**BRIDGES**
**Mathematical Connections**
**in Art, Music, and Science**

# Let the Mirrors Do the Thinking

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## 1. Introduction

Our story begins with a simple example. Suppose that someone asked you to keep a record of your thoughts, exactly, and in terms of the symbols given, when you are making an effort to multiply XVI times LXIV. Also suppose that, refusing to give up, you finally arrive at the right answer, which happens to be MXXIV. We are sure that you would have had a much easier time of it, to solve this problem, if you would have found that 16 times 64 equals 1024.

This example not only looks at what we think and what we write. It also looks at the mental tools, the signs and symbols, that we are using when that thinking and that writing is taking place. How we got these mental tools is a long story, one that includes new developments today.

What follows will run a replay of what happened when Europe took several centuries to go from MXXIV to 1024. This replay in not for numbers: it is for logic. Modern logic starts in the middle 1800s and with George Boole. This means that we have had only about 150 years to establish the symbols we now use for symbolic logic.

These symbols leave a lot to be desired. We hope that we can draw you into taking a look at a lesson in lazy logic. If you follow us all the way, we hope to leave your with a new set of symbols, much better than any you have seen yet. Not only will it be easier for you to use them. Even mirrors will be able to use them.

Fig. 1

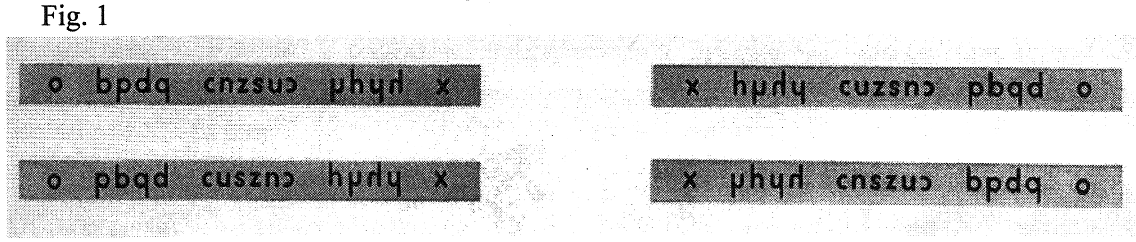


Fig. 2

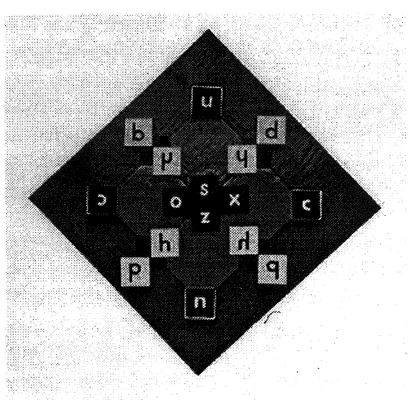
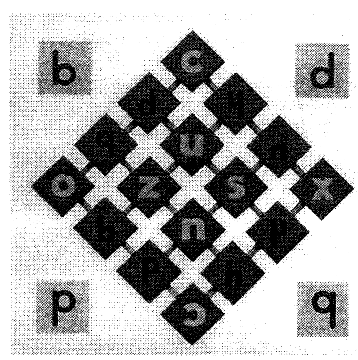


Fig. 3



## 2. Finger Logic

Let us begin with elementary logic and let us do what children do when they learn to count. Use your fingers. Start with any two things (A,B). Then put an A on the left thumb, and a B on the right thumb. Now put one letter on each finger, T for True and F for False. Write the letters so that the four pairs of fingers will be in place and marked from left to right as follows: TT-index, TF-middle, FT-ring, and FF-little.

Finger logic tells us that we have two thumbs (A,B) and four pairs of fingers (TT, TF, FT, FF). The next step is a big step. It is more abstract and it becomes very exact about a fundamental set of relations between the A and B in (A,B). This part of finger logic puts the focus on what are called the 16 binary connectives.

There are 16 ways and only 16 ways that the two fingers in each of the four pairs can touch each other. Call a pair (T)rue when (T)ouching in a pair takes place; if not, (F)alse. For example, (FFFF) has no pairs touching, (TFFF) has the index pair touching, and so forth, for all of the 16 ways. This truth table subdivides into (1 4 6 4 1). It starts with one case of no pairs touching (FFFF) and goes to one case of four pairs touching (TTTT).

