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Mathematical Connections
in Art, Music, and Science

Origami Tessellations

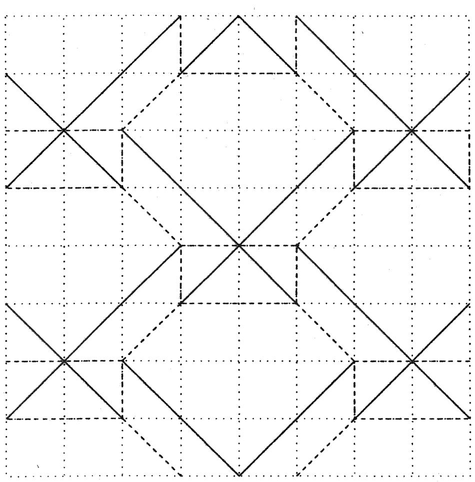
Helena Verrill
Department of Mathematics and Statistics
Queens University, Kingston, K7L 3N6 Canada
E-mail: helena@mast.queensu.ca

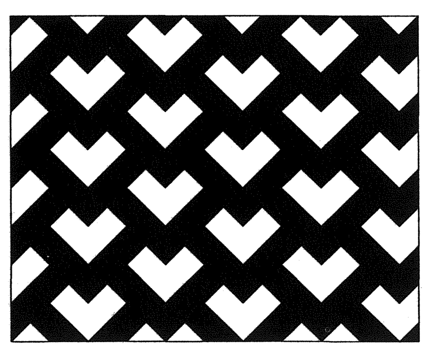
1. Introduction

Origami tessellations are made from a single piece of paper, which is folded in a repeating pattern. Figure 1.a shows an example of a crease pattern for an origami tessellation. This pattern was created with the aim of producing a finished pattern of repeating "heart" shapes.

The problem of finding tessellations which can be folded is difficult because the paper must not be stretched or cut, and must end up as a flat sheet, so this imposes many conditions on the pattern to be folded. An understanding of the geometry of tessellations and of paper folding is required. However, the study of paper folding is still in its infancy, and many questions which have been answered for Euclidean constructions remain unanswered for origami methods; for instance, although impossible with Euclidean geometry, it is simple to trisect an angle using folding techniques, (see [Huz]). However a satisfactory list of axioms, and a list of exactly what is possible, and what is not possible, for origami-geometry constructions has not been found conclusively, though several attempts have been made (See [H4]). On the other hand, though some geometric constructions may be easier with origami, the problem of determining whether a crease pattern can be collapsed to give a flat origami, or even folded at all, is generally very difficult (see [BH] and [K]).

Although tessellations in general have been extensively studied, (see [GS]), little material is available specifically on origami tessellations. This subject was probably first developed by Fujomoto, [F]. For a brief history of origami tessellations consult [L].





Left:Figure 1a: Crease pattern
Above: Figure 1b: Light pattern

FIGURE 1. Example of an origami tessellation ("Hearts")

Here I describe what origami tessellations are, and give various methods of generating these patterns. First, four methods are given which can be used to turn ordinary tessellation patterns into crease patterns. Then modifications of these patterns are considered, ways of adding to an old origami tessellation to create a new result. Next relationships between the methods are considered, and then some simple families are parameterized. Finally we take a look at a different kind of origami tessellation problem, how to fold so that we have a certain number of layers of paper.

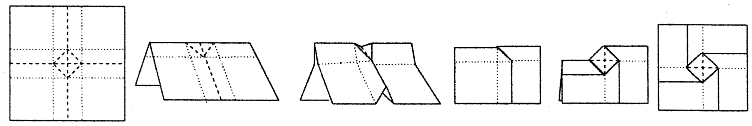


FIGURE 2. Instructions for folding one unit of an origami tessellation

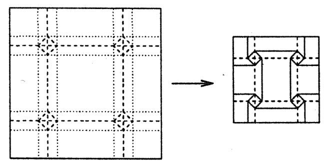


FIGURE 3. From crease pattern to collapsed origami tessellation.

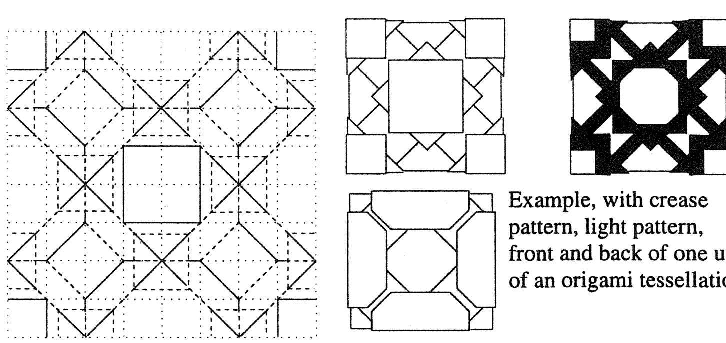
2. Definitions

2.1. Origami Tessellations. By an *origami tessellation*, I mean a flat foldable repeating origami. I use the word *origami* to refer to both the crease pattern and the finished folded object. Technically, we can define an origami tessellation as follows:

Definition: An origami tessellation is a piecewise linear isometry of $\mathbb{R}^2$ which is invariant under some crystallographic group.

2.2. Crease Patterns. It is most common for origami instructions to be given in step by step diagrams, as in Figure 2. However, for origami tessellations it is often simplest to give a crease pattern, for instance that in Figure 3. Carefully fold along the dashed lines in one direction, and the dotted lines in the other direction, and collapse to form a flat origami. Note that the crease pattern in Figure 3 consists of four of the units in Figure 2. We can join together many units, and fold a repeating pattern, to obtain an origami tessellation. Many, but not all origami tessellation patterns can be produced in this way. In [H2] these units are referred to as *orimorphic units*, and their properties are discussed.

