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Mathematical Connections
in Art, Music, and Science

The Mathematics of Steve Reich's *Clapping Music*

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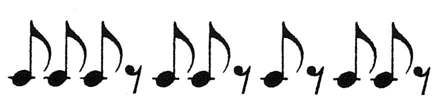
Mathematics and music have been associated for millennia. Among the musical connections are symmetries and patterns used in composition, such as canons and fugues, the use or appearance of the golden mean in the compositions of Dufay [14]and others [10], the appearance of Fibonnaci numbers in the compositions of Bartok [1, 8], tone rows in serial music, the stochastic music of Xenakis, and the recent appearance of fractal music [3]. Mathematics provides insight into the overtone series, a basis for systems of tuning and the timbre of instruments. A deeper, more profound connection is explored in Rothstein's *Emblems of Mind* [13; also see reviews 2, 15], which suggests that mathematics and music may each be approaches to Platonic ideals of truth and beauty. At a foundational level of music, pitch and rhythm are also two mathematical connections to music. Usually both of these constituents appear in a piece. Here, however, we will examine in some detail the mathematics suggested by Steve Reich's *Clapping Music*, a piece based on rhythm alone. (Recordings include *Steve Reich: Early Works*, Elektra/ Nonesuch P 79169, 1987 and *An American Collection*, Harry Christophers, The Sixteen, Collins Classics 12872, 1992.)

Steve Reich wrote *Clapping Music* in 1971 as a reaction to the burden of traveling with thousands of pounds of electronic equipment and musical instruments that were required for performances by his ensemble. This led to his interest in creating a piece of music that could be performed by musicians without instruments, other than their bodies [11]. Two of Reich's earlier pieces were created using tape loops of text spoken by a street preacher, *It's Gonna Rain* (1965), and a member of the Harlem Six, *Come Out* (1966). These achieve their interest by manipulation of the recordings, including playing one copy of the spoken text against another, creating a slippage because of the variation in speeds of the play back, allowing a gradual shift out of phase with one another [12; see also 6].

Clapping Music provides a discrete, human version of creating music via this kind of variation. It is written for two performers who clap the piece. Both performers begin by clapping a rhythm consisting of eight claps and four pauses, arranged in a pattern of three claps, followed in turn by a pause, two claps, a pause, one clap, a pause, two claps, and a pause:



where here, of course, an eighth note ♪ indicates a clap and an eighth rest 𝄾 a pause.

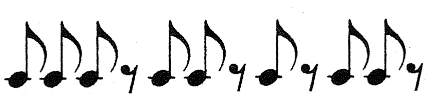
This pattern is repeated for some agreed upon number of times chosen by the performers; in fact, the first performer will continue to clap the same pattern throughout the piece. The second person, however, after some number of repetitions of the original pattern in unison with the first performer, changes patterns by moving the content of the first beat of the pattern to the end, that is, the second performer next claps the following pattern:

The second performer continues to clap this variation of the pattern against the original that is clapped by the first performer for some number of repetitions, after which the second performer again changes the pattern by moving the contents of the first beat of the second pattern to the end, obtaining another variation of the pattern. The piece continues, cycling through all twelve variations of the pattern, each variation obtained by moving the contents of the first beat of the previous variation to the end. The piece ends with a number of repetitions of the unison clapping; the twelfth variation is identical with the original pattern.

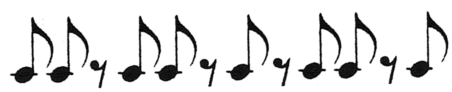
What is a mathematical analysis of this composition? The question is a good beginning, but in fact is too narrow if we seek all the relationships between mathematics and music even in this context. In addition, we may well ask, and will later in the paper, what mathematical questions are suggested by this piece? [See [5] for a discussion of the kinds of question one should expect to ask.] Nevertheless, let us begin with a description of *Clapping Music* in mathematical terms. The piece demonstrates the action of a cyclic permutation acting on the contents of the beats of the original pattern. Since there are twelve beats, the action is by an element σ of S_{12}, the set of permutations of twelve symbols. Applying σ to the original pattern gives the first variation; applying σ some number k of times gives the kth variation, and, because there are twelve beats, applying σ twelve times brings us back to the original pattern. Applying the permutation k times can be described as the result of a related permutation applied just once, namely the permutation σ^k. Applying σ^{12} to the original pattern returns the original pattern to us, thus the action of σ^{12} is the same as applying the identity permutation that leaves the contents of each of the twelve beats fixed. As the application of σ raised to any power less than twelve in fact yields a new variation, σ is in fact a 12-cycle.