Roman numerals start with some of the number values, such as I, V, X, and then the right combinations are used to express any of the in between values, such as IV, VII, IX. Modern logic does the same thing when it starts with a few of the connectives, such as "and" (TFFF), "or" (TTTF), "if" (TFTT), and "equivalence" (TFFT), and then "not" (Negation) is added to this mix to express the other relations in the special 16-set of (A,B) relations (1 4 6 4 1).

Roman numerals are loaded with difficulties because they do not lay bare, in any transparent way, the interrelations among the number values. Notice, instead, that we use Arabic numerals when we build a multiplication table. When modern logic uses "dot, vee, horseshoe" ($\cdot$ v $\supset$) to express "and, or, if," it also does not lay bare the rich web of interrelations that occupy the 16 connectives taken as a total system. Unfortunately, symbolic logic is miles away from coming up with its own multiplication table.

It is obvious that we have such high standards for number symbols; in effect, Arabic numerals. But, squirm as we will, wiggle as we may, it is very odd indeed that the standards we use for logic symbols continue to remain so much lower.

By now the challenge should be clear. It can be put in terms of a working analogy. Roman numerals are to Arabic numerals as the symbols in use for the binary connectives, such as dot-vee-horseshoe, are to what? What follows will introduce you to the "Logic Alphabet". Then comes the part about the mirrors.