An origami tessellation involves four tessellation patterns: (i) the crease pattern, (ii) the pattern of the front when folded, (iii) the pattern of the back when folded, and (iv) the light pattern, obtained by holding the origami up to the light, and looking through it. An example of each of these different kinds of patterns can be seen in Figure 4. The first pattern shows the creases which will be made in the square of paper, and the other pictures show the result after folding. More details of how to fold this pattern can be found at [V1].



Example, with crease pattern, light pattern, front and back of one unit of an origami tessellation.

Generally I simplify the light pattern, and simply distinguish between areas where there are one or more than one layer of paper. For light patterns which show more subtle distinctions see [P1].

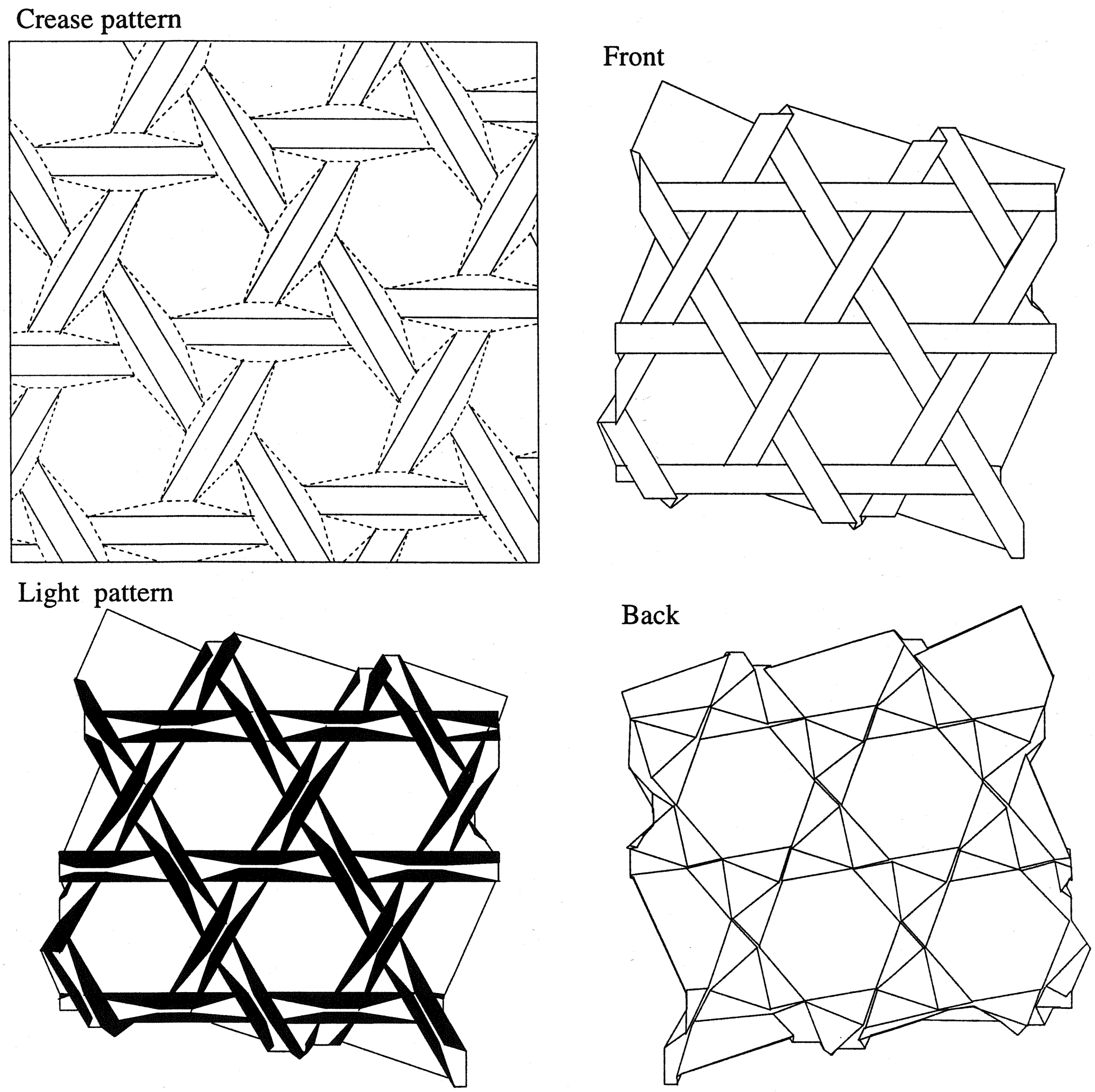


FIGURE 4. Crease Pattern, light pattern, and back and front of folded origami

Also I will often simplify the crease patterns, and not distinguish between the direction (mountain or valley) of the folds.

See [BP],[B], [P1]-[P3], [V1] for examples, [H1] for an introduction to origami geometry, and [KY] for more rigorous mathematical foundations.

3. Methods of producing origami tessellation patterns

In this section we will look at four different methods of producing origami tessellations, the first uses hinged tilings, the second method turns lines to pleats; the third produces weave like patterns, and the fourth produces patterns with star like configurations.

Most of the examples in this section are based on a kind of tiling called *Archimedean tilings*. All the vertices of these tilings are the same, and the edges all have the same length. Given an Archimedean tiling, its *dual tiling* can be constructed by drawing lines joining the mid points of faces of adjacent tiles, as shown in the examples in Figure 6. For more details about Archimedean tilings in general, see [GS].

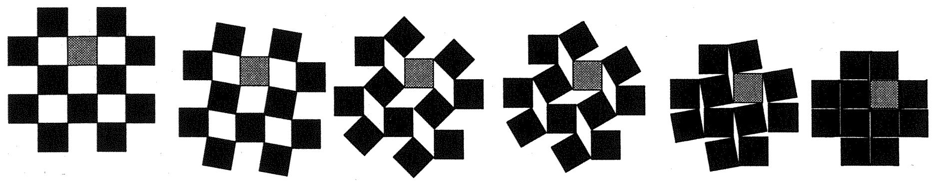


FIGURE 5. An example of a hinged tiling.

