thirds. A tuning for the eight notes, forming the pitches of a major scale that takes the major thirds into account, is one of several scales discussed by Claudius Ptolemy (2nd century AD). This is called the syntonon diatonic scale; the ratios of the frequencies are displayed in the first column of Table 2.

Pitch	Syntonon diatonic scale	Compact tuning	Marpurg tuning
C	2:1	2:1	2:1
B	15: 8	15: 8	15: 8
A#		9:5	9:5
A	5:3	5:3	5:3
G#		8:5	25:16
G	3:2	3:2	3:2
F#		45:32	45:32
F	4:3	4:3	5:3
E	5:4	5:4	5:4
D#		6:5	6:5
D	9:8	9:8	9:8
C#		16:15	25:24
C	1:1	1:1	1:1

Table 2

Figure 4, again using solid horizontal lines for fifths with ratio 3:2, and now with vertical solid lines for major thirds with ratio 5:4, illuminates the relationships of the syntonon diatonic scale.

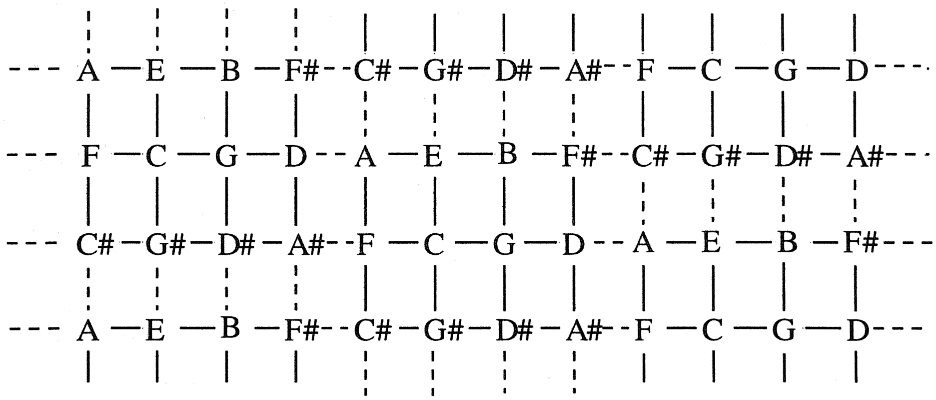


Solid horizontal lines represent perfect fifths; vertical solid lines represent perfect major thirds

Figure 4: Syntonon Diatonic Scale

Compact and Marpurg Tunings. There are now many ways to finish this scale to provide frequencies for all the pitches. The diagram for one possibility, called the compact tuning, is shown in Figure 5, and in abbreviated form in Figure 6. Notice that the configuration of pitches tuned perfectly is symmetric and very compact. The frequencies for the compact tuning appear in the second column of Table 2, filling in the gaps in the syntonon diatonic scale. Notice that in the compact tuning, nine fifths are tuned perfectly, as are eight major thirds.

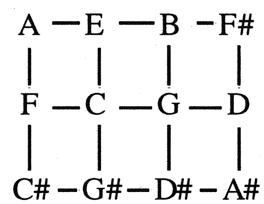
Another proposal to complete the syntonon diatonic scale is one attributed to Friedrich Wilhelm Marpurg (1776). Its abbreviated diagram is shown in Figure 7, and the frequency ratios are displayed in the third column of Table 2. The diagram has a pleasing rotational symmetry, but offers only eight fifths and eight major thirds tuned perfectly.



Solid horizontal lines represent perfect fifths and dotted horizontal lines non-perfect fifths

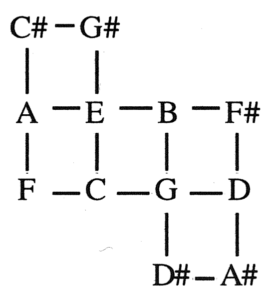
Solid vertical lines represent perfect major thirds and dotted vertical lines non-perfect major thirds

Figure 5: Compact Tuning



Horizontal lines represent perfect fifths and vertical lines perfect major thirds

Figure 6: Compact Tuning, abbreviated diagram



Horizontal lines represent perfect fifths and vertical lines perfect major thirds

Figure 7: Marpurg Tuning, abbreviated diagram

We noted above, in the context of the one-dimensional diagram indicating the Pythagorean tuning in Figure 2, that an identification of the pitches with the common letters created a circle. In other five-limit tunings, those taking into account also major thirds, the similar identification of pitches leads to circles on a torus, the surface of a doughnut; we show all lines as solid for ease in discerning the pattern. The circle representing the Pythagorean cycle of fifths winds three times around the hole of the doughnut; the triangles of major thirds below represent circles passing through the hole; see Figure 8.