Fig. 4

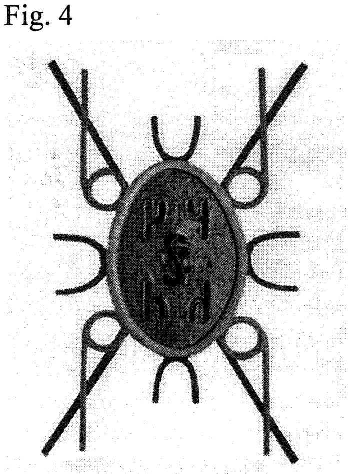


Fig. 5

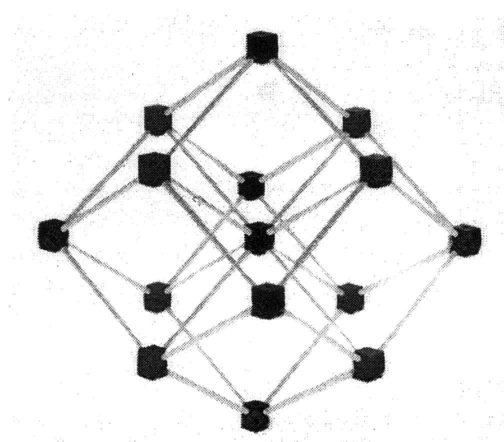


Fig. 6

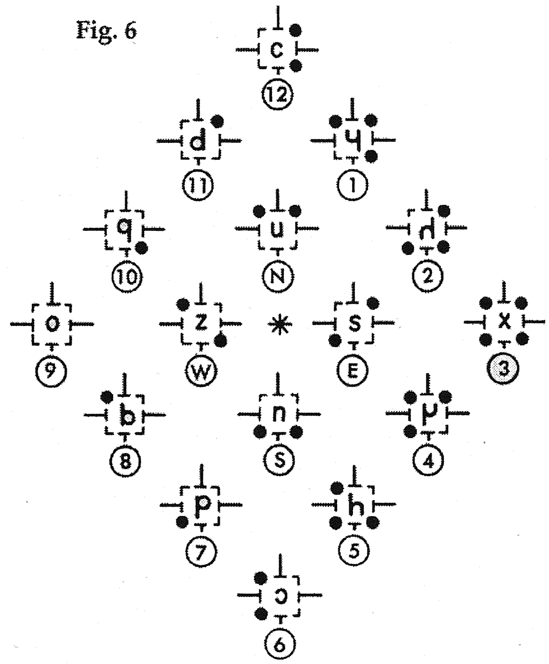



Fig. 7

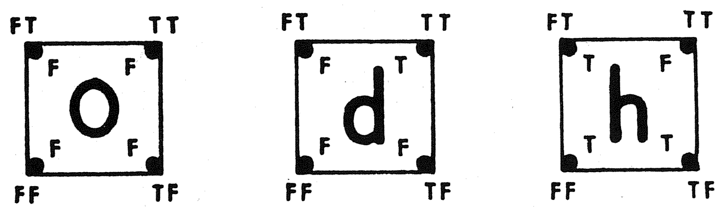



Fig. 8

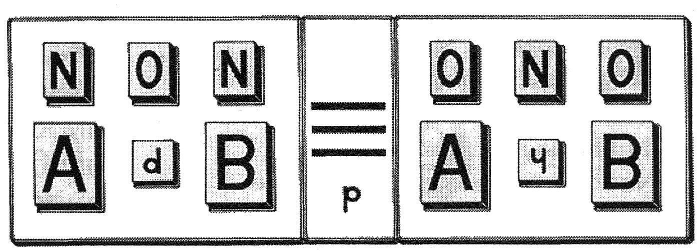
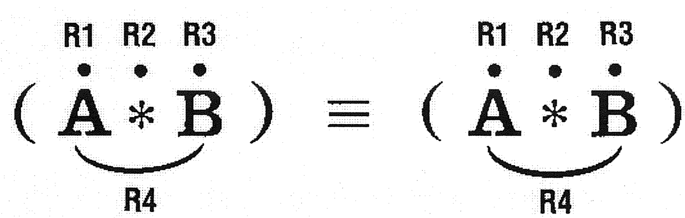



Fig. 9



## 3. Logic Alphabet

In passing notice especially that in 1902 Charles Sanders Peirce (1839-1914), American philosopher and logician, devised a notation that is like the logic alphabet. His manuscript was not published; today it is largely ignored. The second author above devised the logic alphabet 10 years before seeing what Peirce had done.

When we go from Roman numerals to Arabic numerals, the key step is becoming acquainted with the code. Such a key step repeats as follows. Fig. 6 displays the logic alphabet and Fig. 7 specifies the code.

Let LS stand for Letter-Shape and then go directly to (A and B), also written as (A TFFF B). Now use this connective as a model for what we have in mind for all of the 16 connectives when we treat them as a total system, with all parts present and no parts inactive. In (1), begin by seeing that (A and B) is equivalent to itself, (A and B).

(1) (A and B) ≡ (A and B)
(2) (A and B) ≡ (A $\begin{smallmatrix}F&T\\F&F\end{smallmatrix}$ B)
(3) (A and B) ≡ (A ◳ B)
(4) (A and B) ≡ (A d B)
(5) (A * B) ≡ (A * B)

In (2), the pair in which the Touching takes place puts a T in the upper-right corner. In (3), a basic square puts an enlarged dot in the same corner. In (4), the LS has a stem in the same corner. Note especially that (2), (3), and (4) have become the main parts in a triple isomorphism. The "and" in (1) is repeated three times: as a square truth table, as a matching dot-square, and as a corresponding stem-shape. Serving as a simple cursive in a new notation, this iconic d-letter not only stands for conjunction (TFFF). It is easy to remember; it is the last letter of "and."