3.1. Hinged tilings. Some tiling patterns can be hinged (see for example [W]), as shown in Figure 5. It turns out that hinged tilings produced from Archimedean tilings can be folded, assuming the crease directions are assigned correctly, and that the angles between tiles are not too large. This method of producing origami tessellation was developed by Palmer, Barreto, and Bateman.

The procedure is as follows:

1. Begin with an Archimedean tiling.
2. Space out the tiles.
3. Add the dual tiling between the original tiles.
4. Rotate the tiles relative to the dual tiling.
5. Scale the dual relative to the original tiling as desired.

An example is shown in Figure 6. This figure shows how to obtain the hinged tiling; the lines forming the edges of the tiles may be creased to give an origami tessellation. In Figure 7 another example is shown, together with the light pattern of the resulting folded origami. In this case the dual is scaled to be much smaller than the original tiling. By using orimorphic units, Hull shows how the hinged tiling method does indeed give rise to foldable crease patterns. See [BP], [P1]-[P3], [B], [H2] for more detailed explanations.

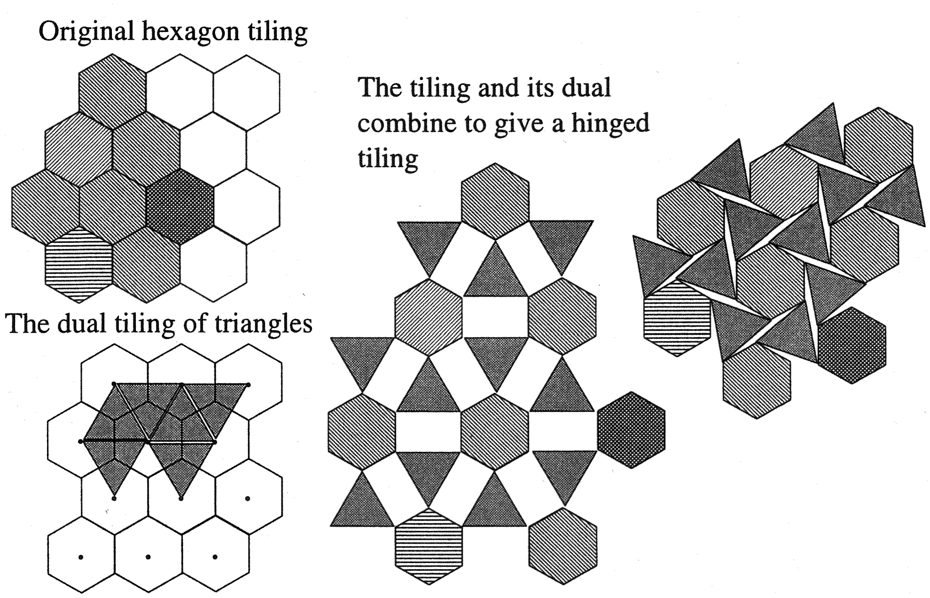


FIGURE 6. How to make a hinged tiling

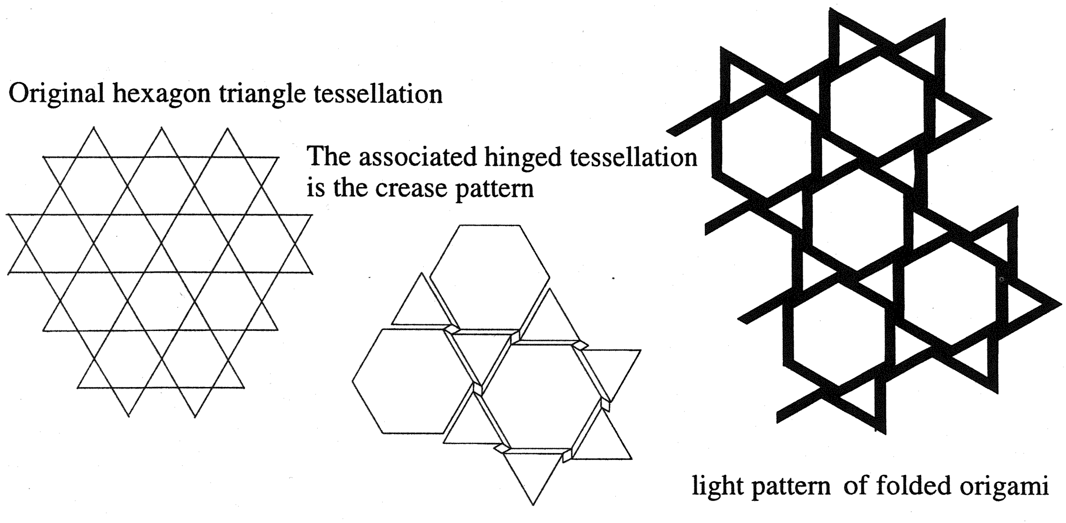


FIGURE 7. From a tiling to a hinged tiling to a light pattern of an origami tessellation

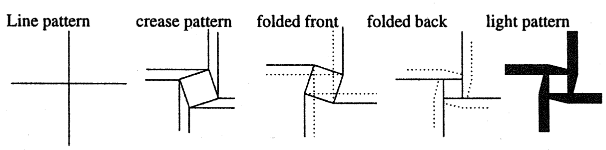


FIGURE 8. From a line pattern to a crease pattern, and the resulting origami.

3.2. Line patterns to pleat patterns. A generalization of the hinged tiling method was developed by Barreto and Palmer (see [BP]). This method can take any line pattern and turn the lines into pleats which can be folded in paper. This method involves adding the dual tiling between the original tiles, and in the case of Archimedean tilings produces the same results as described above. However, there is no restriction on the kind of line pattern this method can be applied to, for instance, Palmer has even folded non periodic, Penrose tilings.

Figure 8 gives an idea of how this method is applied to obtain crease patterns from line patterns. In this example we start with two lines crossing at right angles. The vertex of this line pattern will become a tile of the crease pattern, and the lines become parallel creases as shown. This method, which is described in [BP] can be applied to any line pattern, as long as the lines meet at vertices such that the angles between them is less than 180°. Note, the dotted lines in the third and fourth views of this pattern indicate paper behind, giving an x-ray view. Also strictly speaking the back view would appear as the mirror image of that shown in this diagram, if the front view origami was simply turned over.