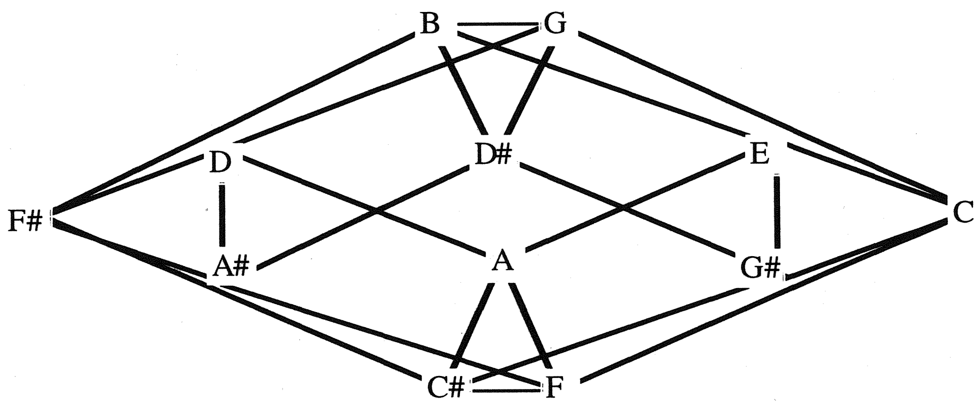
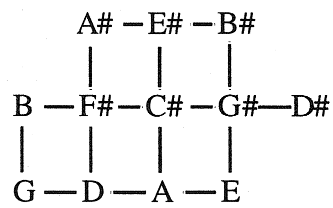


Figure 8: Torus of Fifths and Major Thirds

Riley's Tuning. The tuning used by Terry Riley in *The Harp of New Albion* is a five-limit tuning that is also related to the syntonon diatonic [4, liner notes by Riley]. The tonal center for Riley's tuning is C#; an abbreviated diagram for his tuning is shown in Figure 9, accompanied by a listing of the ratios in Table 3. The names for the pitches are those he uses.

Pitch	Riley's Tunings
C#	2:1
B#	15: 8
B	16: 9
A#	5:3
A	8:5
G#	3:2
G	64:45
F#	4:3
E#	5:4
E	6:5
D#	9:8
D	16:15
C#	1:1

Table 3: Riley's Tuning



Horizontal lines represent perfect fifths and vertical lines perfect major thirds

Figure 9: Riley's Tuning, abbreviated diagram

Mathematical Evaluation of Tuning Systems

What might be mathematical reasons to choose Riley's tuning rather than another? While one might have expected Riley to employ a tuning system with a symmetric diagram, he has not; compare the solid lines above to those in the diagram for the compact tuning or for Marpurg's tuning. Further, in both Riley's and the compact tunings, nine fifths are tuned perfectly, but in Riley's tuning only seven major thirds are tuned perfectly, compared to eight in the compact tuning.

An important consideration, of course, is that a musician will choose one tuning over another for musical reasons, rather than mathematical ones. A musician may simply have liked a piece in one tuning more than another. Or, perhaps, the particular major third tuned perfectly in the compact tuning but not in Riley's may have been irrelevant for a piece. But might there be mathematical reasons to prefer Riley's tuning that haven't yet been considered? After all, the mathematical considerations of numbers of perfectly tuned fifths and thirds and symmetry and compactness of the diagram are rather arbitrary mathematical measures of a tuning system.

One mathematical reason for preferring Riley's tuning to a compact one or to Marpurg's is to count the number of solid lines required to move from the tonal center to any of the notes in the system. In Riley's tuning, only one of the notes, namely G, is at distance 3 from C#; all the others are at distance only one or two from the center. In contrast, in the compact tuning, no matter what center is picked, at least two notes will be at distance 3 from the center; the situation is even worse in Marpurg's.

Another mathematical reason can be given to prefer Riley's tuning. This reason is also based on symmetry; not symmetry of the diagram, but rather symmetry of the ratios themselves about the tonal center. To see this, notice that the ratio of a pitch k (half) steps above C#, times the ratio of the complementary pitch k (half) steps below C#, gives 2:1, the ratio for the octave. For example, the ratio 9:8 for D# times the ratio 16:9 for B gives 2:1. This holds for all the ratios in the octave with one exception, the ratio 64:45 for the infamous interval of a tritone, here the interval C#-G. To have total symmetry, this ratio would have to be $\sqrt{2}$, but as this is an irrational number, no just intonation can have total symmetry in this sense. (The ratio 64:45 is a reasonable (as well as rational) approximation of $\sqrt{2}$ in the five-limit system, as are 45:32, 36:25, and 25:18. In a seven-limit system, 7:5 is an attractive alternative.) In both the compact tuning and Marpurg's tuning, in addition to the unavoidable tritone, there are at least two complementary ratios that fail to multiply to 2:1.

While these considerations of distance and numerical symmetry also have some musical validity, they seem to be unlikely reasons to select Riley's tuning system for *The Harp of New Albion*. Each movement of this piece has its own tonal center. None is centered at C#, the center of the tuning. In fact, the tonal center for the movements "Ascending Whale Dreams" and "Circle of Wolves" is B#; there are pitches at distance 5 from B# in Riley's tuning. These movements depict wild, untamed creatures, calling perhaps for less harmonic tranquillity than the other movements, but nevertheless, the use of this pitch center would seem to mitigate distance and numerical symmetry considerations for the selection of this particular tuning.

In any case, distance and numerical symmetry considerations provide mathematical reasons for preferring Riley's system of just intonation to either a compact one or Marpurg's, even though considerations of geometric symmetry would argue against that preference. We believe that considerations of the mathematical qualities of systems of tunings can provide the growing number of composers exploring just intonations and other alternative tunings with techniques for evaluation of systems and also suggestions for further exploration.

References

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