**Table I**

| A*B | 1<br>o | 2<br>p | 3<br>b | 4<br>q | 5<br>d | 6<br>c | 7<br>u | 8<br>s | 9<br>z | 10<br>n | 11<br>ɔ | 12<br>h | 13<br>μ | 14<br>ᖹ | 15<br>ɥ | 16<br>x |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| OOOO | o | p | b | q | d | c | u | s | z | n | ɔ | h | μ | ᖹ | ɥ | x |
| OONO | o | b | p | d | q | c | n | z | s | u | ɔ | μ | h | ɥ | ᖹ | x |
| NOOO | o | q | d | p | b | ɔ | u | z | s | n | c | ᖹ | ɥ | h | μ | x |
| NONO | o | d | q | b | p | ɔ | n | s | z | u | c | ɥ | ᖹ | μ | h | x |
| NNNO | x | h | μ | ᖹ | ɥ | c | u | z | s | n | ɔ | p | b | q | d | o |
| NNOO | x | μ | h | ɥ | ᖹ | c | n | s | z | u | ɔ | b | p | d | q | o |
| ONNO | x | ᖹ | ɥ | h | μ | ɔ | u | s | z | n | c | q | d | p | b | o |
| ONOO | x | ɥ | ᖹ | μ | h | ɔ | n | z | s | u | c | d | q | b | p | o |
| OOOC | o | p | q | b | d | u | c | s | z | ɔ | n | h | ᖹ | μ | ɥ | x |
| OONC | o | q | p | d | b | u | ɔ | z | s | c | n | ᖹ | h | ɥ | μ | x |
| NOOC | o | b | d | p | q | n | c | z | s | ɔ | u | μ | ɥ | h | ᖹ | x |
| NONC | o | d | b | q | p | n | ɔ | s | z | c | u | ɥ | μ | ᖹ | h | x |
| NNNC | x | h | ᖹ | μ | ɥ | u | c | z | s | ɔ | n | p | q | b | d | o |
| NNOC | x | ᖹ | h | ɥ | μ | u | ɔ | s | z | c | n | q | p | d | b | o |
| ONNC | x | μ | ɥ | h | ᖹ | n | c | s | z | ɔ | u | b | d | p | q | o |
| ONOC | x | ɥ | μ | ᖹ | h | n | ɔ | z | s | c | u | d | b | q | p | o |

The external code is Fig. 7 puts (TT, TF, FT, FF) at the corners of the all-common basic square. The internal code puts a sTem so that it comes close to Touching a corner, when the pair in the same corner is to be counted as True. See that the h-letter (FTTT) has positions that are opposite exactly to where the d-letter (TFFF) does and does not have stems. In (5), as a special act of abstraction and as a key step in all that follows, we come to the asterisk in general (A * B). It has become an algebraic symbol; it covers the full set, any of the 16 connectives in Fig. 6. This blueprint exposes us to the logic alphabet. It has 16 symbols, all cursives, arranged from left to right, in Fig. 1. Again, to repeat the same pattern (1 4 6 4 1), the o-letter has no stems (no pairs touching), and the x-letter has four stems. These LSs have been selected very carefully so that, as a total set and acting as a new notation, they will have shapes that are constucted to encode some fundamental properties in algebra, in geometry, and in symmetry, and so that, to form a good fit, they will be very sensitive to the <u>interrelations</u> among the 16 binary connectives.

In effect, the logic alphabet introduces us to a shape alphabet in a spatial logic, one that does its best to favor what happens when we make logical operations. This is what makes the LSs like Arabic numerals. When we multiply Arabic numerals, we use symbols in such a way that the rules fit the calculations. Likewise, in what follows we will let the logic alphabet focus on <u>four rules</u>. By design, these rules will fit the logical operations. Now we are ready to present the game of <u>flip</u>-<u>mate</u>-<u>flip</u> and <u>flip</u>. This game, also called f-m-f and f, is a shorthand way of saying what the <u>four rules</u> do to each of the LSs in particular, such as (A d B), or to (A * B) in general. R is for Rule, N is for Negation, and C is for Conversion. The C, also called Commuting, reverses the two sides, from (A,B) to (B,A), and vice versa. R1 negates A. The NA <u>flip</u> is from left to right; d changes to b. R2 negates the LS. N* is the <u>mate</u> of any LS because all stem places are reversed; d changes to h. R3 negates B. The NB <u>flip</u> is from top to bottom; d changes to q. R4 converts (A,B). This <u>flip</u> is along the dot-square diagonal that goes from upper-right to lower-left; d remains d because the dot in the d-stem dot-square in (3) stays in place. This example is testing the system. Obviously d must remain d because conjunction in commutative.

All of this forces the same kind of behavior on the logic alphabet. The f-m-f and f rules will become <u>four mirrors</u> and the 16 LSs will be forced to act <u>all alike</u>. This will lead to consequences that are rich and very deep. What follows gives only a small part of the story.