3.3. Weaving patterns. In Figure 7 we saw how the hinged tiling method is applied to the hexagon triangle tessellation to produce a crease pattern. Another method of producing a crease pattern from the same Archimedean tiling was shown in Figure 4. I refer to this as a *weaving pattern*, since this origami tessellation was obtained as a solution to the problem of how to fold something that would produce a woven effect.

More generally the idea is to go from lines to creases in the way indicated by Figure 9. This method is described in more detail in [V4].

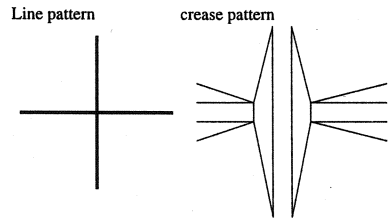


FIGURE 9. Turning a line pattern into a weave like origami crease pattern

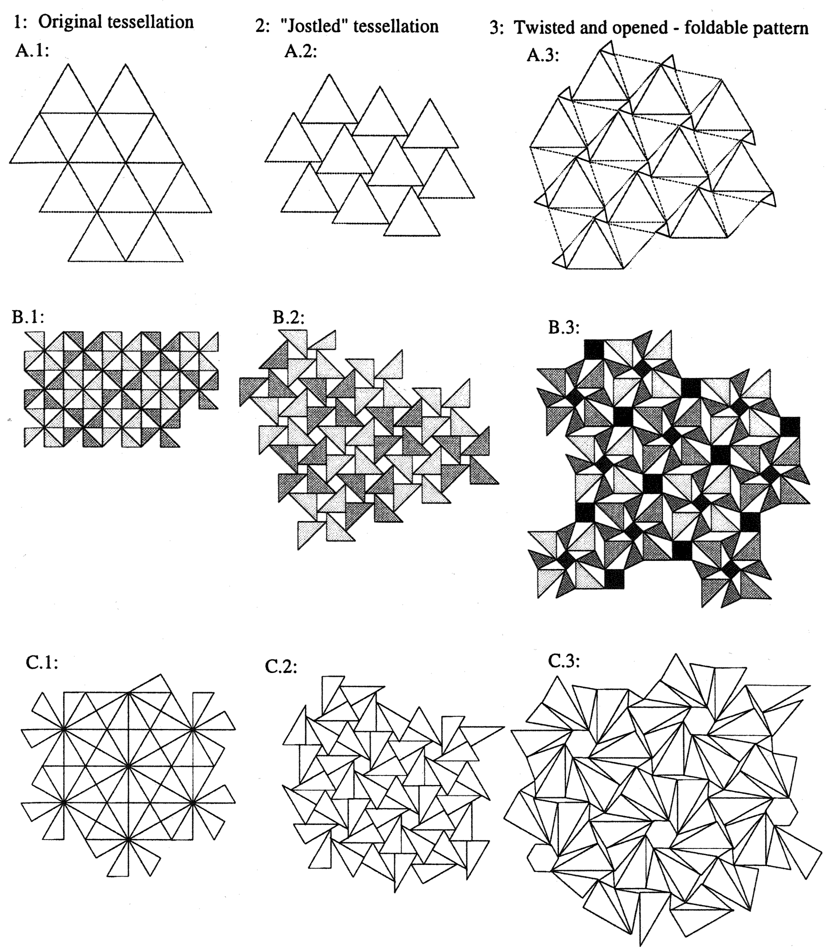


FIGURE 10. Examples of the star method.

3.4. star tessellation. This pattern is produced from certain tilings, in two steps: (1) jostle (2) twist apart. What I mean is shown by example in Figure 10. A more detailed description, and heuristic reason for why this method should give foldable crease patterns, is given in [V4]. I shall refer to this as the *star method*, because of the appearance of the folded results.

4. Additions

Once we have a produced an origami tessellation, more creases may be added, and conversely, some origami tessellations can be reduced to simpler tessellations by removing certain creases. Palmer gives examples in [P2] of combining several simpler origami tessellation patterns to produce more complex designs.

Here we look at several ways creases can be added to the *square twist* origami tessellation, which is the crease pattern produced by the hinged tiling (Figures 2, 3, 5, and 8). With slight modifications these methods should be applicable to other origami tessellation patterns.

The patterns of Figure 11 modify the square twist pattern to produce a spiraling effect on each of the twisted squares.