What does this description bring us? One benefit is that it leads to a partial answer to the question of, "Why this composition?" Steve Reich, even after the conception of a piece constructed along the lines of *Clapping Music*, would have to select some pattern as a starting point. Limiting the question further, how many different compositions could be created from patterns of eight claps and four non-adjacent rests? If we regard the composition as something stretching endlessly in time in both directions, it would be clear that two original patterns lead to the same composition if one is a variation of the other under the application of σ or one of its powers. It would be appropriate then to regard two original patterns as equivalent if they are in the same "orbit" of σ. Combinatorial computations lead to the answer that there are 10 distinct compositions that can be obtained from an eight clap, four non-adjacent rest patterns. Only eight of these allow cycling through twelve distinct variations. The one selected by Reich for this composition is one of only two that has no consecutive repetitions of the numbers of claps between consecutive rests [4].

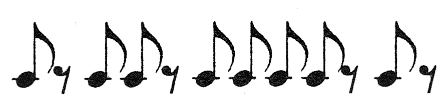
The description in terms of the action of an element of S_{12} also inspires the suggestion of an alternative to the structure of "Clapping Music." Instead of acting on the original pattern with a 12-cycle, could we instead choose to act on the pattern with an element of S_{12} of larger order, that would allow more variations before returning to the original pattern? The answer is yes, in fact, a permutation in S_{12} can have order as large as 60. Consider the permutation τ determined by dividing the original pattern into three sections, the first containing four beats, the second three, and the third the remaining five beats. Allow τ to act on the pattern by moving the content of the first beat to the end of the first section, so that the content of the first beat moves into the fourth position, the content of the first beat of the second section to the end of the second section, and the content of the first beat of the third section to the end of the third section, that is, to the end of the pattern. The diagram below shows the division of the original pattern and the results of applying τ to the pattern. Coincidentally, we obtain the same variation as we do when we apply σ to the original pattern, but a second application of τ shows a new variation:



original



one application

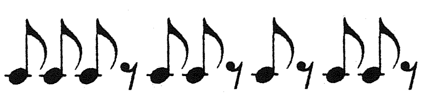


two applications

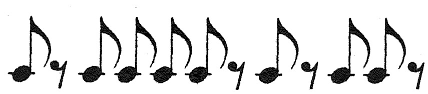
What is the order of τ? That is, how many times must τ be applied before we return to the original pattern? Notice that the content of the first section of four beats returns to its original state after every fourth application of τ, that the content of the second section containing three beats returns after every third application, and the content of the third section of five beats returns after every fifth application. In consequence, the entire pattern does not return to its original state until we have applied τ enough times to reach the least common multiple of 3, 4, and 5, namely 60.

I have made a recording of a piece of music that is constructed from τ instead of from σ. Instead of using two performers clapping, the part of the first performer repeating the original pattern is played on the middle C, while the part of the second performer is played on three different notes. The contents of the first section of four beats is played on F (a fourth above C), of the second section of three beats on E (a third above C), and of the third section of five beats on G (a fifth above G). This provides an auditory realization of such mathematical ideas as the action of a group element (offered by *Clapping Music* as well), the computation of a least common multiple, and the proposition from group theory that the order of the product of two disjoint cycles is the least common multiple of the lengths of the cycles.

Listening to this piece has intrigued students from the middle grades through graduate studies; questions are raised that might not otherwise occur. For example, a variation and the original pattern would naturally be judged quite dissimilar, that is far apart, if no rest (absence of claps) in the variation coincides with a rest in the original pattern. The last variation, the result of applying τ^{59} to the original pattern, is far from the original pattern in this sense, even though it will take only one more application of τ to recover the original pattern. Consider these variations:



original



thirtieth variation

fifty-ninth variation

The thirtieth variation is quite close to the original pattern, yet is as far away from the original pattern in terms of the power of τ as is possible. Conversely, the fifty-ninth variation disagrees with the original pattern in many beats, yet it is only one application of τ removed from the return to the original pattern. The tension between close in the sense of coincidence of the pattern and close in the sense of how close the power of the permutation is to 60, giving the identity permutation, is of course part of the esthetics of the piece that increases the satisfaction at the resolution of the piece.