## 4. Logic Bug, Mirror Logic, and More

Let us begin with Table I. It is the new multiplication table; it goes with the logic alphabet. The (0000) is acting as a combinatory identity. The 16 combinations of the four rules (NNNC), which line up with the 16 combinations of the <u>four mirrors</u>, f-m-f and f in that order, are acting on all of the 16 LSs (A * B) along the top. An easy way to learn about Table I is to notice that Arabic numerals encourage the use of models, such as an abacus and a slide rule. The Romans, in spite of their numerals, did use an abacus; they did not, however, have a slide rule. Modern logic does not have any models that work with the symbols that it uses. The logic alphabet, in contrast, is a notation that stumbles all over itself, so many are the models. The lower-left row of LSs in Fig. 1 occupies the home-row of a Flipstick. It repeats the first row of Table I, and it has another set of LSs on its backside, in see-through placement. R1 flips it to the right, R3 flips it upward, and (R1,R3) rotates it around to the upper-right. Note especially that a flip-stick, taken as a single unit, is acting on all of the 16 LSs, all of them at the same time, such that flipstick logic will be in keeping with a small number of symmetry rules.

Fig. 10

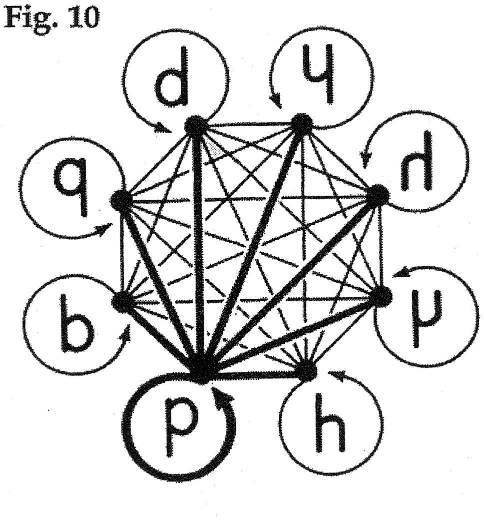


Fig. 11

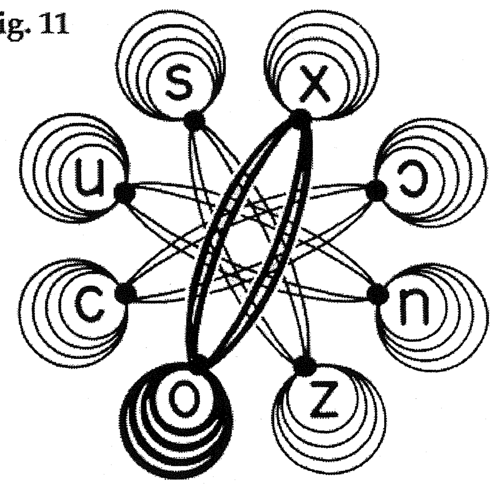



Fig. 12

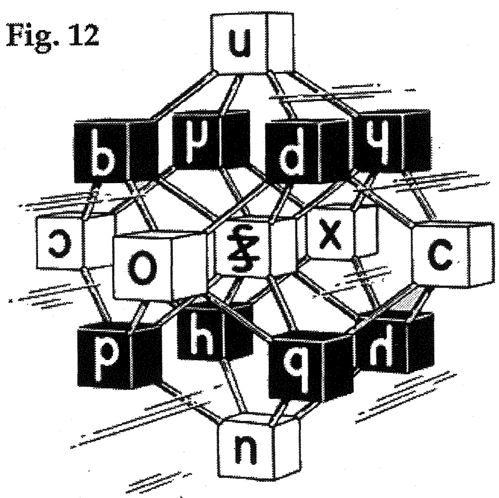
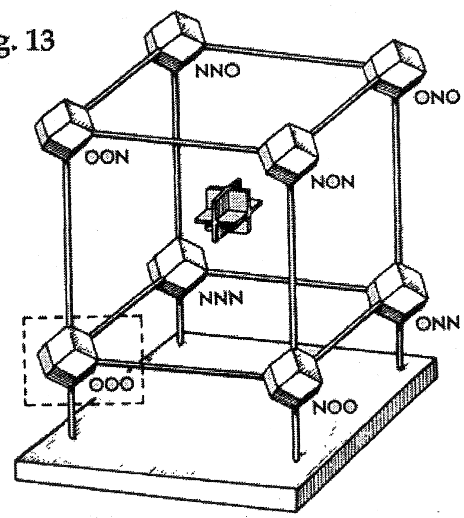



Fig. 13




The total frame in Fig. 2 is a Logic Board. The small blocks also have LSs on both sides. The blocks are single units that are moved to other positions on the board, in keeping with <u>both</u> the positions on the board and the flip-flip-rotate combinations of the small blocks.