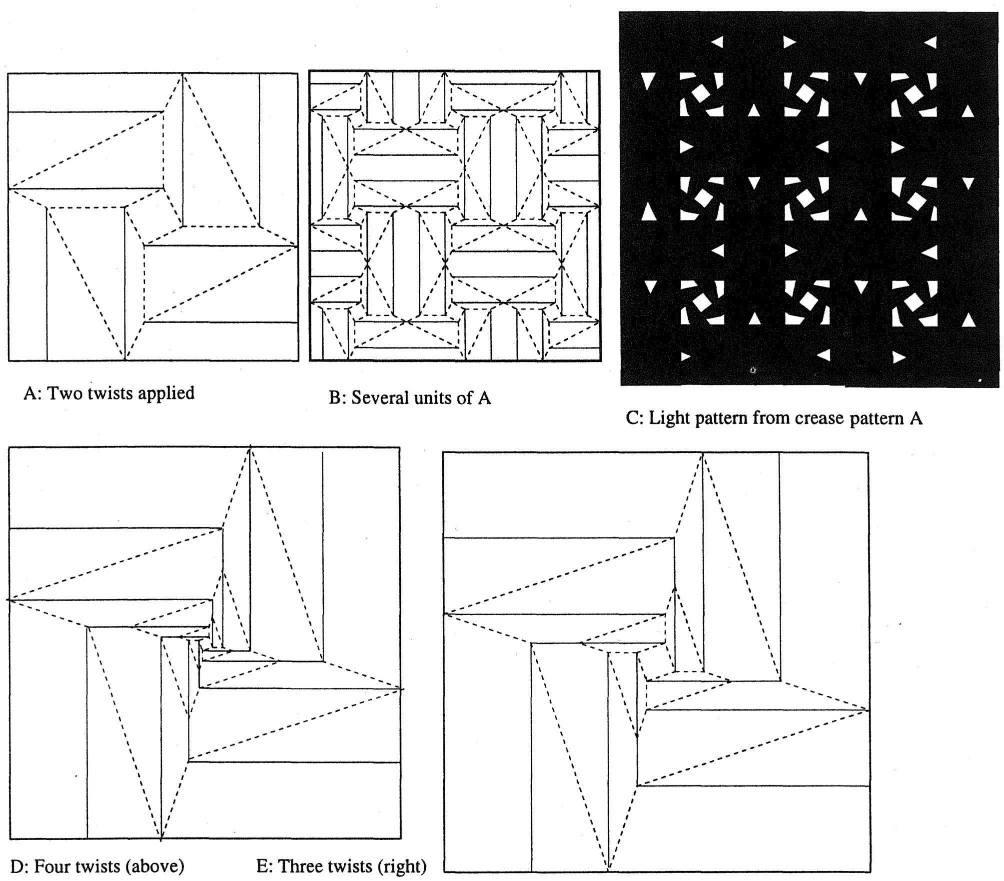


FIGURE 11. Spiral Pattern (A,B,D,E: crease patterns; C: light pattern.)

The spiral method is applied to the tiles of a tessellation. Another possible approach is to modify the pleats between tiles, as in the examples in Figure 12. There are various relationships between the different patterns in this figure, which the reader is invited to investigate.

It would be nice to find a classification of all pleatings of edges of an origami tessellation. The possibilities for the case of parallel creases has been investigated in [BP].

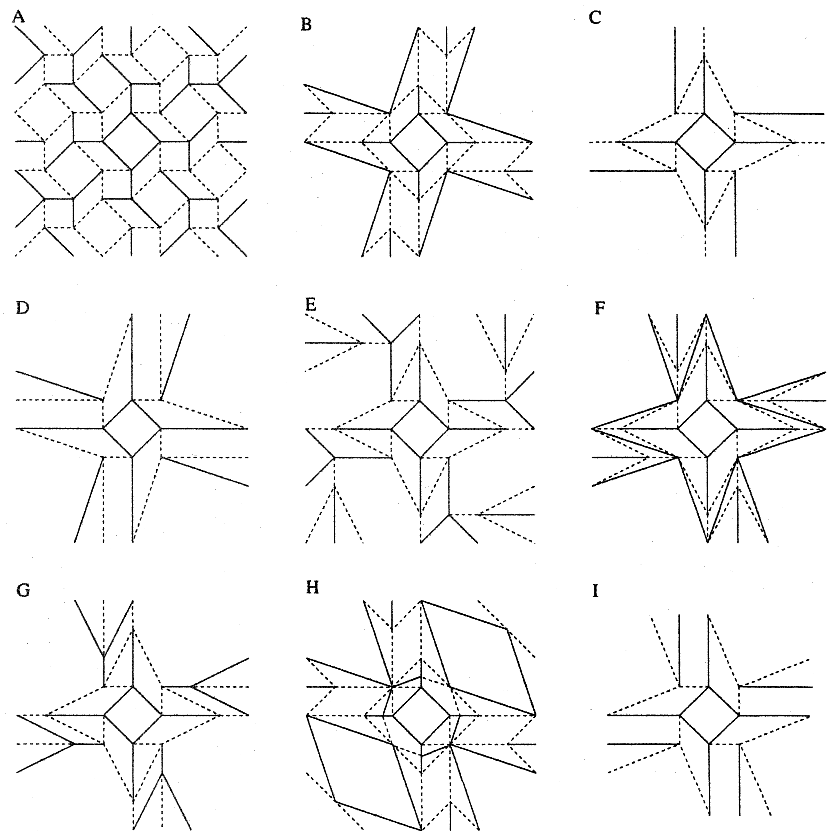


FIGURE 12. Variations of pleats based on the square twist tessellation

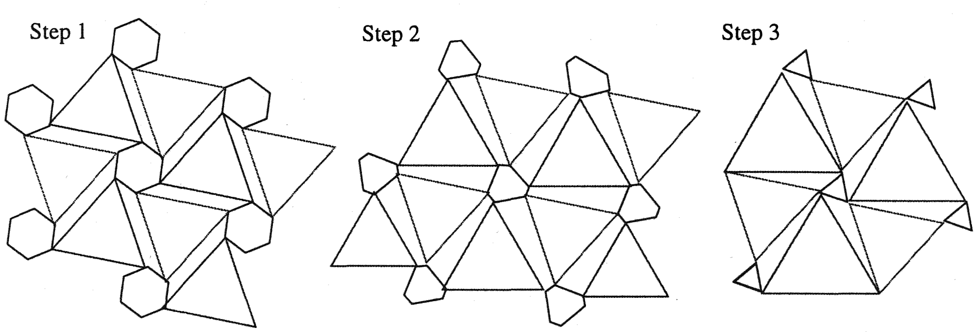


FIGURE 13. From hinged tilings to star patterns.

5. Relationships between the methods of producing origami tessellations

There are relationships between all the methods described in section 3.

For example, the crease pattern in Figure 4 is a distortion of that in Figure 7. The small diamond shapes of the crease pattern in Figure 7 have become the long parallelograms of the crease pattern in Figure 4.

Figure 13 indicates how to distort from a hinged tiling to a star pattern. For certain tessellations, the star method will give rise to exactly the same results as the hinged tiling method.