We have seen that mathematical considerations can lead to a (slightly) new musical composition. It is also true that mathematical questions can arise from the music. A natural one in this context is to inquire about the maximal order for a permutation on n objects, i.e., for an element of S_n, for different values of n. It may come as a surprise that this sequence is not strictly increasing; for example, an element of largest order in S_{13} has order 60, as is the case for S_{12}. In fact, it is the case that there are arbitrarily many consecutive values of n for which the largest order in S_n remains the same, without increasing [9]. There is no closed formula for the maximal orders, so that this provides then a fertile ground for investigation by undergraduate students. The sequence of maximal orders clearly (though some students give this a little thought!) goes to infinity. At what rate? The answer is most easily expressed as

$$\log(\textit{largest order in } S_n) \sim \sqrt{n \log(n)},$$

so that the sequence proceeds to infinity faster than any power of n [7]. As another example, we can simplify the problem of maximal order in general by restricting our attention to permutations that act on exactly two disjoint sections of a pattern of n objects. In this case, the maximal order for any value of n can be explicitly computed, but depends on whether n is odd, divisible by 4, or even and not divisible by 4. This sequence goes to infinity at the same rate as n squared.

In conclusion, a mathematical consideration of *Clapping Music* leads to interesting developments in both music and mathematics. The interaction between mathematics and music is indeed strong and productive for both disciplines.

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http://www.research.att.com/~njas/sequences/index.html

for reference 9.

References

[1] Tibor Bachmann and Peter J. Bachmann, *An Analysis of Bela Bartok's Music through Fibonaccian Numbers and the Golden Mean*, The Musical Quarterly, Vol. 45, pp. 72-82, 1979

[2] Dan Fitzgerald and Joel Haack, *Book Review: Emblems of Mind: The Inner Life of Music and Mathematics by Edward Rothstein*, Humanistic Mathematics Network Journal, Vol. 14, pp. 34-35, November 1996

[3] Martin Gardner, *White, Brown, and Fractal Music*, Scientific American, Vol. 238, pp. 16-32, April, 1978, and reprinted in *Fractal Music, Hypercards and More . . .*, New York: W. H. Freeman and Company, pp. 1-23, 1992

[4] Joel K. Haack, *Clapping Music--A Combinatorial Problem*, The College Mathematics Journal, Vol. 22, pp. 224-227, May 1991
[5] Joel K. Haack, *Connections: Mathematics and the Humanities*, Humanities Education, Vol. 6, pp. 3-6, Spring, 1989
[6] Douglas R. Hofstadter, *Parquet Deformations: A Subtle, Intricate Art Form*, Scientific American, Vol. 249, pp. 14-20, July, 1983, and reprinted in *Metamagical Themas: Questing for the Essence of Mind and Pattern*, New York: Basic Books, pp. 191-212, 1985
[7] Edmund Landau, *Handbuch der Lehre von der Verteilung der Primzahlen*, Leipzig and Berlin: B. G. Teubner, 1909
[8] Erno Lendvai, *Béla Bartók: an analysis of his music*, London: Kahn & Averill, 1979
[9] Jean-Louis Nicolas, *Ordre Maximal d'un Élément du Groupe S_n des Permutations et 'Highly Composite Numbers'*, Bulletin de la Société Mathématique de France, Vol. 97, pp. 129-191, 1969
[10] Clive B. Pascoe, *Golden Proportion in Musical Design*, DME thesis, University of Cincinnati, 1973
[11] Steve Reich, *Liner Notes*, Steve Reich: Early Works, Elektra/Nonesuch P 79169, 1987
[12] Steve Reich, *Writings about Music*, Halifax, NS: The Press of the Nova Scotia College of Art and Design, 1974
[13] Edward Rothstein, *Emblems of Mind: The Inner Life of Music and Mathematics*, New York: Random House, Inc., 1995
[14] Margaret Vardell Sandresky, *The Golden Section in Three Byzantine Motets of Dufay*, Journal of Music Theory, Vol. 25, pp. 291-306, 1981
[15] Sandra Z. Zeith, *Book Review: Emblems of Mind, The Inner Life of Music and Mathematics by Edward Rothstein*, Humanistic Mathematics Network Journal, Vol. 14, pp. 30-33, November 1996