The (4 x 4) pattern shown in Fig. 3 is a Clock-Compass. This repeat of Fig. 6 has 12 LSs around the outside, at the positions of a clock, and 4 LSs in the center, at the positions of a compass. The 8 odd-stemmed LSs are tall and black. The 8 even-stemmed LSs are squat and white. Fig. 3 also shows what happens when the four combinations of (R1, R3) act on the four LSs (p,b,q,d). The composite display in Fig. 4 is a Logic Bug. Unlike Fig. 1 but like Fig. 3, it is a 2-D arrangement of the 16 LSs. Like Fig. 1, it can also be subjected to the flip-flip-rotate symmetry changes.

Now we can make a statement for R2. Look symmetrically across the center in these models to find the mate of any LS. This applies not only to Figs. 1-4, but also, as we will see, to Figs. 5, 12, and 13.

Figs. 10 and 11 go with the top half of Table I. Fig. 10 is for the odd-stemmed LSs; Fig. 11 is for the even-stemmed LSs. In Fig. 12, the LSs from Fig. 10 occupy a cube and those from Fig. 11 an octahedron. These two geometric solids interpenetrate in Figs. 5 and 12.

The 16 LSs in Fig. 12 are at the vertices of a Logical Garnet. In a league all by itself, this model is a shadow rhombic dodecahedron. The 16 vertices of a 4-cube have been shadowed, or projected, into 3-D. This model absorbs all of the f-m-f and f symmetry properties, the same ones that fall in place, that line up with, that are isomorphic to the interrelations that inhabit the 16 connectives.

Now let us look at mirror logic. R2 goes through the center of Fig. 12. The three mirrors (R1, R2, R3) could also be put at the right angles to each other. The unmarked garnet in Fig. 5 works either way. Both ways generate the 8-cell of garnets shown in Fig. 13. This is for the top half of Table I. R4 includes the bottom half of Table I. It adds a diagonal mirror to these figures. Like the 8 garnets in Fig. 13, a model for the 16 rows in Table I would have 16 garnets. A model that allows (* A B) and (A B *), along with (A * B), would have 48 garnets. This carries us head on into the crystallography of logic.

Mirror logic can also be cast as transformational logic. Fig. 8 is made from a set of movable blocks that form and resolve equivalences. It shows one of De Morgan's laws. In "dot,vee", (NA • NB) ≡ N(A v B). In words, it says (Not-A and Not-B) is equivalent to (A Not-or B). In Fig. 8, use the mirrors to show that it is "balanced" (valid). When we rotate on the left (NON) and mate on the right (ONO), both sides become (A p B): in words, (neither A nor B); in finger pairs, (FFFT). Again both as a special act of abstraction and as a key step in the present approach, Fig. 9 feeds combinations of the mirrors to a generalized equivalence. R1-R2-R3 are active when Ns enter and leave the over-dots. R4 is active when the under-arcs reverse the (A,B) order. Fig. 9 is a master equivalence, one that covers 4096 atomic equivalences. All and exactly all of these substitutions count as well-formed formulas, when f-m-f and f act on both sides of Fig 9.

How about an optical computer? It is easy enough to say, "Let the fingers do the walking." By now it is obvious, we hope, that, when we construct the logic symbols with great care, it will be easy to say, "Let the mirrors do the thinking."

One more nudge. Suppose that you have lived all of your life back in the age of Roman numerals. Also suppose that one day quite by accident you read an 7-page summary of Arabic numerals. How would you have reacted to that much change? We suggest that what you now know about the logic alphabet leaves you very much in the same position.

Photographs were taken by Fred Cockrill and drawings were made by Warren D. Tschantz. Figs. 6, 7, 12, and 13 are repeats from Semiotica, 1982, 38, 17-54.

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