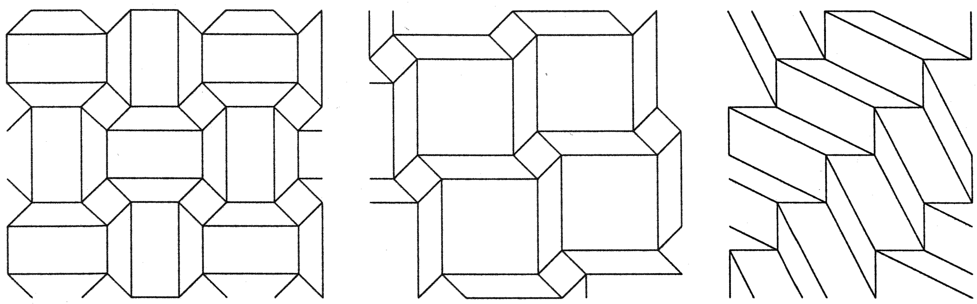


FIGURE 14. Members of three families of crease patterns

6. Parameterizing families of origami tessellations.

In this section we will look at how some families of origami tessellations can be parameterized. We restrict our attention to three examples, in which all the tiles are quadrilaterals, all the vertices have degree four, and all the vertices are identical (up to rotation and reflection). Figure 14 shows representatives of the three families we consider. (Note, it should be possible to obtain a family for each of the 17 wall paper patterns, with all vertices the same, using (modifications of) patterns from [GS].)

In Figure 15 the orbifolds of these tessellations are shown, with the crease pattern marked. In this figure, first the crease pattern is drawn, with a *fundamental unit* marked. This unit will give the whole tessellation pattern if it is repeated over and over. below the tessellation, that unit is given, with identifications marked at the boundaries, and order of rotations marked at corners. This indicates how the fundamental unit is to be repeated. If the fundamental unit is glued to itself according to these identifications, we obtain what is known as an *orbifold*. The orbifold is sketched below the fundamental unit for two of the examples. In the first case it's something like a cone; in the second case something like a Danish pastry, and in the third something impossible to draw in three dimensional space! For more details of orbifolds see [M].

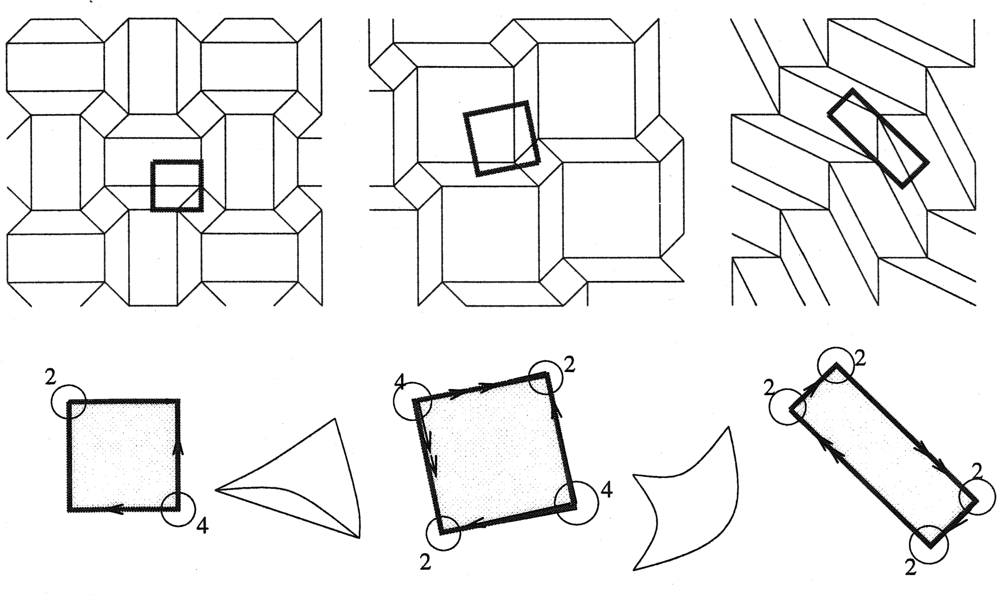


FIGURE 15. orbifolds for crease patterns

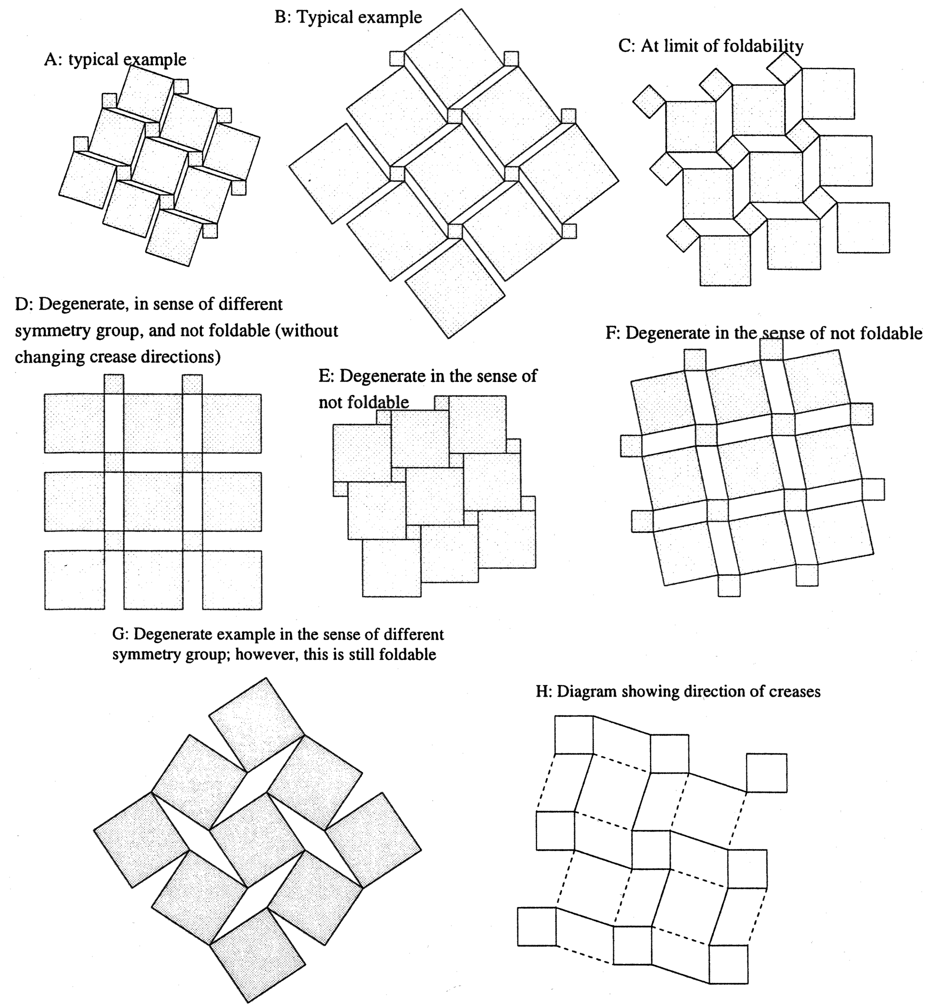


FIGURE 16. Members of the square twist family.

In looking at these families, if two crease patterns are the same up to some rotation, reflection or scaling, etc, then they are considered the same. For all these families, we can use the orbifold to give an explicit description of the geometry of the variations in the family. Here we only consider the first example in detail. This is the hinged tiling of squares, the square twist pattern. For details of the other examples see [V3].

Figure 16 shows several of the members of the first family. In this figure, the first two examples are typical of the foldable examples of this family. Example C is at the limit of foldability, in that further twisting produces non foldable examples, like that in example F. Figure D is not foldable in the same way as the typical examples (though if different crease directions are chosen it is foldable; generally I am not showing crease directions, and thinking about symmetries without regard to this matter, H of this figure shows possible crease directions, marked as different kinds of lines for the two possible different directions.) Example G is foldable, but has a different symmetry group.

In figure 17 the pattern is marked on the orbifold. This is made by drawing squares, with centers at the opposite corners of the square used to make the orbifold. The squares meet at a point on the orbifold. Thus this pattern is determined by just this one point. So the pattern is parameterized by a subset of the orbifold. Since various points give the same pattern (up to allowed symmetries), as shown in Figure 18, we can identify various points of the orbifold, and we get a

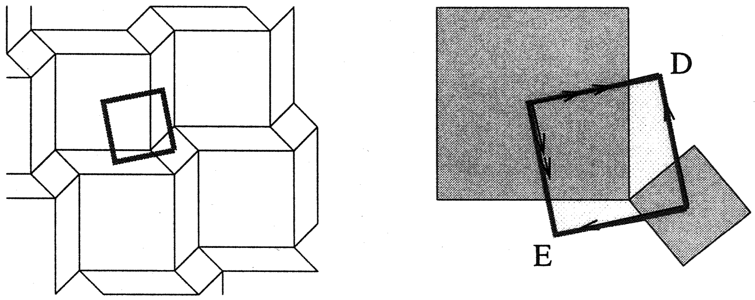


FIGURE 17. Pattern marked on the orbifold

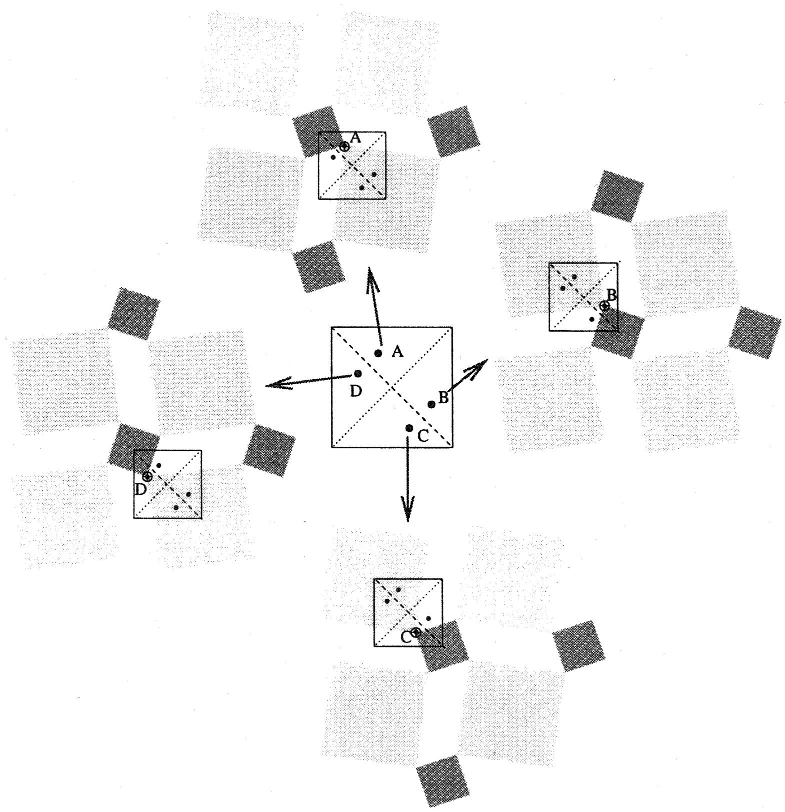


FIGURE 18. Example of how four points of the orbifold correspond to the same crease pattern up to symmetry

manifold to parameterize this kind of origami crease pattern, which comes from a quotient of the orbifold, which generally identifies four points, and is a quarter of the square shown in Figure 19.

In Figure 19, various parts of the manifold for parameterizing this kind of crease pattern are pointed out. Interesting features are are follows:

One) Degenerate cases where both the angle θ and $180° - \theta$, marked in Figure 19, are bigger than 45 degrees. This condition is given in [BP], and is illustrated by Figure 16 examples D and F. This is the shaded region of the diagram in Figure 19.

Two) Degenerate cases where the symmetries are not the generic case; these occur on the line DE of Figure 19. This is exemplified by Example G of Figure 16.

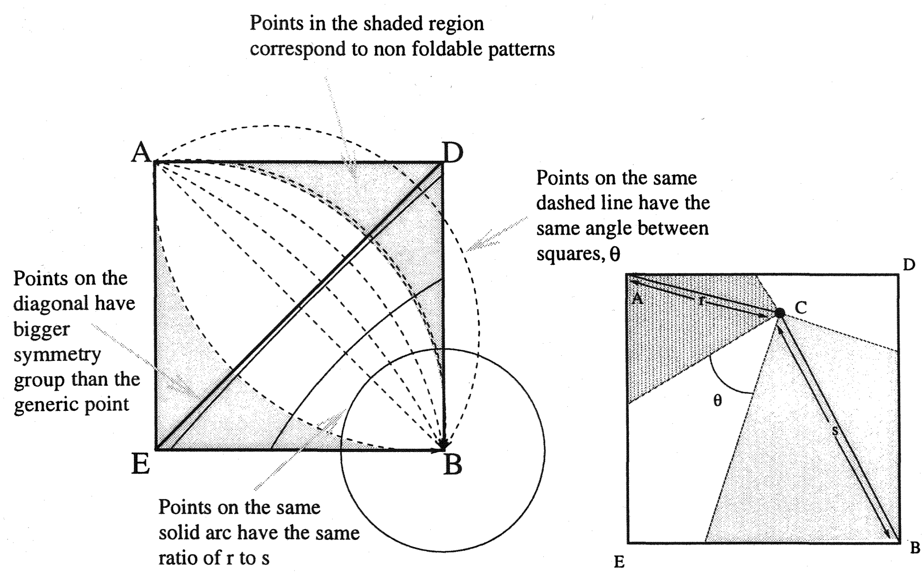


FIGURE 19. Left: Interesting parts of parameterization. Right: Lengths and angles.

Three) Points where the angle θ is constant. Loci of such points are arcs of circles, since the angle between lines AC and AB are constant. They are given by the dashed lines in Figure 19.

Four) Points where the ratio of r to s is constant. Loci of such points are also circles, and are given by the non broken arcs.

Note that in this diagram, the whole of the orbifold is shown for clarity as to what's going on; really only a quarter of this is the space parameterizing this family of crease patterns.

7. Everywhere equal number of layers

Finally, a different kind of origami tessellation puzzle is presented.

This problem is a generalization and modification of one posted on origami-l mailing list by John Smith [Sm].

PROBLEM 1. *(Finite version) For a given fixed n, fold a square so that after folding, any point is identified with exactly $n-1$ other points, (except for points over crease lines).*

Figure 20 shows solutions for $n = 2$ and 3.

PROBLEM 2. $P_\infty(n)$ *(Tessellation version)*

For a given n, fold an origami tessellation so that in the resulting flat folded origami there are exactly n layers of paper everywhere (counting points on creases with appropriate fractional multiplicity). In other words, find a piecewise linear isometry of $\mathbb{R}^2$, invariant under some crystallographic group, which is n to 1 almost everywhere.

Figure 21 shows solutions for $n = 9$ and 15.

THEOREM 7.1. *There is no solution to $P_\infty(n)$ for n even.*

PROOF. See [Sh]. □

CONJECTURE 7.2. There are no non trivial solutions to $P_\infty(n)$ for $n < 7$.

Here trivial refers to solutions where all the lines are parallel to each other. Such "trivial" solutions reduce the problem to that of "fractal dragon curves"; see [H5] for references. In this way, we can also find an almost trivial solution for any composite, such as that for $n = 15$, shown in Figure 21. It would be interesting to find other solutions, especially for primes.

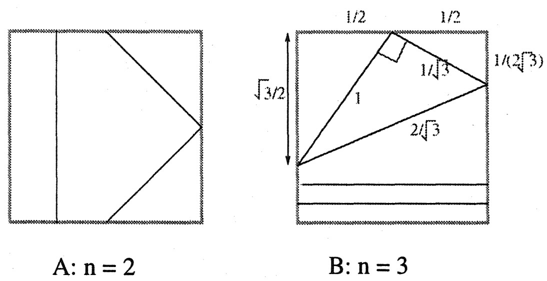


FIGURE 20. Solutions to Problem 1 for n=2,3

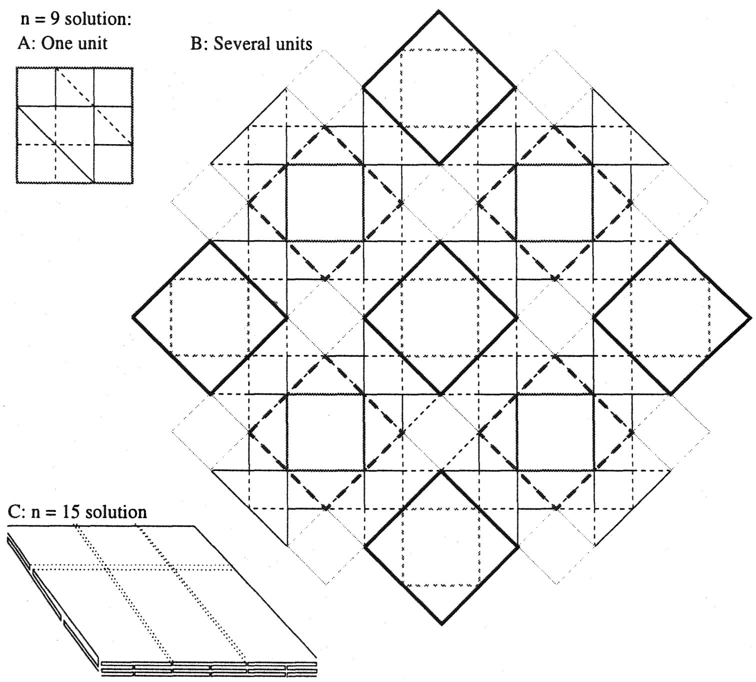


FIGURE 21. Solutions to Problem 2 for n=9,15

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